Intelligence of Low-dimensional Topology 2024

Grid homology and the connected sum of knots

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1 A very short survey of grid homology

• Grid homology (Manolescu, et.al.,'07) is a combinatorial reconstruction of knot Floer homology. There are some versions of grid homology for a knot $K \in S^3$:

- the bigraded $\mathbb{F}[U]$-module $GH^-(K)$,
- the bigraded $\mathbb{F}$-vector space $\widehat{GH}(K), \widetilde{GH}(K)$[*1].

$$GH^-(K) \cong HFK^-(S^3, K), \quad \widehat{GH}(K) \cong \widehat{HFK}(S^3, K).$$

Q. Can we prove the results of HFK in the framework of GH?

Q. Can we give new invariants using GH?

[*1] $\widetilde{GH}(K) \cong \bigotimes_{2^n} \widehat{GH}(K)$, where n denotes the size of the grid diagram.

2 A very short survey of grid homology

Some properties of HFK are shown in the framework of GH, e.g.

- GH is a categorification of the Alexander polynomial.

$$\sum_{d,s\in\mathbb{Z}}(-1)^d\cdot t^s\cdot\dim_{\mathbb{F}}\widehat{GH}_d(K,s)=\Delta_K(t).$$

- There is a skein exact sequence

$$\cdots\to GH^-(K_+)\to GH^-(K_-)\to GH^-(K_0)\to GH^-(K_+)\to\cdots.$$

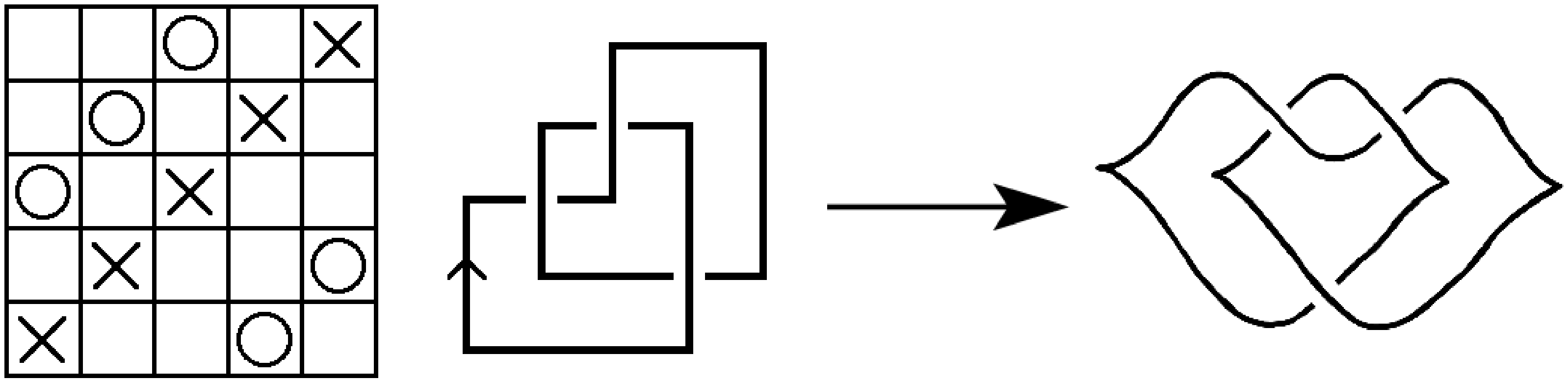
- If K_1, K_2 are connected by a genus g cobordism, then

