Problems on Low-dimensional Topology, 2019

Edited by T. Ohtsuki[1]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 22–24, 2019.

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1 Crossing numbers of composite knots and spatial graphs

(Benjamin Bode)[2]

This section is a collection of problems that relate to the question of the additivity of the crossing number under the connected sum operation, *i.e.*, for two knots K_1 and K_2 is $c(K_1\#K_2) = c(K_1) + c(K_2)$? This problem is extremely difficult, but the related problems I am going to discuss could give some insights on the crossing number of $K_1\#K_2$ without us having to tackle the conjecture itself.

Malyutin [23] showed that if $c(K_1\#K_2) \geq c(K_1)$ for all knots K_1, K_2 or if $c(K_1\#K_2) \geq \frac{2}{3}(c(K_1) + c(K_2))$ for all knots K_1, K_2, then hyperbolic knots are not generic, meaning that the percentage of hyperbolic knots amongst all of the prime knots of n or fewer crossings approaches 100 as n approaches infinity, a conjecture that was widely believed to be true until then. A positive answer to some of the questions below would disprove this conjecture.

Let $S \subset S^3$ be diffeomorphic to the standard 2-sphere S^2 and denote the two balls that are bounded by S by B_1 and B_2. Let A and B be distinct points on S. For a fixed projection direction we are interested in the number of crossings of a simple path from A to B that lies completely in one of the B_is. We say a path $\gamma : [0,1] \to B_i$ with $\gamma(0) = A$, $\gamma(1) = B$ and $\gamma(s) \neq \gamma(t)$ for all $s \neq t \in [0,1]$ is *minimal* in B_i if $c(\gamma) \leq c(\gamma')$ for all such paths γ'. A minimal path γ_i in B_i might not be unique (not even up to isotopy). Figure 1a) shows an example of S and minimal paths γ_i in B_i. We can see that one of them, γ_2, is isotopic to a path in S.

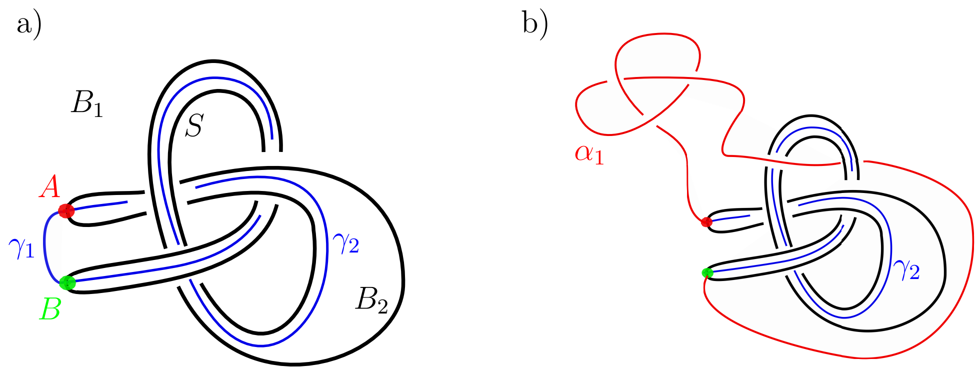

Figure 1: a) The path γ_1 is minimal in B_1 with $c(\gamma_1) = 0$. The path γ_2 is minimal in B_2 with $c(\gamma_2) = 3$. The path γ_2 is isotopic to a path in $S = \partial B_2 = \partial B_1$, while γ_1 is not. b) The path γ_2 is minimal in B_2 with respect to α_1 with $c(\gamma_2 \cup \alpha_1) = 6$ and is isotopic to a path in S.

Question 1.1 (B. Bode). *Given S, A, B and a projection direction, is it always true that there is an $i \in \{1,2\}$ and a path γ_i that is minimal in B_i and that is isotopic (with fixed endpoints) to a path in S?*

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This question becomes more relevant to the additivity of the crossing number if we change the situation slightly.

In addition to S, A, B and the fixed projection direction we are now also given two simple paths, $\alpha_i : [0,1] \to B_i$ from A to B. We say a simple path $\gamma : [0,1] \to B_i$ from A to B is *minimal in* B_i *with respect to* α_j, $j \neq i$ if $c(\gamma \cup \alpha_j) \leq c(\gamma' \cup \alpha_j)$ for all such paths γ'. An example is depicted in Figure 1b).

Question 1.2 (B. Bode)**.** *Given S, A, B, α_i and a projection direction, is it always true that there is an $i \in \{1,2\}$ and a path γ_i that is minimal in B_i with respect to α_j, $j \neq i$ and that is isotopic (with fixed endpoints) to a path in S?*

A positive answer to this question would imply that $c(K_1 \# K_2) \geq \min\{c(K_1), c(K_2)\}$ and would therefore show that hyperbolic knots are not generic in the sense of [23].

