Problems on Low-dimensional Topology, 2024

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 22–24, 2024.

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1 Braids, tangles and quandle colorings

(Kodai Wada)

Let $m \geq 2$ be an integer. We consider m-braids with downward orientation. The set $\mathcal{B}_m$ of m-braids in a cylinder forms a group, called the *m-braid group*, under multiplication given by stacking of m-braids. This group is generated by $m-1$ elements $\sigma_1, \ldots, \sigma_{m-1} \in \mathcal{B}_m$ called the *standard generators*, where σ_i represents the m-braid with only one positive crossing between the ith and $(i+1)$th strings (all the other strings go vertically down). The *pure m-braid group* $\mathcal{PB}_m$ is the subgroup of $\mathcal{B}_m$ consisting of pure m-braids.

A *quandle* is a set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying three axioms, which algebraically encode the Reidemeister moves in knot theory. The set Q^m has an action of $\mathcal{B}_m$ from the right defined by, for an element $v = (a_1, \ldots, a_m) \in Q^m$,

$$v \cdot \sigma_i = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i \triangleleft a_{i+1}, a_{i+2}, \ldots, a_m).$$

The pure m-braid group $\mathcal{PB}_m$ acts on Q^m from the right as well as $\mathcal{B}_m$ does. We denote by $\overset{\mathcal{B}}{\sim}$ and $\overset{\mathcal{PB}}{\sim}$ two equivalence relations on Q^m induced from the actions of $\mathcal{B}_m$ and $\mathcal{PB}_m$, respectively.

Problem 1.1 (K. Wada)**.** *Let Q be a quandle.*

(i) *Find a necessary and sufficient condition for two elements v and w in Q^m to be $v \overset{\mathcal{B}}{\sim} w$ (or $v \overset{\mathcal{PB}}{\sim} w$).*

(ii) *Find the orbit decomposition of Q^m under the action of $\mathcal{B}_m$ (or $\mathcal{PB}_m$).*

In our talk, we settle this problem for the case $Q = \mathbb{Z}$ with a binary operation $\triangleleft$ defined by $a \triangleleft b = 2b - a$.

Roughly speaking, an *(m,m)-tangle* (or an *m-string link*) in a cylinder is an m-braid (or a pure m-braid) without monotone property. Let $(Q, \triangleleft)$ be a quandle and D a diagram of an (m,m)-tangle. A *Q-coloring* of D is a map $\mathcal{C} : \{\text{arcs of } D\} \to Q$ satisfying

$$\mathcal{C}(a) \triangleleft \mathcal{C}(b) = \mathcal{C}(c)$$

at each crossing of D, where b is the over-arc of the crossing and a, c are the under-arcs such that the normal orientation of b points from a to c. We remark that the above two actions of $\mathcal{B}_m$ and $\mathcal{PB}_m$ correspond to Q-colorings for m-braid diagrams and pure m-braid diagrams, respectively.

For two elements v and w in Q^m, we write $v \overset{\mathcal{TG}}{\sim} w$ (or $v \overset{\mathcal{SL}}{\sim} w$) if there is a diagram D of an (m,m)-tangle (or an m-string link) admitting a Q-coloring such that the m points on a horizontal line at the top of D receive v, and the m points at the bottom of D receive w. As a generalization of Problem 1.1(i), we propose the following:

Problem 1.2 (K. Wada)**.** *For a given quandle Q, find a necessary and sufficient condition for two elements v and w in Q^m to be $v \overset{\mathcal{TG}}{\sim} w$ (or $v \overset{\mathcal{SL}}{\sim} w$).*

In the setting of virtual knot theory, we can consider similar problems to Problems 1.1 and 1.2.

