Classical invariants and rack coloring invariants of Legendrian knots

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Contents

Definition

M : a 3-manifold.
A contact structure ξ *on* M *is a plane field on* M *satisfying, when* $\xi = \ker \alpha$ *for a local 1-form* α *on* M, $\alpha \wedge d\alpha \neq 0$.
Then (M, ξ) *is called a contact 3-manifold.*

Example

(x, y, z) : *the standard coordinate on* $\mathbb{R}^3$.
$\alpha_{std} := dz + xdy$.
$\xi_{std} := \ker \alpha_{std} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \rangle_{\mathbb{R}}$.
Then ξ_{std} *is a contact structure on* $\mathbb{R}^3$.
ξ_{std} *is called the standard contact structure on* $\mathbb{R}^3$.

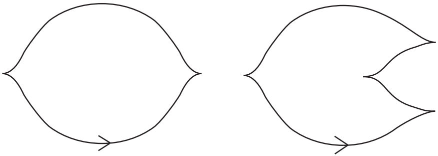
Definition

(M, ξ) : *a contact 3-manifold.*
A smooth knot K *in* (M, ξ) *is called Legendrian if* $T_pK \subset \xi_p$ *for any* $p \in K$.

Definition

K_0, K_1 : Legendrian knots in a contact 3-manifold (M, ξ).
K_0 is said to be Legendrian isotopic to K_1 if there exists an isotopy φ_t of M $(t \in [0,1])$ such that φ_0 is id_M, $\varphi_1(K_0)$ is K_1 and $\varphi_t(K_0)$ is Legendrian for any $t \in [0,1]$.

Legendrian isotopic is more strict equivalence relation than ambient isotopic. Actually, two Legendrian unknots below are not Legendrian isotopic.



We consider the classification of Legendrian knots up to Legendrian isotopy.

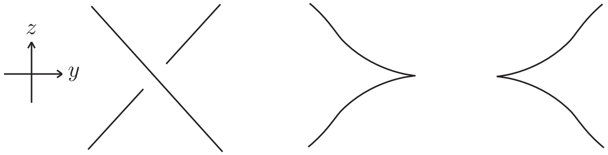
We only consider Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.

$$\mathbb{R}^3 \ni (x, y, z) \mapsto (y, z) \in \mathbb{R}^2 : \text{the front projection.}$$

Front diagrams of Legendrian knots have the following features.
$\gamma(t) = (x(t), y(t), z(t))$: a parametrization of a Legendrian knot K.
Since $\alpha_{std}(\gamma'(t)) = 0$,

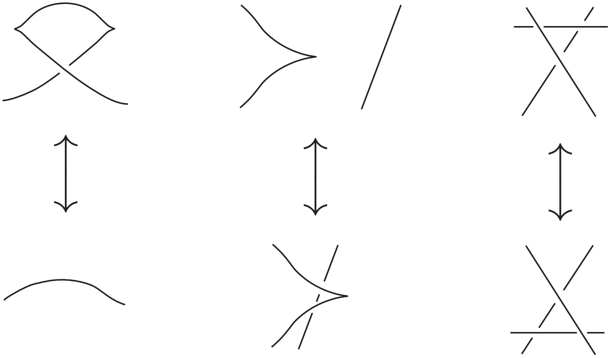
$$z'(t) + x(t)y'(t) = 0.$$

(1) Each point $(y(t), z(t))$ in $\mathbb{R}^2$ with $y'(t) = 0$ is a singular point called a cusp.
(2) Due to $x(t) = -\frac{dz}{dy}(t)$, at each crossing the slope of the overcrossing is smaller than that of the undercrossing.



Theorem

K_0, K_1 : Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.
D_i : the front diagram of K_i $(i = 0, 1)$.
Then K_0 and K_1 are Legendrian isotopic if and only if D_0 and D_1 are related by a finite sequence of the following three types of local moves.



The moves are called the Legendrian Reidemeister moves.