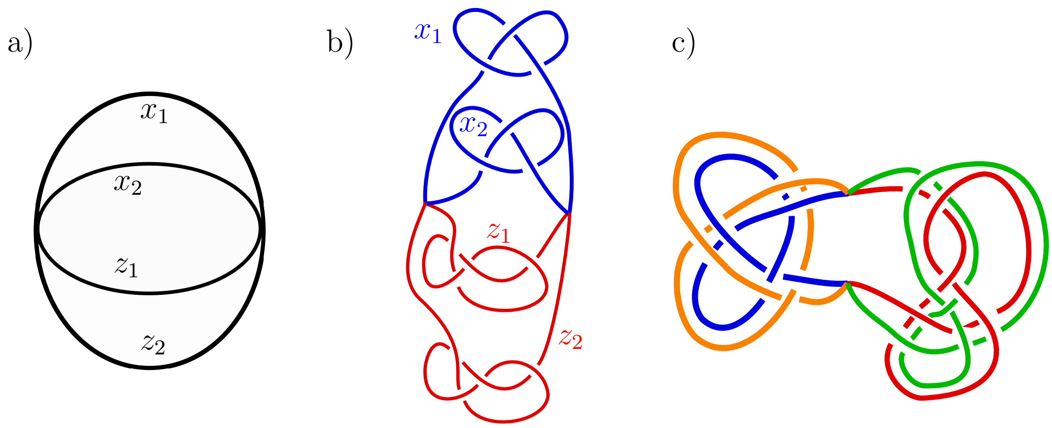


Figure 2: a) The planar graph θ^2. b) The spatial graph $\theta^2_{3_1,4_1}$. c) An element of $\Omega^2_{3_1,4_1}$, which is not $\theta^2_{3_1,4_1}$. Here the constituent knots are $3_1\#4_1$ and 3_1 (each twice).

The next question aims at connections between the crossing numbers of composite knots and spatial graphs as in [4]. Let θ^2 be the planar graph with two vertices that are connected by four edges, shown in Figure 2a). We label the edges by x_1, x_2, z_1 and z_2 in no particular order. Let $\theta^2_{K_1,K_2}$ be the spatial graph that results from tying the x-edges into the knot K_1 and the z-edges into K_2 as in Figure 2b). The unions of any pair of distinct edges of this spatial graph form knots, the constituent knots of the spatial graph, namely either $K_1\#K_2$ (x_i with z_j), $K_1\#K_1$ (x_i with x_j) or $K_2\#K_2$ (z_i and z_j). Let $\Omega^2_{K_1,K_2}$ be the set of isotopy classes of embeddings of θ^2 such that its constituent knots are as follows:

- $x_i \cup z_j = K_1\#K_2$ for all $i, j = 1, 2$,
- $x_i \cup x_j = K_1\#K_1$ and $z_i \cup z_j = K_2\#K_2$ or both are prime. (In particular, they are not unknots.)

From this definition it is clear that $\theta^2_{K_1,K_2} \in \Omega^2_{K_1,K_2}$.

Question 1.3 (B. Bode)**.** *Is $c(\theta_{K_1,K_2}) \leq c(\Gamma)$ for all $\Gamma \in \Omega^2_{K_1,K_2}$?*

It follows from [4] that a positive answer to this question would imply that

$$c(K_1\#K_2) \geq \frac{1}{4}(c(K_1\#K_1) + c(K_2\#K_2)). \tag{1}$$

Note that we do not know $c(\theta^2_{K_1,K_2})$, since we do not know if the diagram in Figure 2b) is minimal.

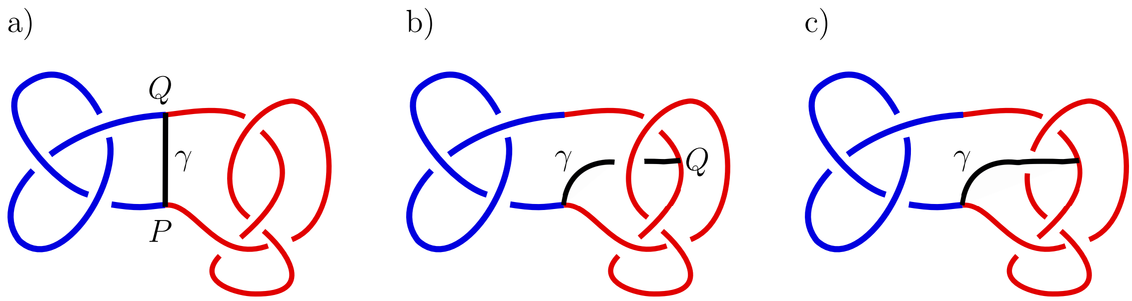


Figure 3: a) For this choice of P and Q the black arc γ minimizes $c(D')$ because it does not add any crossings to D. The constituent knots of this theta curve are $3_1\#4_1$, 3_1 and 4_1. b) For a different choice of P and Q the black curve has to intersect D somewhere. The shown arc does so the minimal number of times possible. The constituent knots of this theta curve are $3_1\#4_1$, 3_1 and 4_1. c) Changing the signs of crossings between the black arc and the diagram D does not change the fact that γ minimzes $c(D')$ for this choice of endpoints P and Q. It can change the constituent knots however. Here they are $3_1\#4_1$, 3_1 and the unknot.

Let D be a knot diagram of a composite knot $K_1\#K_2$. Pick an arbitrary point P on D. For any additional point Q on D any simple path γ from P to Q that does not intersect $K_1\#K_2$ turns the knot diagram into a diagram $D' = D \cup \gamma$ of a spatial graph (a theta-curve θ), one of whose constituent knots is $K_1\#K_2$.

Let γ' be a path that minimizes $c(D')$ among all simple paths from P to Q that do not intersect $K_1\#K_2$. Again this minimizer is usually not unique. An example for two different choices of P and Q, where D is the minimal diagram of $3_1\#4_1$ is given in Figure 3.