2 Topological 4-genus of torus knots

(Sebastian Baader)

About 40 years ago, Rudolph observed that Freedman's disk theorem [7] caused a drop in the genus of torus knots, when passing from the smooth to the (locally flat) topological category [15]. Interestingly, this was before the resolution of the Thom conjecture by Kronheimer and Mrowka [11]. The (almost) precise value of the topological 4-genus is currently known for torus knots of braid index $2, 3, 4$ and 6 only [1]. In these cases, the topological 4-genus $g_4(K)$ (almost) coincides with half the maximal value of the signature function

$$\widehat{\sigma}(K) = \max_{\omega \in S^1} |\sigma_\omega(K)|.$$

Question 2.1 (S. Baader)**.** *Does $2g_4(K) = \widehat{\sigma}(K)$ hold for all torus knots K?*

A positive answer to the above question would imply the following asymptotic behaviour of the topological 4-genus:

$$\lim_{n\to\infty} \frac{g_4(T(n, n+1))}{n^2} = \frac{1}{4}.$$

Comparing this with the smooth 4-genus $g_s(T(n, n+1)) = \frac{1}{2}n(n-1)$ would thus imply the asymptotic equation $g_4 = \frac{1}{2}g_s$ for torus knots with large parameters. Recent developments based on McCoy's nullhomologous twisting method [14] imply the asymptotic inequality $g_4/g_s \le \frac{14}{27}$, which is only $\frac{1}{54}$ away from one half. This motivates the following asymptotic version of the above question.

Question 2.2 (S. Baader)**.** *Does $\lim_{n\to\infty} \frac{g_4(T(n, n+1))}{n^2} = \frac{1}{4}$ hold?*

This asymptotic equality has a surprising consequence, when combined with the main result in [2]: every additive topological concordance invariant $\rho(K)$ with the two properties $\rho(K) \le g_4(K)$ and $\rho(T(2, N)) = g_4(T(2, N))$ satisfies

$$\lim_{n\to\infty} \frac{\rho(T(n, n+1))}{g_4(T(n, n+1))} = 1.$$

Indeed, the limit in Question 2.2 together with Corollary 3 in [2] implies asymptotically $\frac{\rho(T(n,n+1))}{g_4(T(n,n+1))} \ge 1$ (caveat, in that paper g_4 stands for the slice genus). In short, all additive topological concordance invariants, which are bounded above by the topological 4-genus, and which are maximal on two-strand torus knots, essentially coincide with the topological 4-genus on torus knots with large parameters. This includes the classical signature invariant divided by two.

3 On the volume conjecture for Turaev-Viro invariants of 3-manifolds

(Renaud Detcherry)

For M a compact oriented 3-manifold, let $\mathrm{Vol}(M)$ be its simplicial volume, that is, the sum of the hyperbolic volumes of its JSJ pieces. Moreover, a sequence of quantum invariants, the Turaev-Viro invariants $TV_r(M,q)$ can be computed from any ideal triangulation of M. The Turaev-Viro invariants depend on an odd integer $r \geq 3$, and a root of unity q such that q^2 is a primitive r-th root of unity. The Chen-Yang volume conjecture asserts a connection between those two invariants:

Conjecture 3.1 (Chen-Yang's volume conjecture [3])**.** *For any compact oriented* 3*-manifold* M, *one has*

$$\lim_{\substack{r\to\infty \\ r\ odd}} \frac{2\pi}{r}\log TV_r(M, e^{\frac{2i\pi}{r}}) = \mathrm{Vol}(M).$$

This conjecture is closely related to the *Kashaev-Murakami-Murakami volume conjecture*, which was formulated earlier and replaces M with a knot complement in S^3 and TV_r with some evaluations of the normalized colored Jones polynomials $J_{K,n}$ of K.

To this date, either conjecture is only known for a handful of 3-manifolds or knots. However, more is known about the qualitative behavior of the exponentially growth rate of the TV_r invariants. The author and Kalfagianni introduced and studied the quantity

$$LTV(M) = \limsup_{\substack{r\to\infty \\ r\,\mathrm{odd}}} \frac{2\pi}{r}\log TV_r(M, e^{\frac{2i\pi}{r}}).$$

In [4], we proved the following:

Theorem ([4]). Let M be a compact oriented 3-manifold with empty or toroidal boundary. Then

(1) If M' is the complement of a link in M, then $LTV(M) \leq LTV(M')$.