$$|\tau(K_1)-\tau(K_2)|\leq g.$$

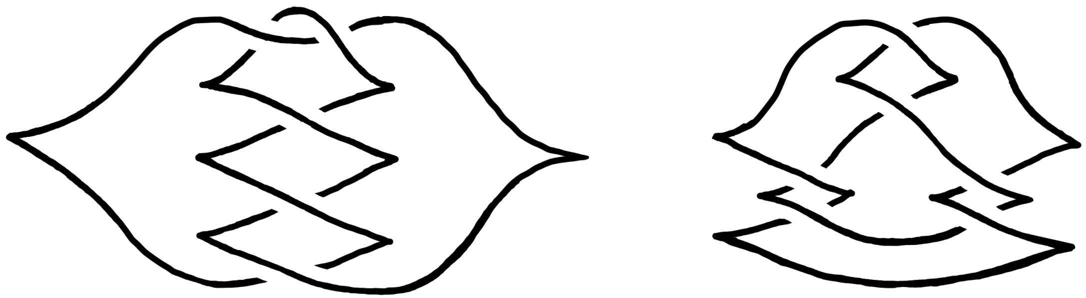
- $\widehat{GH}$ of a quasi-alternating knot K is determined by $\Delta_K(t)$ and $\sigma(K)$.
- GH with coefficients in $\mathbb{Z}$.

3 Applications of GH

• The Legendrian grid (GRID) invariants $\lambda^{\pm}$ (Ozsváth, et.al.,'08)
-the homology classes of GH^{-} represented by the canonical cycles.



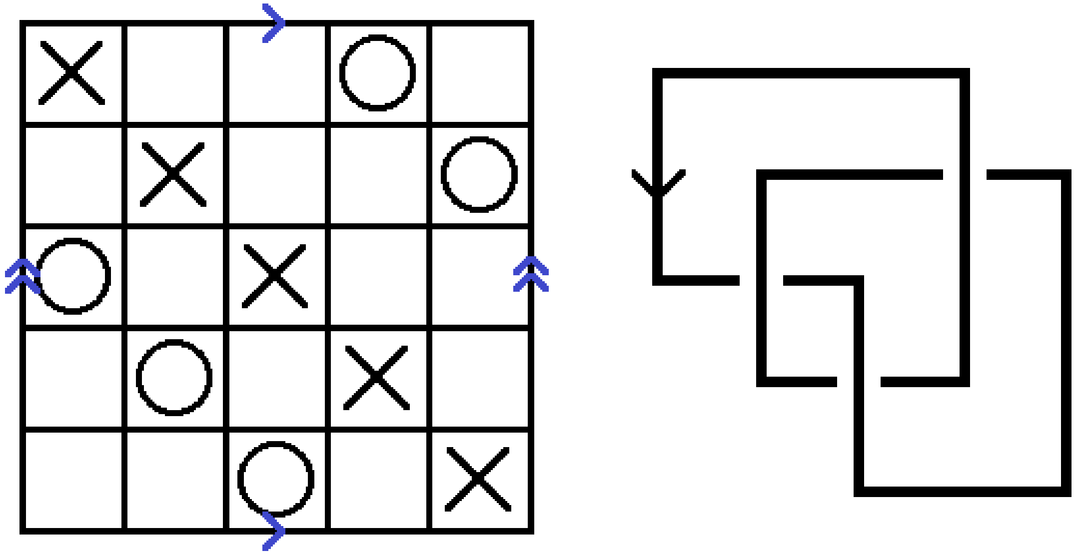
The following two Legendrian knots with topological type $m(5_2)$ are distinguished by $\lambda^{\pm}$. Both knots have $\mathrm{tb} = 1$ and $\mathrm{r} = 0$.



• $\lambda^{\pm}$ are equivalent to the LOSS invariant (Baldwin, et.al., '13).

4 Grid diagram

Definition. A grid diagram g is an $n \times n$ grid of squares on the torus, some of which are decorated with O- (sometimes O^*-) and X-markings such that
- Each row and column has just one O and one X,
- No square is marked with both an O and an X.