Question 1.4 (B. Bode)**.** *Given any diagram D of $K_1\#K_2$, does there exist a choice of P and Q such that there is a simple path γ' from P to Q that does not intersect $K_1\#K_2$, that minimizes $c(D')$ and such that the constituent knots of θ are $K_1\#K_2$, K_1 and K_2?*

A positive answer to this question would imply that $c(K_1\#K_2) > \frac{2}{3}(c(K_1) + c(K_2))$ and therefore also show that the conjecture on the genericity of hyperbolic knots is false [23].

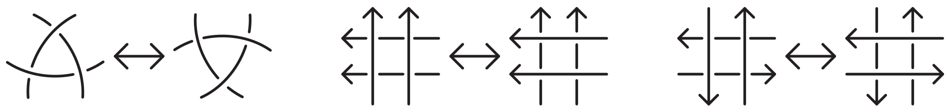
2 Correspondence between local moves and invariants of virtual knots

(Shin Satoh[3])

The set of welded knots is a quotient of the set of virtual knots, by the move "overcrossings commute". It is known (see [37]) that the set of classical knots is a proper subset of the set of welded knots, and hence a proper subset of virtual knots.

Some invariants of classical knots correspond to some local moves in the sense that two classical knots have the same invariant if and only if they are related by a finite sequence of the local moves; see *e.g.* [34, Section 2.8]. For example, the Arf invariant of a classical knot corresponds to pass moves [19, 20]. It is important to study such correspondence because it reveals a relationship between algebraic and geometric structures in classical knot theory.

From this point of view, there are many problems according to which family of knots, which invariant of knots, and which local moves for knots we choose. For example, the delta move, pass move, and sharp move are known to be unknotting operations for welded knots [40], but not unknotting operations for virtual knots [41].



Problem 2.1. *Find invariants of a μ-component virtual link ($\mu \geq 1$) corresponding to the delta move.*

Problem 2.2. *Find invariants of a μ-component virtual link ($\mu \geq 1$) corresponding to the pass move.*

Problem 2.3. *Find invariants of a μ-component virtual link ($\mu \geq 1$) corresponding to the sharp move.*

Problem 2.4. *Find invariants of a μ-component welded link ($\mu \geq 2$) corresponding to the delta move.*

Problem 2.5. *Find invariants of a μ-component welded link ($\mu \geq 2$) corresponding to the pass move.*

Problem 2.6. *Find invariants of a μ-component welded link ($\mu \geq 2$) corresponding to the sharp move.*

In virtual knot theory, there are several invariants such as the n-writhe ($n \neq 0$), the writhe polynomial, the odd writhe, the r-covering ($r \geq 0, r \neq 1$) [29], and the Jones polynomial.

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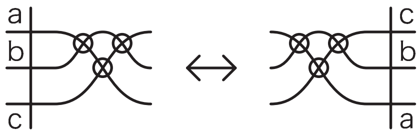
Problem 2.7. *Find local moves for a virtual knot corresponding to the n-writhe* $(n \neq 0)$.

Problem 2.8. *Find local moves for a virtual knot corresponding to the r-covering* $(r \geq 0, r \neq 1)$.

Problem 2.9. *Find local moves for a μ-component virtual link* $(\mu \geq 1)$ *corresponding to the Jones polynomial.*

It is known that the odd writhe of a virtual knot corresponds to a local move called the Ξ-move [41] and the writhe polynomial corresponds to a local move called the shell move (given in our talk of this conference). We find several invariants corresponding to the shell move in the case of a 2-component virtual link [30]. In the figure, the real crossings with the same label have the same crossing information.



Ξ-move

Problem 2.10. *Find invariants for a μ-component virtual link* $(\mu \geq 2)$ *corresponding to the Ξ-move.*

Problem 2.11. *Find invariants for a μ-component virtual link* $(\mu \geq 3)$ *corresponding to the shell move.*

There is known no skein relation for the odd writhe and writhe polynomial of a virtual knot.

Problem 2.12. *Find a skein relation for the odd writhe of a virtual knot.*

Problem 2.13. *Find a skein relation for the writhe polynomial of a virtual knot.*

3 Rectilinear spatial complete graphs

(Ryo Nikkuni)[4]

An embedding f of a finite graph G into $\mathbb{R}^3$ is called a *spatial embedding* of G, and the image $f(G)$ is called a *spatial graph* of G. We call a subgraph γ of G homeomorphic to the circle a *cycle* of G, and also call a *k-cycle* if it contains exactly k edges. We denote the set of all k-cycles of G by $\Gamma_k(G)$, and the set of all pairs of two disjoint cycles of G consisting of a k-cycle and an l-cycle by $\Gamma_{k,l}(G)$. For a cycle γ (resp. a pair of disjoint cycles λ) and a spatial embedding f of G, $f(\gamma)$ (resp. $f(\lambda)$) is none other than a knot (resp. a 2-component link) in $f(G)$. For a cycle γ of G containing all vertices of G, we call $f(\gamma)$ a *Hamiltonian knot* in $f(G)$.