Several invariants of Legendrian knots computed from diagrams

1. Classical invariants

2. Legendrian contact homology (Chekanov-Eliashberg DGA)
(vanishing for stabilized Legendrian knots)

3. Ruling polynomial
(vanishing for stabilized Legendrian knots)

4. Rack coloring invariants

The classical invariants of Legendrian knots

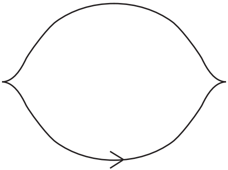
(1) the Thurston-Bennequin number $tb(K) \in \mathbb{Z}$
(an invariant of unoriented Legendrian knots)
(2) the rotation number $rot(K) \in \mathbb{Z}$
(an invariant of oriented Legendrian knots)

$tb(K)$ is the twisting number of the contact plane ξ relative to a Seifert surface of K along K.
$rot(K)$ is the rotation number of the tangent vector of K on the contact plane ξ along K.

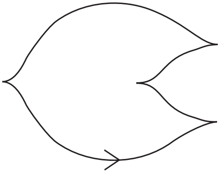
$$tb(K) = w(D) - \frac{1}{2}c(D),$$
$$rot(K) = \frac{1}{2}(dc(D) - uc(D)),$$

where $w(D)$: the writhe of D,
$c(D)$: the number of the cusps of D,
$dc(D)$: the number of the downward cusps of D,
$uc(D)$: the number of the upward cusps of D.

Note that $rot(-K) = -rot(K)$,
where $-K$ is the same knot as K with the reverse orientation.



$tb = -1$
$rot = 0$



$tb = -2$
$rot = -1$

These two Legendrian unknots are not Legendrian isotopic.

K : a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ with the knot type $\mathcal{K}$.

Theorem (Bennequin (1983))

$tb(K) + |rot(K)| \leq 2g(\mathcal{K}) - 1.$

Theorem (Rudolph (1997))

$tb(K) + |rot(K)| \leq 2g_s(\mathcal{K}) - 1.$

Definition

A knot type $\mathcal{K}$ is called Legendrian simple if the following holds for any two Legendrian knots K_0 and K_1 of the knot type $\mathcal{K}$:
if $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$, then K_0 and K_1 are Legendrian isotopic.

In other words, Legendrian simplicity of $\mathcal{K}$ means that the pair of the classical invariants completely classifies the Legendrian isotopy classes of $\mathcal{K}$.

Theorem (Eliashberg-Fraser (2009))

The unknot is Legendrian simple.

Theorem (Etnyre-Honda (2001))

Each torus knot is Legendrian simple.

Theorem (Etnyre-Honda (2001))

The figure eight knot is Legendrian simple.

Definition

$(X, *)$ is called a rack if X is a set with a binary operation $*$ satisfying the following conditions for all $x, y, z \in X$:

$$*x : X \to X \text{ is a bijection,}$$
$$(x * y) * z = (x * z) * (y * z).$$

A rack which satisfies $x * x = x$ for all $x \in X$ is called a quandle.

Since the axioms of a quandle correspond to the Reidemeister moves, quandle colorings of diagrams bring knot invariants.

We consider the axioms correspond to the Legendrian Reidemeister moves in order to obtain invariants of Legendrian knots.

Definition (K.)

$(X, *, f, g)$ *is called a bi-Legendrian rack if* $(X, *)$ *is a rack and* f *and* g *are maps on* X *satisfying the following conditions for all* $x, y \in X$:

$$\begin{aligned} f \circ g &= g \circ f, \\ fg(x * x) &= x, \\ f(x * y) &= f(x) * y, \\ g(x * y) &= g(x) * y, \\ x * f(y) &= x * y, \\ x * g(y) &= x * y. \end{aligned}$$

Example

$(G, *)$: *a conjugation quandle, i.e.* G *is a group and* $x * y = y^{-1}xy$ *for* $x, y \in G$. *Take* $z \in Z(G)$ *and define* $f(x) := zx$.
Then $(G, *, f, f^{-1})$ *is a bi-Legendrian quandle.*

Example

X : a set.
f, g : bijections on X such that they are commutative.
Define

$$x * y := (f \circ g)^{-1}(x).$$

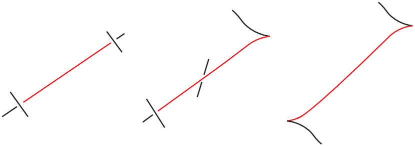
Then $(X, *, f, g)$ is a constant bi-Legendrian rack.

