On the volume conjecture for Turaev-Viro invariants of 3-manifolds

Intelligence of Low-dimensional Topology,
Kyoto University,
May 24, 2024

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joint works with Giulio Beletti, Effie Kalfagianni and Tian Yang

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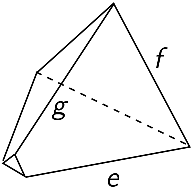
- For M a manifold closed or with boundary, $r \geqslant 3$ odd, $TV_r(M)$ is the Turaev-Viro invariant of M at level r and root $q = e^{\frac{2i\pi}{r}}$.

State sum definition of Turaev-Viro invariants



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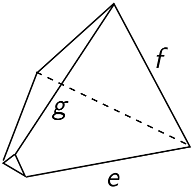

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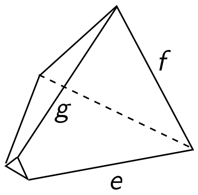

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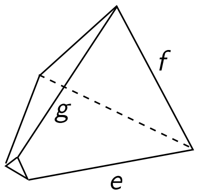

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$c : E \to \{0, 2, \ldots r-3\} \in A_r(\tau)$ the set of r-admissible colorations if:

- $c_e \leqslant c_f + c_g$
- $c_e + c_f + c_g \leqslant 2(r-2)$

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$$\{n\}! = \{n\}\{n-1\}\ldots\{1\}.$$

Moreover, for $c \in A_r(\tau)$ and $e \in E$, we write $|e|_c = (-1)^{c(e)} \frac{\{c(e)+1\}}{\{1\}}$.

State sum definition of TV_r

$$TV_r(M) = 2^{b_2 - b_0}\ \eta_r^{2|V|} \sum_{c \in A_r(\tau)} \prod_{e \in E} |e|_c \prod_{\Delta \in \tau} |\Delta|_c,$$

Volume conjectures

Volume conjecture 1 (Kashaev 97, Murakami-Murakami 00)

If K is an hyperbolic knot then $\lim\limits_{n\to\infty} \frac{2\pi}{n} \log |J_n(K, e^{\frac{2i\pi}{n}})| = \mathrm{Vol}(K)$

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Definition: Simplicial Volume

$\mathrm{Vol}(M) = \sum \mathrm{Vol}(H_i)$, for H_i hyperbolic piece in the prime+JSJ decomposition.

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- If M is a Dehn-filling of M', then $\mathrm{Vol}(M) \leqslant \mathrm{Vol}(M')$.

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- An infinite family $M_L(k,l)$ of 3-manifolds of volume $2(k+2l)v_8$ and $k+l$ JSJ pieces (Kumar-Melby 2021)

Relationship with the original volume conjecture

Formula for TV_r of a link complement (D-Kalfagianni-Yang 2017)

For the complement of a link L in S^3 with n components:

$$TV_r(S^3 \setminus L) = 2^{n-1}(\eta_r)^2 \sum_{1 \leqslant \mathbf{i} \leqslant \frac{r-1}{2}} |J_{\mathbf{i}}(L, e^{\frac{4i\pi}{r}})|^2$$

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The growth rate of $TV_r(M)$, with $r = 2n+1$, is the same as that of the last term, $J_n(L, e^{\frac{2i\pi}{n+1/2}})$.

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Remark: True in all known cases and in numerical experiments

The growth rate LTV of TV_r invariants

Definition

For M a 3-manifold, define

$$LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log |TV_r(M)|$$

$$\text{and } lTV(M) = \liminf_{r \to \infty} \frac{2\pi}{r} \log |TV_r(M)|.$$

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Theorem (D-Kalfagianni 17)

For 3-manifolds M with empty or toroidal boundary:

- LTV (and lTV) decrease under Dehn fillings and cutting along tori
- There exists a constant $C > 0$ such that for any M, one has

$$LTV(M) \leq C \cdot \mathrm{Vol}(M).$$

Sketch of proof

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- $RT_r(\emptyset) = \mathbb{C}$, $RT_r(\Sigma_1 \coprod \Sigma_2) = RT_r(\Sigma_1) \otimes RT_r(\Sigma_1)$, and the form on $RT_r(T^2)$ is definite positive

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- If $M = M_1 \underset{\Sigma}{\cup} M_2$ then $RT_r(M) = \langle RT_r(M_1), RT_r(M_2) \rangle$.

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Follows from state sum formula + asymptotic bounds for 6j-symbols (Belletti-D-Kalfagianni-Yang 2018).

A weak form of the conjecture

Weak Chen-Yang volume conjecture

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Remark: If $ITV(M) > 0$, then $ITV(M \setminus L) > 0$ for any link L in M.

A weak form of the conjecture

Weak Chen-Yang volume conjecture

Let M a compact 3-manifold. Then $ITV(M) > 0$ if and only if M has a hyperbolic JSJ piece.

Remark: If $ITV(M) > 0$, then $ITV(M \setminus L) > 0$ for any link L in M.

Kalfagianni and Melby (2023) used this property to verify the weak Chen-Yang conjecture for many knots with low crossing number.

Relationship with the AMU conjecture

Definition: Quantum representations

The TQFTs RT_r induce for each surface Σ, a representation

$$\rho_r : \mathrm{MCG}(\Sigma) \longrightarrow \mathrm{PAut}(RT_r(\Sigma))$$

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Recall $f \in \mathrm{MCG}(\Sigma)$ can be of finite order, reducible, or pseudo-Anosov

Conjecture of Andersen-Masbaum-Ueno

$f \in \mathrm{MCG}(\Sigma)$ has a pseudo-Anosov part $\Leftrightarrow$ $\rho_r(f)$ has infinite order for any $r >> 0$.

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Sketch of proof: For M_f the mapping cylinder of f, one has

$$TV_r(M_f) = ||RT_r(M_f)||^2 = ||\mathrm{Tr}(\rho_r(f))||^2$$

If the latter grows exponentially, since $\dim RT_r(\Sigma)$ grows polynomially in r, then $\rho_r(f)$ has an eigenvalue of modulus > 1 for $r >> 0$.

Special cases of the AMU conjecture

Theorem

(1) (D-Kalfagianni 17)For any $n \geq 2$ and $g \geq \max(3, n)$, there exists a pseudo-Anosov element in $\mathrm{MCG}(\Sigma_{g,n})$ that satisfies the AMU conjecture.

(2) (D-Kalfagianni 20) For any $g \geq 9$, there is a pseudo-Anosov element in $\mathrm{MCG}(\Sigma_{g,1})$ that satisfies the AMU conjecture.

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Remark: (1) uses homogeneous links to show there are hyperbolic fibered links with a figure eight component.
(2) uses open book decomposition techniques to show that there is a hyperbolic fibered knot in the hyperbolic Dehn filling of the figure eight.

Open problem

There are still no known examples of the AMU conjecture for pseudo-Anosov maps on a closed surface.

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Problem

Prove the (weak) Chen-Yang volume conjecture for some closed hyperbolic 3-manifolds, in order to get the first examples of the AMU conjecture which are pseudo-Anosov maps on a closed surface.

Conclusion

Thank you!