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Let K_n be the *complete graph* on n vertices, that is the graph consisting of n vertices such that each pair of its distinct vertices is connected by exactly one edge. A spatial embedding $f_{\rm r}$ of K_n is said to be *rectilinear* if for any edge e of K_n, $f_{\rm r}(e)$ is a straight line segment in $\mathbb{R}^3$. Such an embedding can be constructed by taking n vertices of K_n on the moment curve (t, t^2, t^3) in $\mathbb{R}^3$ and connecting every pair of two distinct vertices by a straight line segment, see Figure 4 for $n = 6, 7$ (we say such a rectilinear spatial graph of K_n is *standard*). As a consequence of generalizations of the Conway-Gordon theorems [8], Morishita-Nikkuni showed the following formula.

Theorem ([26]). *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding $f_{\rm r}$ of K_n, we have*

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\rm r}(\gamma)\right) = \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\rm r}(\lambda)\right)^2 - \binom{n-1}{5}\right),$$

where lk *denotes the linking number, and a_2 denotes the second coefficient of the Conway polynomial.*

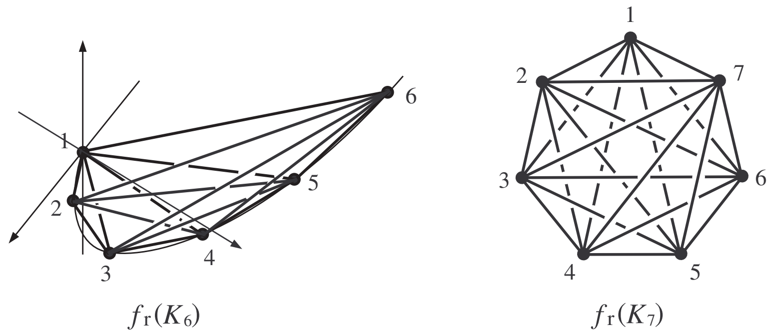

Figure 4: Standard rectilinear spatial graphs of K_n $(n = 6, 7)$

Note that every polygonal 2-component link with exactly six sticks is either a trivial link or a Hopf link. Thus $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\rm r}(\lambda)\right)^2$ coincides with the number of "triangle-triangle" Hopf links in $f_{\rm r}(K_n)$. The original Conway-Gordon theorem for K_6 implies that $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\rm r}(\lambda)\right)^2$ is greater than or equal to the number of subgraphs of K_n isomorphic to K_6, that is equal to $\binom{n}{6}$. On the other hand, it is known that every rectilinear spatial graph of K_6 contains at most three Hopf links [17, 16, 31]. This implies that $\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\left(f_{\rm r}(\lambda)\right)^2$ is less than or equal to $3\binom{n}{6}$. Thus by the theorem, we have the following evaluations of the "algebraic" number of Hamiltonian knots in every rectilinear spatial graph of K_n.

Corollary ([26]). *Let $n \geq 6$ be an integer. For any rectilinear spatial embedding $f_{\rm r}$ of K_n, we have*

$$\frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!} \leq \sum_{\gamma \in \Gamma_n(K_n)} a_2\left(f_{\rm r}(\gamma)\right) \leq \frac{3(n-2)(n-5)(n-1)!}{2 \cdot 6!}.$$

The lower bound in the corollary is sharp for arbitrary $n \geq 6$ [26]. Actually the standard rectilinear spatial graph of K_n realizes the sharp lower bound. In the case of $n = 6$, the upper bound 1 is also sharp. On the other hand, the upper bound in the case of $n = 7$ is 15, but according to a computer search in [18], there seems to be no rectilinear spatial embedding f_{r} of K_7 such that $\sum_{\gamma \in \Gamma_7(K_7)} a_2(f_{\mathrm{r}}(\gamma)) = 13, 15$. This strongly suggests that the upper bound in the corollary is not sharp.

Problem 3.1 (R. Nikkuni)**.** *Determine the sharp upper bound of* $\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_{\mathrm{r}}(\gamma))$ *for all rectilinear spatial embeddings* f_{r} *of* K_n *for each* $n \geq 7$*.*

By the above mentioned theorem, Problem 3.1 is equivalent to the following problem.

Problem 3.2 (R. Nikkuni)**.** *Determine the maximum number of triangle-triangle Hopf links in* $f_{\mathrm{r}}(K_n)$ *for all rectilinear spatial embeddings* f_{r} *of* K_n *for each* $n \geq 7$*.*

4 Instanton Floer theory for 3-manifolds and the homology cobordism group of integral homology 3-spheres

(Yuta Nozaki, Kouki Sato, Masaki Taniguchi)

Instanton Floer theory

The instanton Floer homology group $I_*(Y)$ is an invariant of an oriented integral homology 3-sphere Y introduced by Floer [13]. The group $I_*(Y)$ is an analog of infinite dimensional Morse homology with respect to the Chern-Simons functional. We denote by $\Omega^1(Y) \otimes \mathfrak{su}(2)$ the set of $\mathfrak{su}(2)$-valued 1-forms. The Chern-Simons functional $cs(a)$ is given by

$$cs(a) := \frac{1}{8\pi^2} \int_Y \mathrm{Tr}\,(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

If we fix a Riemann metric g on Y, one can consider the formal gradient of cs with respect to an L^2-metric. There is a large symmetry on $\Omega^1(Y) \otimes \mathfrak{su}(2)$ which is called null-homotopic gauge symmetry defined by

$$\mathcal{G}_Y := \left\{ g \in \mathrm{Map}(Y, SU(2)) \,\middle|\, \deg(g) = 0 \right\},$$

where the degree is the mapping degree. The action is given by $a \cdot g := g^{-1}dg + g^{-1}ag$. One can see the map cs descends to the map

$$cs : \mathcal{B}_Y := \big(\Omega^1(Y) \otimes \mathfrak{su}(2)\big)/\mathcal{G}_Y \to \mathbb{R}.$$

The set of solution to $F(a) = 0$ in $\mathcal{B}_Y$ is denoted by R_Y. Some parts of R_Y correspond to critical values of cs. In a good situation, the chains are generated by some part of R_Y. The differential is defined by counting the solution to the gradient flow of cs. The gradient flows correspond to solutions to ASD-equation on $Y \times \mathbb{R}$.