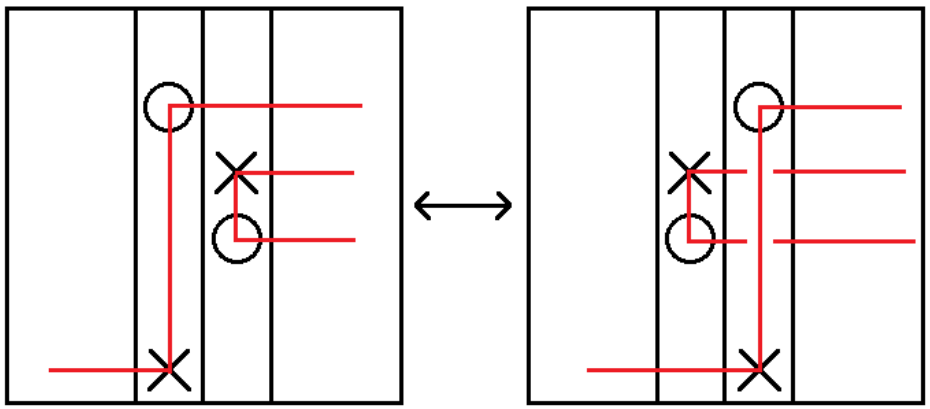


Let $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ be horizontal and vertical circles on g.

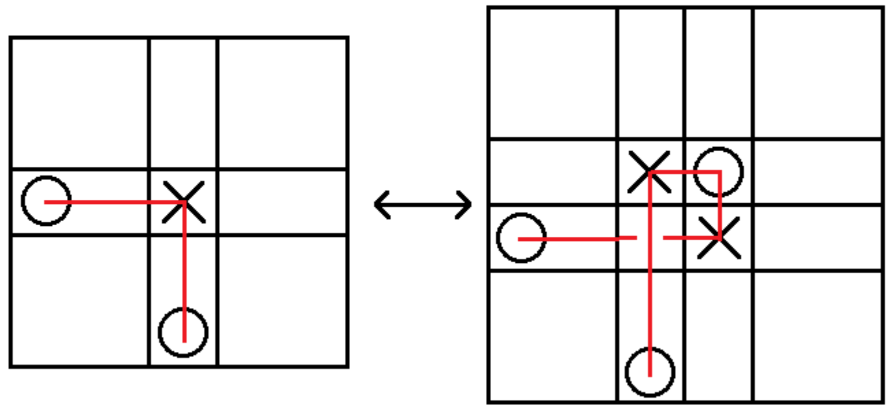
5 Grid moves

Fact. Every knot is represented by grid diagrams.

Fact. Two grid diagrams representing the same knot are connected by a finite sequence of commutations and (de-)stabilizations.



commutation

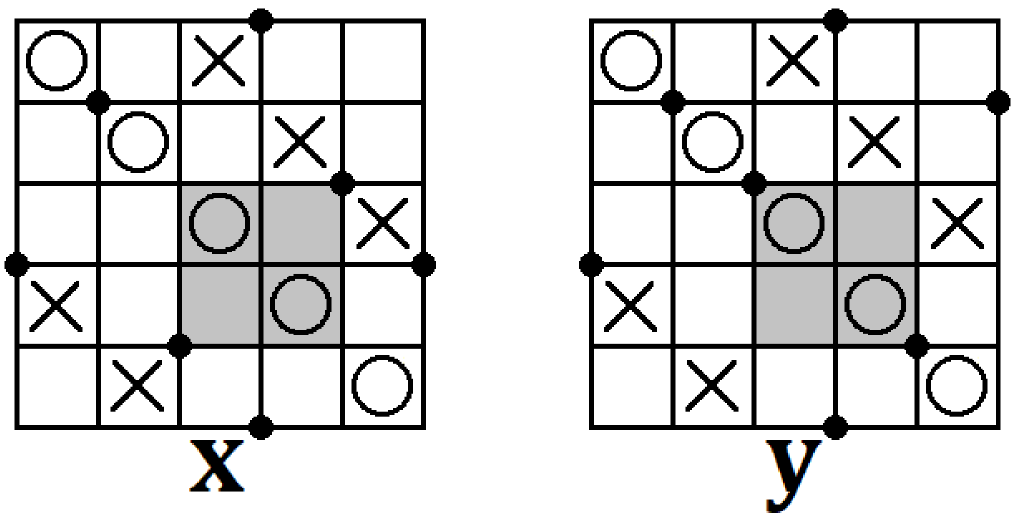


(de-)stabilization

6 A state, a rectangle

Definition. • A state of g is an n-tuple of points on the torus such that each horizontal and vertical circle contains a point.
• Let $\mathbf{S}(g)$ be the set of states of g.