Example

$(\mathbb{Z}_8, *, f, g)$ is a bi-Legendrian rack if we define

$$x * y = 3x + 2y,$$
$$f(x) = x + 4,$$
$$g(x) = 5x + 4.$$

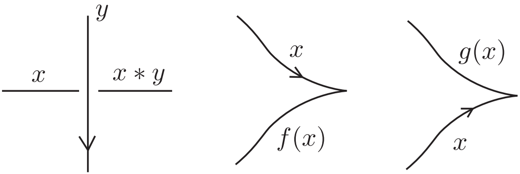
An arc of a front diagram means a part of the diagram each of whose end is either an undercrossing or a cusp and which contains neither undercrossings nor cusps in its interior.



D : the front diagram of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$.
$(X, *, f, g)$: a bi-Legendrian rack.
An $(X, *, f, g)$-coloring c of D is a map

$$c : \{\text{arcs of } D\} \to X$$

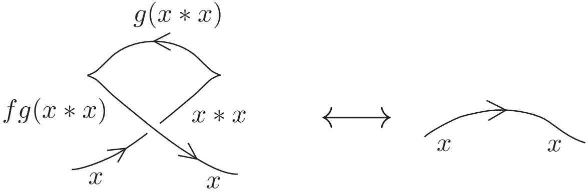
such that at each crossing and each cusp the relations below hold.

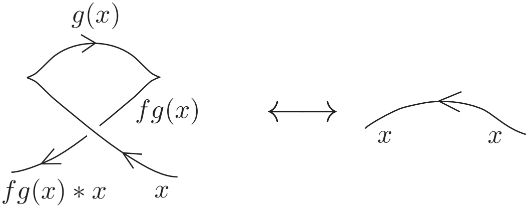


$\mathrm{Col}(D, X) := \{(X, *, f, g)\text{-colorings of } D\}$.

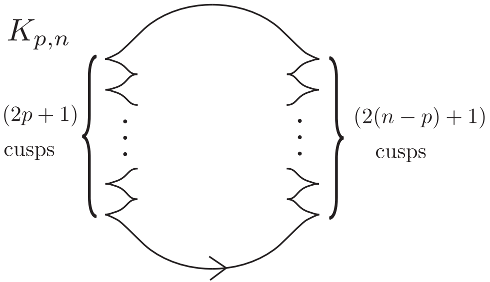
Proposition (K.)

$\#\mathrm{Col}(D, X)$ is invariant under the Legendrian Reidemeister moves. Namely, $\#\mathrm{Col}(D, X)$ is an invariant of a Legendrian knot K, denoted by $\#\mathrm{Col}(K, X)$.





It is known that any Legendrian unknot is Legendrian isotopic to $K_{p,n}$ for some $p, n \in \mathbb{Z}_{\geq 0}$ $(0 \leq p \leq n)$.



$tb(K_{p,n}) = -1 - n,\ rot(K_{p,n}) = 2p - n.$

Theorem (K.)

For any $n \in \mathbb{Z}_{\geq 0}$*, there exists a bi-Legendrian rack* $(X_n, *, f, g)$ *such that* $n+1$ *Legendrian unknots* $K_{p,n}$ $(0 \leq p \leq n)$ *are simultaneously distinguished by* $\#\mathrm{Col}(K_{p,n}, X_n)$.

Theorem (K.)

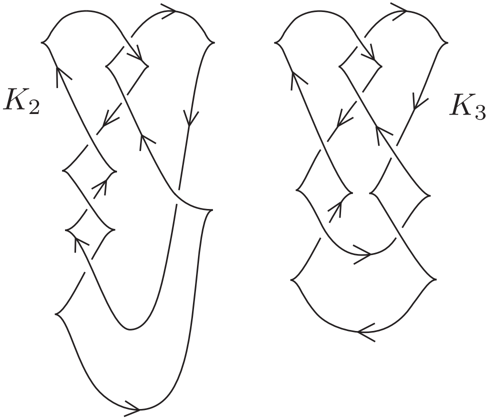
K_0, K_1 : *Legendrian knots in* $(\mathbb{R}^3, \xi_{std})$.
If K_0 *and* K_1 *are of the same knot type,* $tb(K_0) = tb(K_1)$ *and* $rot(K_0) = rot(K_1)$,
then $\#\mathrm{Col}(K_0, X) = \#\mathrm{Col}(K_1, X)$ *for any bi-Legendrian quandle* $(X, *, f, g)$.