Although the group $I_*(Y)$ is the first example of Floer homology groups for 3-manifolds, even the following fundamental problem is still open.

Problem 4.1. *Construct a well-defined equivariant instanton Floer homology for $SU(2)$-bundles on all 3-manifolds.*

The main problems are to deal with the reducible solutions and the dependence of perturbations. For example, the dependence of perturbations made in [2] is still open. We also mention a problem related to Floer homotopy types introduced in [7]. It is known that several Floer theoretical invariants of 3 or 4-manifolds are obtained as the singular homology of some topological objects, and the stable homotopy types of the topological objects themselves are invariants of 3 or 4-manifolds. Thus, the homotopy type is called *the Floer homotopy type* ([24, 22]). For the group $I_*(Y)$, its Floer homotopy type has been unknown.

Problem 4.2. *Construct a Floer homotopy type of $I_*(Y)$.*

The main problems to define an instanton Floer homotopy type are related to the bubble phenomena and the existence of structures of manifolds with corners on the compactification of moduli spaces of trajectories and the framings. If the problem is solved, we can apply a generalized cohomology theory and obtain a family of invariants.

Homology cobordism group

Two oriented integral homology 3-spheres Y_1, Y_2 are *homology cobordant* if there exists a cobordism W from Y_1 to Y_2 with $H_*(W;\mathbb{Z}) \cong H_*(S^3 \times I;\mathbb{Z})$. This is an equivalence relation on the set of oriented integral homology 3-spheres, and the quotient set Θ^3 equipped with the connected sum operation is an abelian group called the *homology cobordism group*. It is known [12, 14] that Θ^3 contains a $\mathbb{Z}^\infty$ subgroup, which is generated by Seifert homology 3-spheres. The group Θ^3 has further been studied since various Floer theory for 3-manifolds were established, while we still have several elementary open problems. For instance, the following question is open.

Question 4.3. *Denote by Θ^3_S the subgroup of Θ^3 generated by Seifert homology 3-spheres. Then, is the quotient group Θ^3/Θ^3_S non-trivial?*

Here we mention that the above question is related to our invariant $r_+ \colon \Theta^3 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$; for details of r_+, see [32]. In fact, the value $r_+(Y)$ is contained in $cs(R_Y)$, and if Y is a linear combination of Seifert homology 3-spheres, then $cs(R_Y) \subset \mathbb{Q}$. These imply that if a homology 3-sphere Y has irrational r_+, then its homology cobordism class $[Y]$ is not contained in Θ^3_S. On the other hand, by Mathematica, the authors estimated the value $r_+(S^3_{1/2}(5_2^*))$ with an error of at most 10^{-46}, where $S^3_{1/2}(5_2^*)$ denotes the 3-manifold obtained by 1/2-surgery on the mirror of the knot 5_2, noting that $S^3_{1/2}(5_2^*)$ is a hyperbolic 3-manifold (see [5]). The result seems to imply that $r_+(S^3_{1/2}(5_2^*))$ is irrational. If the value $r_+(S^3_{1/2}(5_2^*))$ is truly irrational, then we can conclude that $[S^3_{1/2}(5_2^*)] \notin \Theta^3_S$.

Question 4.4 (Y. Nozaki, K. Sato, M. Taniguchi)**.** *Is the value $r_+(S^3_{1/2}(5_2^*))$ irrational?*

The method of our computation is based on Kirk and Klassen's formula of cs given by the integration along a path in the space of irreducible $SL(2,\mathbb{C})$-representations. To obtain the approximate value of r_+, we use a description of the space of $SL(2,\mathbb{C})$-representations of $\pi_1(S^3 \setminus 5_2)$, as in [36], in terms of a *Riley polynomial* $\phi(t,u) \in \mathbb{Z}[t^{\pm 1}, u]$ with $\deg_u \phi = 3$. Then we can explicitly solve the equation $\phi(u,t) = 0$ with respect to u and use the solutions to compute r_+. However, Riley polynomials $\phi(t,u)$ of 2-bridge knots K might be of degree larger than 4. In this case, one cannot solve $\phi(t,u) = 0$ in general.

Problem 4.5 (Y. Nozaki, K. Sato, M. Taniguchi)**.** *In the case* $\deg_u \phi > 4$*, give a method to compute an approximate value of* $r_+(S^3_{1/n}(K))$.

Note that, in principle, we can compute approximate values by dividing a path into shorter paths.