(2) If M' is obtained from M by cutting along a torus T embedded in M, then $LTV(M) \leq LTV(M')$.

(3) There exists a universal constant C, not depending on M, such that

$$LTV(M) \leq C \cdot \mathrm{Vol}(M).$$

The properties (1) and (2) of decreasingness under Dehn filling and subadditivity under gluing along tori are also true for the simplicial volume. Hence the above theorem provides evidence for the connection between Turaev-Viro invariants and simplicial volume.

The simplicial volume has however more well-known properties with respect to operations on 3-manifolds. In particular, it is known that $\mathrm{Vol}(\widehat{M}) = d\mathrm{Vol}(M)$ when

$\widehat{M}$ is a d-fold cover of M. Moreover, while simplicial volume decreases under Dehn-fillings, the simplicial volume of a Dehn filling of a 3-manifold M tends to $\mathrm{Vol}(M)$ when the Dehn filling slopes are long.

Question 3.2 (R. Detcherry)**.**

(1) *If $\widehat{M}$ is a d-fold cover of M, can one compare $LTV(\widehat{M})$ and $LTV(M)$?*

(2) *If M_n is a sequence of Dehn fillings of M along slopes s_n whose lengths tend to ∞, is it true that $LTV(M_n) \longrightarrow LTV(M)$?*

Perhaps question (2) would be more approachable for some specific family of examples, for instance link complements. An interesting family to study is the family of *fundamental shadow links*, which is a family of hyperbolic links in connected sums $(S^2 \times S^1)^{\# k}$. Those links have the following striking properties: any manifold with empty or toroidal boundary is a Dehn filling of one of those links, and moreover, their quantum invariants are easy to express, being a product of quantum $6j$-symbols.

Another approach to Conjecture 3.1, or to the Kashaev volume conjecture, is to relate it with another conjecture, the AJ conjecture. Colored Jones polynomials of knots have the property of forming a q-holonomic sequence by a theorem of Garoufalidis and Le [9], that is, for any knot, the colored Jones polynomials satisfy a recurrence relation of the form

$$a_d(q, q^n) J_{K,n+d}(q) + \ldots + a_0(q, q^n) J_{K,n}(q) = 0,$$

for some coefficients $a_i(q, Q) \in \mathbb{Z}[q^{\pm 1}, Q^{\pm 1}]$. A minimal such recurrence relation gives rise to the non-commutative A-polynomial of K, which is the following polynomial in non-commutative variables Q and E :

$$\hat{A}_K(q, Q, E) = a_d(q, Q) E^d + \ldots + a_0(q, Q).$$

The AJ-conjecture posists the following identity:

Conjecture 3.3 (AJ conjecture)**.** *For any knot K, one has*

$$\hat{A}_K(q = 1, Q, E) = A_K(Q, E),$$

up to factors depending only on Q, where A_K is the classical A-polynomial of K, which describes the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $S^3 \setminus K$.

One may wonder whether it is a common property of q-holonomic sequences to grow exponentially at roots of unity. We ask:

Question 3.4 (R. Detcherry)**.**

(1) *Let $f_n(q) \in \mathbb{Z}[q^{\pm 1}]$ be a q-holonomic sequence. Does the limit*

$$\lim_{n \to \infty} \frac{2\pi}{n} \log f_n(e^{\frac{2i\pi}{n}})$$

exists ? If yes, how can it be related to the recurrence relations satisfied by f?