Definition. For $\mathbf{x}, \mathbf{y} \in \mathbf{S}(g)$ with $\#(\mathbf{x} \cap \mathbf{y}) = n - 2$, an (empty) rectangle r from $\mathbf{x}$ to $\mathbf{y}$ is a rectangular region on g such that
(i) The NE and SW corners are $\mathbf{x} \setminus \mathbf{x} \cap \mathbf{y}$ and the NW and SE corners are $\mathbf{y} \setminus \mathbf{x} \cap \mathbf{y}$, and (ii) r contains no points of $\mathbf{x} \cup \mathbf{y}$ in its interior.



Let $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ be the set of empty rectangles from $\mathbf{x}$ to $\mathbf{y}$.

7 The definition of the grid chain complex $GC^-(g)$

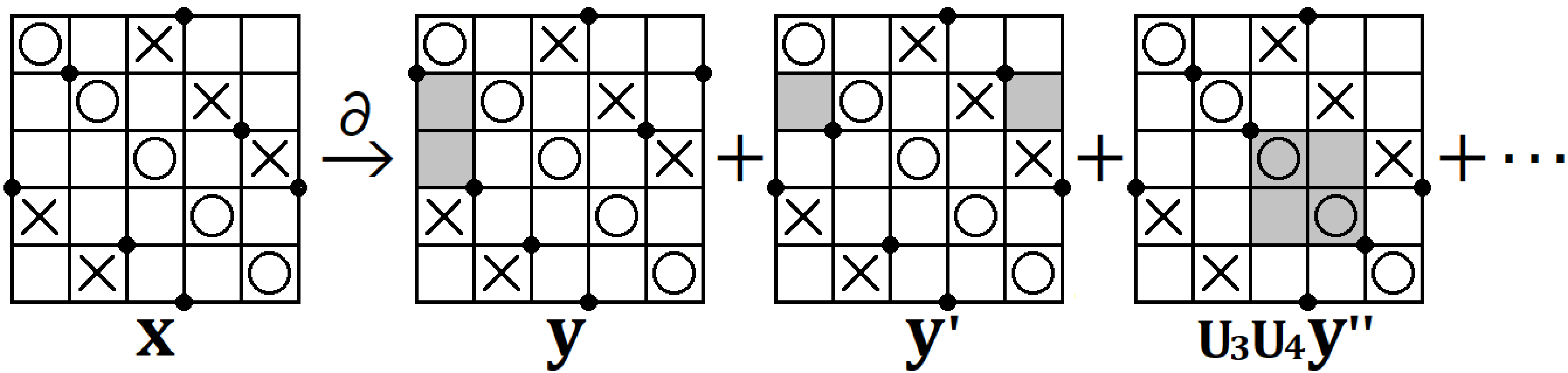
Number the markings: $\mathbb{O} = \{O_i\}_{i=1}^n$, $\mathbb{X} = \{X_i\}_{i=1}^n$.

Definition. • Let $GC^-(g)$ be the $\mathbb{F}[U_1, \ldots, U_n]$-module freely generated by $\mathbf{S}(g)$.

• Define the differential ∂ algorithmically by

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(g)} \sum_{\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r \cap \mathbb{X} = \emptyset\}} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \mathbf{y},$$