5 The AMU Conjecture for self-homeomorphisms of surfaces and the volume conjecture for 3-manifolds

(Tian Yang)

According to Nielsen-Thurston's classification of the elements of the mapping class group of surfaces, every irreducible orientation preserving self-homeomorphism of a surface of finite type is either periodic (of finite order) or pseudo-Anosov (preserving two transverse measure laminations). Here a self-homeomorphism being irreducible means that it does not restrict of a proper subsurface. In [1], Andersen-Masbaum-Ueno made the following

Conjecture 5.1 (J. E. Andersen, G. Masbaum, K. Ueno [1])**.** *Let* Σ *be a orientable surface of finite type, let* ϕ *be a pseudo-Anosov self-homeomorphism of* Σ*, and let* $\{\rho_r\}_r$ *be the sequence of the Turaev-Viro representations of the mapping class group of* Σ*. Then for* r *sufficiently large,* $\rho_r([\phi])$ *is a linear transformation of infinite order.*

Combined with the fact that the image of a finite order element under any group representation is of finite order, the AMU Conjecture essentially claims that the sequence of Turaev-Viro representations of the mapping class groups respects the Nielsen-Thurston classification. The similar conjecture can be made for the Reshetikhin-Turaev representations, which are a sequence of projective representations of mapping class group of surfaces. The AMU conjecture is known to be true for punctured spheres [1, 11] and the once-punctured torus [38]. Recently, Marché-Santharoubane [25] related the Turaev-Viro representations to representations of the fundamental group of surfaces, and provide an efficient algorithm of determining whether an element of the fundamental group can be represented by a simple closed curve on the surface, assuming that the AMU Conjecture is true.

Observed by Santharoubane [39] (see also Detcherry-Kalfagianni [10]), the AMU Conjecture is a consequence of the following a weaker version of the Volume Conjecture of Chen-Yang [6].

Conjecture 5.2 (a weaker version of the volume conjecture of Q. Chen and T. Yang [6])**.** *Let M be a hyperbolic 3-manifold with finite volume, and let $TV_r(M;q)$ be its r-th Turaev-Viro invariant at the root of unity q. Then for r running over all the odd integers,*

$$\liminf_{r\mapsto\infty} \frac{1}{r} \ln TV_r(M; e^{\frac{2\pi i}{r}}) > 0.$$

The relationship between the two conjectures mentioned above is given by the underlying TQFTs. Roughly speaking, the mapping cylinder MC_ϕ of ϕ can be considered as a cobordism from Σ to itself. Hence for each r, the Turaev-Viro TQFT assigns MC_ϕ a linear map, which is exactly $\rho_r([\phi])$ by the construction of the Turaev-Viro representation. By the TQFT axioms, the trace of $\rho_r([\phi])$ equals to the Turaev-Viro invariant $TV_r(M_\phi)$ of the mapping torus M_ϕ of ϕ. Since ϕ is pseudo-Anosov, Thurston's result shows that M_ϕ is hyperbolic. Then Conjecture 5.2 implies that $TV_r(M)$ grows at least exponentially at particular roots of unity. On the other hand, if $\rho_r([\phi])$ was of finite order, the each of its eigenvalues should be a root of unity. As a consequence, the trace of $\rho_r([\phi])$ is at most the dimension of the TQFT vector space of Σ, which by the Verlinde formula is only a polynomial in r. That is a contradiction.

In a recent work [9], Detcherry-Kalfagianni showed that the behavior of the Turaev-Viro invariant is "similar to" that of the hyperbolic volume, in the sense that it does not increase under Dehn-fillings. Therefore, if one could prove Conjecture 5.2 for a 3-manifold M, then Conjecture 5.2 is automatically true for all the 3-manifolds obtained from M by removing a link inside it. Recently, Ohtsuki [35] and Belletti-Detcherry-Kalfagianni-Yang [3] proved the Volume Conjecture of Chen-Yang for infinite families of 3-manifolds, including the closed hyperbolic ones obtained by doing integral Dehn-fillings along the figure-8 knot and the fundamental shadow link complements. Therefore, Conjecture 5.2 hold for all the 3-manifolds obtained from the examples mentioned above by remove a link inside them, and the AMU Conjecture holds for the fibered ones obtained from those examples by doing the same operation. From the discussions above, one sees that a solution to the follow problem will give a final solution to the AMU Conjecture, at least for all the punctured surfaces.

Problem 5.3 (T. Yang)**.** *Find a family of 3-manifolds for which Conjecture 5.2 holds, and by removing links from which one gets all the pseudo-Anosov mapping torus of all punctured surfaces.*

6 The mapping class group of a surface and the quantum invariants of integral homology 3-spheres

(Shunsuke Tsuji)

Let $\Sigma_{g,1}$ be a surface of genus 1 with a connected non-empty boundary. We consider the lower central series $\{F^n\pi_1(\Sigma_{g,1},*)\}_{n\geq 1}$ of $\pi_1(\Sigma_{g,1},*)$ where $*\in\partial\Sigma_{g,1}$, defined by $F^1\pi_1(\Sigma_{g,1},*) \stackrel{\text{def.}}{=} \pi_1(\Sigma_{g,1},*)$ and $F^{n+1}\pi_1(\Sigma_{g,1},*) \stackrel{\text{def.}}{=} [\pi_1(\Sigma_{g,1},*), F^n\pi_1(\Sigma_{g,1},*)]$.