(2) *Let K be a hyperbolic knot that satisfies the AJ conjecture. Can one deduce that the volume conjecture is true for K? Or at least, that K is q-hyperbolic, that is*
$$\liminf_{n\to\infty} \frac{2\pi}{n} \log J_{K,n}(e^{\frac{2i\pi}{n}}) > 0?$$

4 Embeddings of simplicial complexes in closed manifolds

(Makoto Ozawa)

Throughout this section we work in the piecewise linear category, consisting of simplicial complexes and piecewise-linear maps. For two simplicial complexes X and Y, X is said to be *critical* for Y if X cannot be embedded in Y, but for any point $p \in X$, $X - p$ can be embedded in Y. In this article, the polyhedron $|X|$ is expressed directly using X. Hereafter, we assume the connectivity of simplicial complexes for simplicity. Let $\Gamma(Y)$ denote the set of critical complexes for Y.

Problem 4.1 (M. Ozawa)**.** *Characterize* $\Gamma(Y)$ *for a closed n-manifold Y.*

It is shown that:

(1) $\Gamma(S^1) = \emptyset$ ([6])

(2) $\Gamma(S^2) = \{K_5, K_{3,3}\}$ ([6])

(3) $\Gamma(F_g) = \{F_0, \ldots, F_{g-1}\} \cup \Omega(F_g)$ for $g > 0$, where F_g denotes a closed orientable surface of genus g and $\Omega(F_g)$ denotes the set of forbidden graphs for F_g ([6])

(4) Some family of multibranched surfaces $X_1, X_2, X_3, X_g(p_1, \ldots, p_n) \in \Gamma(S^3)$ ([5], [12])

Let $\mathcal{C}$ denote the set of all connected simplicial complexes. X, $Y \in \mathcal{C}$ are *equivalent*, denoted by $X{\sim}Y$, if X can be embedded in Y and Y can be embedded in X. We denote by $[X] \subseteqq [Y]$ if X can be embedded in Y. Then $(\mathcal{C}/{\sim}, \subseteqq)$ is a partially ordered set. For $[X]$, $[Y] \in \mathcal{C}/{\sim}$, $[X]$ is said to be *critical* for $[Y]$ if $[X] \not\subseteqq [Y]$, but for any $[X'] \subsetneqq [X]$, $[X'] \subseteqq [Y]$. Put

$$\Gamma([Y]) = \{[X] \in \mathcal{C}/{\sim} \ \mid [X] \text{ is critical for } [Y]\}$$

It follows that if $X \in \Gamma(Y)$, then $[X] \in \Gamma([Y])$.

Problem 4.2 (M. Ozawa)**.** *Characterize* $\Gamma([Y])$ *for a closed n-manifold Y.*

We denote the quotient space obtained from the n-ball B^n and the closed interval $[0, 1]$ by identifying a point p in intB^n and $\{0\}$ by $B^{n\perp}$. It is shown that:

(1) $\Gamma([S^1]) = \{B^{1\perp}\}$ ([6])

(2) $\Gamma([S^2]) = \{[K_5], [K_{3,3}], [B^{2\perp}]\}$ ([13])

(3) $\Gamma([F_g]) = \{[F_0], \ldots, [F_{g-1}], [B^{2\perp}]\} \cup \{[G] \mid G \in \Omega(F_g)\}$ for $g > 0$ ([6])

5 Complexified tetrahedral decomposition of knot complement

(Jun Murakami)

It is known that there is an ideal tetrahedral decomposition corresponds to the potential function of the colored Jones polynomial. For each crossing of the knot diagram, an ideal octahedron is assigned, which is decomposed into five ideal tetrahedrons. These ideal tetrahedrons have one to one correspondence to the quantum factorials in the quantum R-matrix assigned to the crossing to obtain the colored Jones polynomial of the knot. By constructing the volume potential function from the colored Jones polynomial, and get the saddle point corresponding to the geometric representation of the fundamental group of the complement, then we can get the shape parameters of ideal tetrahedrons.