Theorem (K.)

K_2, K_3 : *the Chekanov knots.*
Then $\#\mathrm{Col}(K_2, X) = \#\mathrm{Col}(K_3, X)$ *for any bi-Legendrian rack* $(X, *, f, g)$.

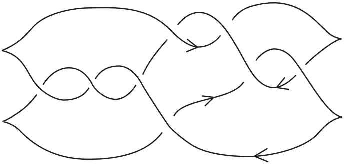
Theorem (K.)

K_4, K_5 : *Legendrian knots with the knot type* 6_3.
Then $\#\mathrm{Col}(K_4, X) = \#\mathrm{Col}(K_5, X)$ *for any bi-Legendrian rack* $(X, *, f, g)$.

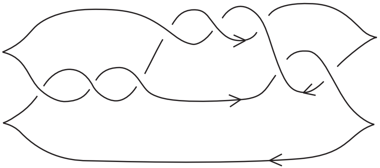


K_2 and K_3 are of the same knot type $m(5_2)$, $tb(K_2) = tb(K_3) = 1$ and $rot(K_2) = rot(K_3) = 0$.
However, Chekanov (2002) proved K_2 and K_3 are not Legendrian isotopic by using Legendrian contact homology.
K_2 and K_3 are called the Chekanov knots.

K_4



K_5



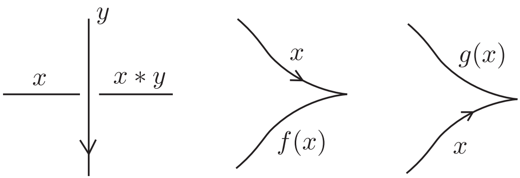
K_4 and K_5 are of the same knot type 6_3, $tb(K_4) = tb(K_5) = -4$ and $rot(K_4) = rot(K_5) = 1$.
However, Ng (2003) proved K_4 and K_5 are not Legendrian isotopic by using Legendrian contact homology.

Karmakar, Saraf and Singh defined the fundamental bi-Legendrian rack of a Legendrian knot as an analogue of the fundamental quandle of a knot. [arXiv:2301.06854].

D : the front diagram of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$.

Definition (Karmakar-Saraf-Singh)

The fundamental bi-Legendrian rack $bLR(K)$ of K is generated by the arcs of D and crossings and cusps give the relations.



D : the front diagram of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$.
$(X, *, f, g)$: a bi-Legendrian rack.

Remark

An $(X, *, f, g)$-coloring of D is regarded as a homomorphism from the fundamental bi-Legendrian rack $bLR(K)$ of K to $(X, *, f, g)$.
Hence the fundamental bi-Legendrian rack $bLR(K)$ of K is the universal invariant for bi-Legendrian rack coloring numbers $\#\mathrm{Col}(K, X)$.

K : a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ with the knot type $\mathcal{K}$.

Remark

The fundamental quandle of $\mathcal{K}$ is obtained from the fundamental bi-Legendrian rack $bLR(K)$ of K by adding the relation $f(x) = g(x) = x$ for any generator x of $bLR(K)$.

Theorem (Karmakar-Saraf-Singh (2023))

Legendrian unknots are completely classified by the fundamental bi-Legendrian rack $bLR(K)$*.*

Theorem (Karmakar-Saraf-Singh (2023))

Legendrian left-handed trefoil knots are completely classified by the fundamental bi-Legendrian rack $bLR(K)$*.*

We introduce a 4-Legendrian rack as a generalization of a bi-Legendrian rack.

Definition (K.)