We denote by $\mathcal{M}(\Sigma_{g,1})$ the mapping class group of $\Sigma_{g,1}$ and by $\mathcal{I}(\Sigma_{g,1})$ the Torelli group, which is the kernel of the action of $\mathcal{M}(\Sigma_{g,1})$ on the homology group of $\Sigma_{g,1}$. We can define two filtrations $\{\mathcal{I}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$ and $\{\mathcal{M}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$. The first $\{\mathcal{I}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$ is the lower central series, defined by $\mathcal{I}^{(1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \mathcal{I}(\Sigma_{g,1})$ and $\mathcal{I}^{(n+1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} [\mathcal{I}(\Sigma_{g,1}), \mathcal{I}^{(n)}(\Sigma_{g,1})]$. The second $\{\mathcal{M}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$ is the Johnson filtration, satisfying $\mathcal{M}^{(n)}(\Sigma_{g,1})$ is the kernel of the action of $\mathcal{M}(\Sigma_{g,1})$ on $F^1\pi_1(\Sigma,*)/F^{n+1}\pi_1(\Sigma,*)$.

We denote by

$$z^{\mathrm{sl}_2}(M) = 1 + z_1^{\mathrm{sl}_2}(M)(q-1) + z_2^{\mathrm{sl}_2}(M)(q-1)^2 + \cdots$$

the invariant of an integral homology 3-sphere M defined by T. Ohtsuki [33]. We fix a Heegaard splitting of $S^3 = H_g^+ \cup_\iota H_g^-$, where H_g^+ and H_g^- are handle bodies of genus g and ι is a diffeomorphism from ∂H_g^+ to ∂H_g^-. We denote $M(\psi) \stackrel{\text{def.}}{=} H_g^+ \cup_{\psi\circ\iota} H_g^-$ for $\psi \in \mathcal{M}(\Sigma_{g,1})$, where we consider $\Sigma_{g,1}$ as a submanifold of ∂H_g^+.

Then we obtain $z_i^{\mathrm{sl}_2}(M(\psi)) = 0$ if $\psi \in \mathcal{I}^{(2i+1)}(\Sigma_{g,1})$ for any i. In the case of the Johnson filtration $\{\mathcal{M}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$, there exists $\psi \in \mathcal{M}^{(3)}(\Sigma_{g,1})$ satisfying $z_1^{\mathrm{sl}_2}(M(\psi)) \neq 0$. S. Morita [27] constructs the core of the Casson invariant $d : \mathcal{M}^{(2)}(\Sigma_{g,1}) \to \mathbb{Z}$ where $z_1^{\mathrm{sl}_2}(M(\psi)) = 0$ if $\psi \in \mathcal{I}^{(3)}(\Sigma_{g,1}) \cap \ker d$. In other words, we can define $z_1^{\mathrm{sl}_2}$ using the Johnson homomorphisms and the core of the Casson invariant.

Conjecture 6.1 (S. Morita)**.** *For any $i \in \mathbb{Z}_{\geq 1}$, $z_i^{\mathrm{sl}_2}(M(\psi)) = 0$ if $\psi \in \mathcal{M}^{(2i+1)}(\Sigma_{g,1}) \cap \ker d$.*

By definition, if $i = 1$, the conjecture is true. Morita–Sakasai–Suzuki [28] prove that the conjecture is true if $i = 2, 3$.

We can also define

$$z^{\mathrm{sl}_N}(M) = 1 + z_1^{\mathrm{sl}_N}(M)(q-1) + z_2^{\mathrm{sl}_N}(M)(q-1)^2 + \cdots$$

using the sl_N-quantum group in [21].

Conjecture 6.2 (S. Morita, S. Tsuji)**.** *Fix an integer N larger than 2. For any $i \in \mathbb{Z}_{\geq 1}$, $z_i^{\mathrm{sl}_N}(M(\psi)) = 0$ if $\psi \in \mathcal{M}^{(2i+1)}(\Sigma_{g,1}) \cap \ker d$.*

By definition, if $i = 1$, the conjecture is true. Morita-Sakasai-Suzuki [28] also prove that the conjecture is true if $i = 2, 3$.

We introduce an approach of the conjectures using skein algebras. Let $\{\mathcal{M}^{(n)}_{\mathrm{Kauffman}}(\Sigma_{g,1})\}_{n\geq 1}$ be a filtration of the Torelli group defined using the Kauffman bracket skein algebra. For any $i \in \mathbb{Z}_{\geq 1}$, we have $z_i^{\mathrm{sl}_2}(M(\psi)) = 0$ if $\psi \in \mathcal{M}^{(2i+1)}_{\mathrm{Kauffman}}(\Sigma_{g,1})$. If $\mathcal{M}^{(i)}(\Sigma_{g,1}) \cap \ker d \subset \mathcal{M}^{(i)}_{\mathrm{Kauffman}}(\Sigma_{g,1})$ for any i, the first conjecture is true. Let $\{\mathcal{M}^{(n)}_{\mathrm{HOMFLY-PT}}(\Sigma_{g,1})\}_{n\geq 1}$ be a filtration of the Torelli group defined using the HOMFLY-PT skein algebra. For any N and any $i \in \mathbb{Z}_{\geq 1}$, we have $z_i^{\mathrm{sl}_N}(M(\psi)) = 0$ if $\psi \in \mathcal{M}^{(2i+1)}_{\mathrm{HOMFLY-PT}}(\Sigma_{g,1})$. We remark that $\mathcal{M}^{(i)}(\Sigma_{g,1}) \cap \ker d \supset \mathcal{M}^{(i)}_{\mathrm{HOMFLY-PT}}(\Sigma_{g,1})$. If $\mathcal{M}^{(i)}(\Sigma_{g,1}) \cap \ker d = \mathcal{M}^{(i)}_{\mathrm{HOMFLY-PT}}(\Sigma_{g,1})$ for any i, the second conjecture is true for any N.