However, there are various ways to express the colored Jones invariant. One of the universal way to construct the colored Jones polynomial other than from the quantum R-matrix is the face model introduced by Kirillov-Reshetikhin [10]. In this construction, state is assigned to the regions of the knot diagram, and the quantum $6j$-symbol is assigned to each crossing. The six parameters of the quantum $6j$-symbol are the state of the four regions around the vertex and the colors of the strand to make the crossing. By constructing the volume potential function from this expression of the colored Jones invariant, we may get some information about the hyperbolic structure of the knot complement as for the case starting from the quantum R-matrix, which may be given as follows.

Problem 5.1 (J. Murakami)**.** *Construct a complexified tetrahedral decomposition corresponding the volume potential function of the colored Jones polynomial constructed through the face model in [10].*

Such tetrahedron may correspond to the saddle point of the potential function, and the parameter corresponds to the edge may relate the eigenvalue of certain element of the fundamental group of the complement.

6 Graph complexes and the space of embeddings of a manifold into $\mathbb{R}^n$

(Leo Yoshioka)[2]

Write $\mathrm{Emb}(L, \mathbb{R}^n)$ for the space of smooth embeddings of a manifold L into $\mathbb{R}^n$. In 2020, Fresse, Turchin and Willwacher showed the following deep result by developing a homotopy theory called embedding calculus.

Theorem ([8]) Let $n - j \geq 2$. Assume M is a complement of a compact manifold in $\mathbb{R}^j$, and $M \neq \mathbb{R}^j$ when $n - j = 2$. Let A be a rational model of $M \cup \infty$. Write

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$\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$ for the homotopy fiber of $\mathrm{Emb}(M, \mathbb{R}^n) \hookrightarrow \mathrm{Imm}(M, \mathbb{R}^n)$. Then we have

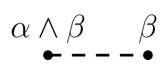
$$\overline{\mathrm{Emb}}(M, \mathbb{R}^n) \simeq_{\mathbb{Q}} \mathrm{MC}_\bullet(HGC_{\overline{A},n}).$$

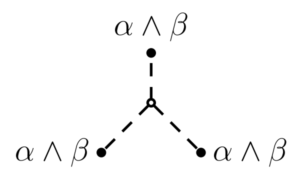
Here, $HGC_{\overline{A},n}$ is a L_∞ algebra and "MC" stands for Maurer-Cartan elements. Generators of $HGC_{\overline{A},n}$ are hairy graphs whose hairs are decorated by elements of the augmentation ideal $\overline{A}$.

Example ([8]) Since $S^1 \times S^2$ are embeddable to $\mathbb{R}^4$, we can apply the above theorem to $\overline{\mathrm{Emb}}(S^1 \times S^2, \mathbb{R}^6)$. Take $A = \mathbb{Q}[\alpha, \beta]/(\alpha^2, \beta^2) \oplus \mathbb{Q}[1]$ ($|\alpha| = 1, |\beta| = 2$). As shown in [8],

$$\pi_0 \mathrm{MC}_\bullet(HGC_{\overline{A},n}) = \{\lambda L_\beta + \mu T_{\alpha\wedge\beta} \mid \lambda, \mu \in \mathbb{Q}, \mu = 0 \text{ or } \lambda = 0\}.$$

Here,

L_β =


and $T_{\alpha\wedge\beta}$ =


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Example ([8]) When M is $\mathbb{R}^j$, we can take $\mathbb{Q}[\omega]/\omega^2$ ($|\omega| = j$) as A. Then the L_∞ structure is trivial. Write $HGC_{n,j}$ for the graph complex. By the above theorem, we have

$$\pi_* \overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n) \otimes \mathbb{Q} \cong H_*(HGC_{n,j}) \quad (n - j \geq 3).$$

On the other hand, Sakai and Watanabe [16, 17] gave another graph complex $\widetilde{GC}_{n,j}$ and a map $I : \widetilde{GC}_{n,j} \to A^*_{dR}\, \overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$ for $n - j \geq 2$, by configuration space integrals. This approach may clarify a geometric meaning of the above example.