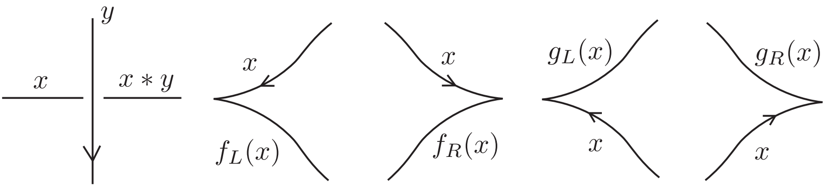
$(X, *, f_L, f_R, g_L, g_R)$ *is called a* 4*-Legendrian rack if* $(X, *)$ *is a rack and* f_L, f_R, g_L *and* g_R *are maps on* X *satisfying the following conditions for all* $x, y \in X$:

$$\begin{aligned}
f_L \circ g_R = g_R \circ f_L = f_R \circ g_L &= g_L \circ f_R, \\
f_L g_R(x * x) &= x, \\
f_L(x * y) &= f_L(x) * y, \\
f_R(x * y) &= f_R(x) * y, \\
g_L(x * y) &= g_L(x) * y, \\
g_R(x * y) &= g_R(x) * y, \\
x * f_L(y) = x * f_R(y) &= x * y, \\
x * g_L(y) = x * g_R(y) &= x * y.
\end{aligned}$$

D : the front diagram of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$.
$(X, *, f_L, f_R, g_L, g_R)$: a 4-Legendrian rack.
An $(X, *, f_L, f_R, g_L, g_R)$-coloring c of D is a map

$$c : \{\text{arcs of } D\} \to X$$

such that at each crossing and each cusp the relations below hold.



$\text{Col}(D, X) := \{(X, *, f_L, f_R, g_L, g_R)\text{-colorings of } D\}$.

Proposition (K.)

$\#\text{Col}(D, X)$ is invariant under the Legendrian Reidemeister moves. Namely, $\#\text{Col}(D, X)$ is an invariant of a Legendrian knot K, denoted by $\#\text{Col}(K, X)$.

We define the fundamental 4-Legendrian rack of a Legendrian knot.

D : the front diagram of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$.

Definition

The fundamental 4-Legendrian rack $4LR(K)$ of K is generated by the arcs of D and crossings and cusps give the relations.

The fundamental 4-Legendrian rack $4LR(K)$ is the universal invariant for 4-Legendrian rack coloring numbers.

Remark

The fundamental bi-Legendrian rack $bLR(K)$ of K is obtained from the fundamental 4-Legendrian rack $4LR(K)$ of K by adding the relation $f_L(x) = f_R(x)(= f(x))$ and $g_L(x) = g_R(x)(= g(x))$ for any generator x of $4LR(K)$.

Theorem (K.)

K_0, K_1 : Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.
If K_0 and K_1 are of the same knot type, $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$,
then $\#\mathrm{Col}(K_0, X) = \#\mathrm{Col}(K_1, X)$ for any 4-Legendrian quandle $(X, *, f_L, f_R, g_L, g_R)$.

Theorem (K.)

K_2, K_3 : the Chekanov knots.
Then the fundamental 4-Legendrian racks $4LR(K_2)$ and $4LR(K_3)$ are isomorphic.

Theorem (K.)

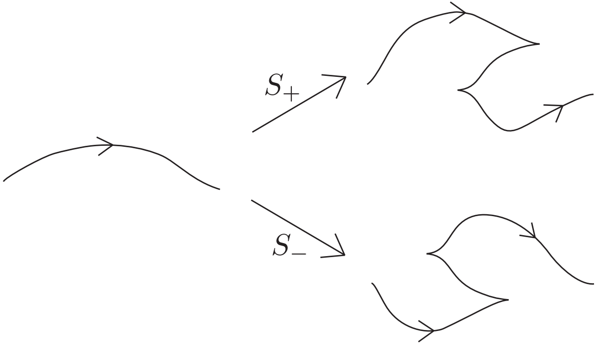
K_4, K_5 : Legendrian knots with the knot type 6_3.
Then the fundamental 4-Legendrian racks $4LR(K_4)$ and $4LR(K_5)$ are isomorphic.

Future works (problem)

1. Can rack coloring invariants distinguish Legendrian knots with the same classical invariants?

2. Are the classical invariants recovered from rack coloring invariants?

3. Relation to Legendrian contact homology or rulings?



A positive (or negative) stabilization S_+ (or S_-) is an operation represented by adding two downward (or upward) cusps to a trivial arc for the front diagram. $S_\pm$ changes the Legendrian isotopy class and does not change the knot type. Note that $S_+S_- = S_-S_+$.

$$tb(S_\pm(K)) = tb(K) - 1,$$
$$rot(S_\pm(K)) = rot(K) \pm 1.$$