7 Positive flow-spines and contact 3-manifolds

(Ippei Ishii, Masaharu Ishikawa, Yuya Koda, Hironobu Naoe)

In this section, M always denotes a closed, oriented, smooth 3-manifold.

A (*positive*) *contact structure* on M is a transversely orientable 2-plane field on M, given as the kernel of a 1-form (called a *contact form*) α on M, where α satisfies $\alpha \wedge d\alpha > 0$. The pair (M, ξ) is called a *contact* 3*-manifold*. Two contact structures ξ_0 and ξ_1 are said to be *isotopic* if there exists a 1-parameter family of contact structures connecting them. For a contact form α, the *Reeb vector field* R_α is defined by $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. We also call R_α a Reeb vector field of the contact structure $\xi = \ker \alpha$. The flow generated by R_α is called the *Reeb flow* of α (or a Reeb flow of ξ). A contact structure ξ is said to be *overtwisted* if there exists a disk D embedded in M such that ∂D is everywhere tangent to ξ and the framing of D along ∂D coincides with that of ξ. Otherwise ξ is said to be *tight*.

A 2-dimensional polyhedron P in M is called a *flow-spine* if

(1) P is a *spine*, that is, $M \setminus P$ is an open 3-ball; and

(2) there exists a non-singular flow $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$ on M such that for each point of P, there exists a positive chart $(U; x, y, z)$ of M around the point such that $(U, U \cap P)$ is diffeomorphic (by an orientation-preserving diffeomorphism) to one of the four models shown in Figure 5, where the flow Φ on U is generated by the vector field $\partial/\partial z$.

Further, a flow-spine P is said to be *positive* if P has at least one point of the model of Figure 5 (iii) and has no point of the model of Figure 5 (iv). In the above setting, we say that the flow Φ is *carried* by P. A contact structure ξ on M is said to be *supported* by a flow-spine P if a Reeb flow of ξ is carried by P.

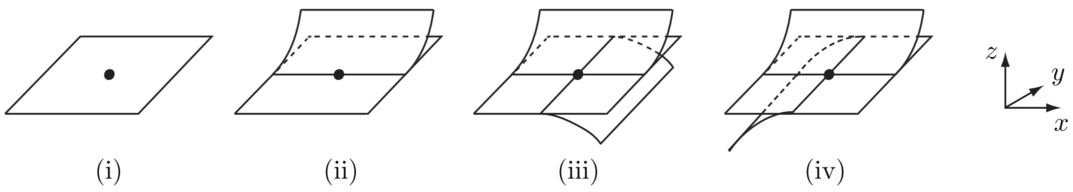

Figure 5: The local models of a flow-spine.

Theorem (I. Ishii, M. Ishikawa, Y.Koda, H. Naoe). *The map*

$$\{\textit{positive flow-spines of } M\}/\textit{isotopy} \to \{\textit{contact structures on } M\}/\textit{isotopy}$$

that takes a positive flow-spine P (up to isotopy) to a contact structure ξ (up to isotopy) whose Reeb flow is carried by P is a well-defined surjective map.

Problem 7.1 (I. Ishii, M. Ishikawa, Y.Koda, H. Naoe)**.** *Find moves for positive flow-spines so that the map*

$$\{\textit{positive flow-spines of } M\}/\textit{moves} \;\to\; \{\textit{contact structures on } M\}/\textit{isotopy}$$

induced from the surjection in the theorem is a bijection.

Apparently, an answer to the above problem completes to give a counterpart of the famous Giroux correspondence [15]. In the Giroux correspondence, it is known that a contact 3-manifold (M, ξ) is Stein fillable if and only if (M, ξ) admits a supporting open book decomposition whose monodromy is a product of right-handed Dehn twists. In particular, ξ is tight in this case.

Problem 7.2 (I. Ishii, M. Ishikawa, Y.Koda, H. Naoe)**.** *Give a criterion for the tightness or Stein fillability of contact structures in terms of supporting positive flow-spines.*

It is known that for any non-singular flow Φ on M, there exists a flow-spine carrying Φ. Further, by the above mentioned theorem, a certain Reeb flow of any contact manifold (M, ξ) is carried by a positive flow-spine.

Question 7.3 (I. Ishii, M. Ishikawa, Y.Koda, H. Naoe)**.** *Is any Reeb flow of any contact manifold (M, ξ) carried by a positive flow-spine?*

A point of a flow-spine P whose neighborhood is shaped on the model (iii) in Figure 5 is called a *vertex* of P. The *complexity* $c(M, \xi)$ of a contact 3-manifold (M, ξ) is defined to be the minimum number of vertices of any positive flow-spine supporting ξ. Note that c is finite-to-one. The classification of contact 3-manifolds of complexity up to 3 is now in progress.

Problem 7.4 (I. Ishii, M. Ishikawa, Y.Koda, H. Naoe)**.** *Classify the contact* 3*-manifolds of complexity* 4*.*

In our classification, it seems that any positive flow-spine with at most 3 vertices supports a tight contact structure. On the other hand, there exists a positive flow-spine of S^3 with 5 vertices supporting an overtwisted contact structure. It is interesting to determine whether there is a positive flow-spine with 4 vertices supporting an overtwisted contact structure or not.

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