Problem 6.1 (L. Yoshioka)**.** *Define a graph complex $\widetilde{GC}_{\overline{A},n}$ and establish configuration space integrals $I : \widetilde{GC}_{\overline{A},n} \to A^*_{dR}\, \overline{Emb}(M, \mathbb{R}^n)$. Give a quasi-isomorphism $p : \widetilde{GC}_{\overline{A},n} \to \widetilde{HGC}_{\overline{A},n}$ to the dual of $HGC_{\overline{A},n}$. Show $H^*(I)(H^*(p))^{-1}$ is injective.*

Example. The integral associated to the graph L_β will be defined by

$$I(L_\beta) = \int_{(x_1,x_2)\in \mathrm{Conf}(L_\beta)} \omega(x_1, x_2)$$

where $\mathrm{Conf}(L_\beta) = \big((S^1 \times S^2) \times (*_{S^1} \times S^2)\big) \setminus \mathrm{diag} \subset (S^1 \times S^2)^2$, and $\omega(x_1, x_2)$ is the pullback of the standard volume form on S^5. Show this integral gives an invariant of $\mathrm{Emb}(S^1 \times S^2, \mathbb{R}^6)$, by checking $\int_{\partial \overline{\mathrm{Conf}}(L_\beta)} \omega(x_1, x_2) = 0$.

References

[1] Baader, S., Banfield, I., Lewark, L., *Untwisting 3-strand torus knots*, Bull. London Math. Soc. **52** (2020) 429–436.

[2] Baader, S., Feller, P., *Asymptotics of the smooth A_n-realization problem*, arXiv:2303.15434.

[3] Chen, Q., Yang, T., *Volumes conjectures for the Reshetikhin-Turaev and Turaev-Viro invariants*, Quantum Topol. **9** (2018) 419–460.

[4] Detcherry, R., Kalfagianni, E., *Gromov norm and Turaev-Viro invariants of 3-manifolds,* Ann. Sci. Éc. Norm. Supér. (4) **53** (2020) 1363–1391.

[5] Eto, K., Matsuzaki, S., Ozawa, M., *An obstruction to embedding* 2*-dimensional complexes into the* 3*-sphere*, Topol. Appl. **198** (2016) 117–125.

[6] Eudave-Muñoz, M., Ozawa, M., *Forbidden complexes for the 3-sphere*, arXiv:2403.18279.

[7] Freedman, M. H., *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982) 357–453.

[8] Fresse, B., Turchin, V., Willwacher, T., *On the rational homotopy type of embedding spaces of manifolds in* $\mathbb{R}^n$, arXiv.2008.08146.

[9] Garoufalidis, S., Lê, T. Q. T., *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005) 1253–1293.

[10] Kirillov, A. N., Reshetikhin, N. Yu., *Representations of the algebra* $\mathcal{U}_q(sl(2))$*, q -orthogonal polynomials and invariants of links,* Adv. Ser. Math. Phys. **7**, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989, 285–339.

[11] Kronheimer, P. B., Mrowka, T. S., *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994) 797–808.

[12] Matsuzaki, S., Ozawa, M., *Genera and minors of multibranched surfaces*, Topol. Appl. **230** (2017) 621–638.

[13] Mardešić, S., Segal, J., *A note on polyhedra embeddable in the plane*, Duke Math. J. **33** (1966) 633–638.

[14] McCoy, D., *Null-homologous twisting and the algebraic genus*, arXiv:1908.04043.

[15] Rudolph, L., *Some topologically locally-flat surfaces in the complex projective plane*, Comment. Math. Helv. **59** (1984) 592–599.

[16] Sakai, K., *Configuration space integrals for embedding spaces and the Haefliger invariant.* J. Knot Theory Ramifications **19** (2010) 1597–1644.

[17] Sakai, K., Watanabe, T., 1*-loop graphs and configuration space integral for embedding spaces*, Math. Proc. Cambridge Philos. Soc. **152** (2012) 497–533.