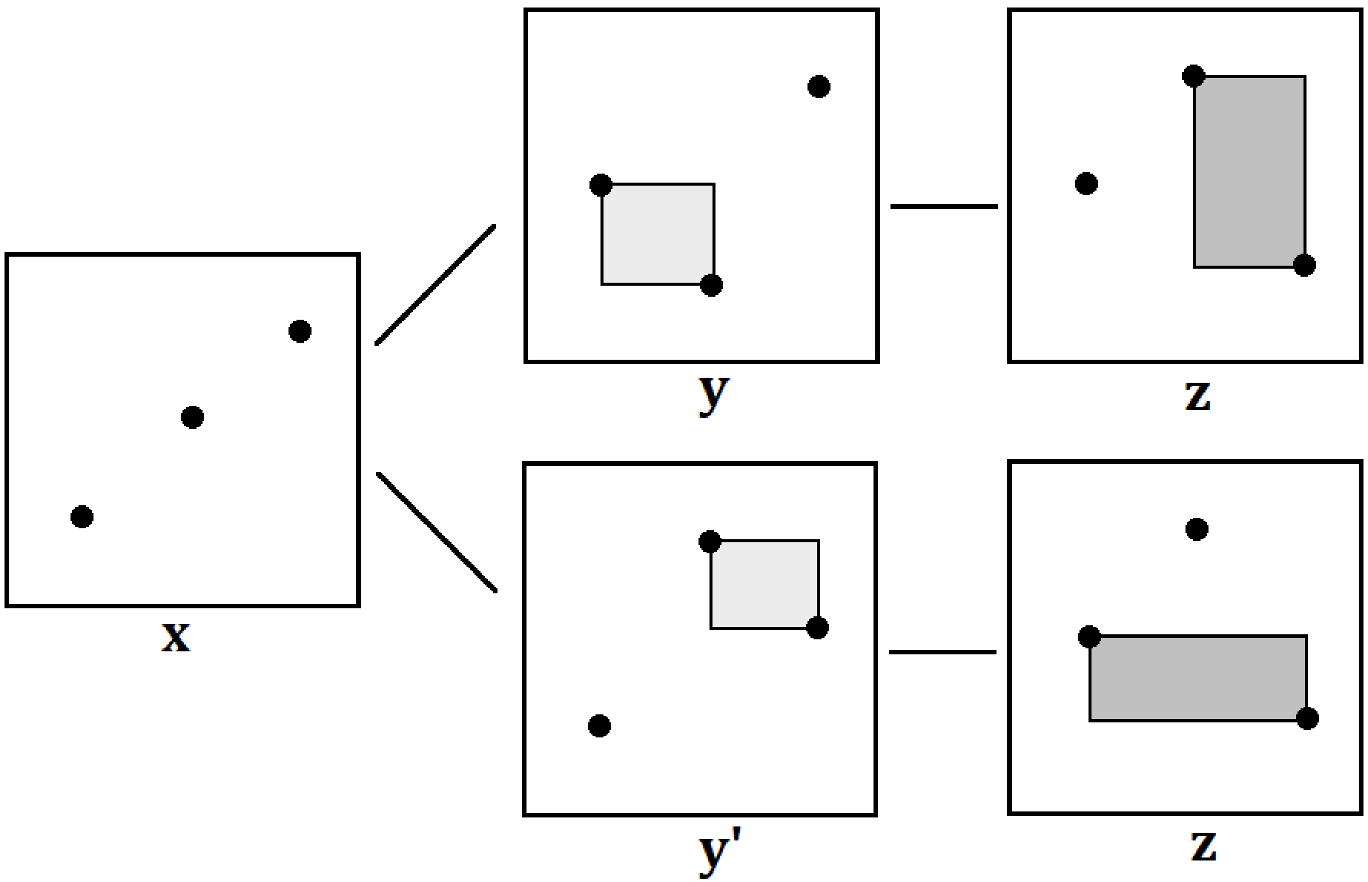
where $U_i(r) = 1$ if $O_i \in r$ and $O_i(r) = 0$ if $U_i \notin r$



8 The proof that $\partial \circ \partial = 0$

Proposition. $\partial \circ \partial = 0$.

Proof. Every state in $\partial \circ \partial(\mathbf{x})$ appears two times.



□

9 The absolute gradings

$GC^-(g)$ has two absolute gradings, Maslov grading and Alexander grading.

- There are two combinatorial function $M, A\colon \mathbf{S}(g) \to \mathbb{Z}$.

If there is a rectangle $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$, we have

$$M(\mathbf{x}) - M(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}),$$
$$A(\mathbf{x}) - A(\mathbf{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O}).$$

- $GC^-(g)$ has two gradings M, A defined by

$$M(U_1^{k_1} \cdots U_n^{k_n} \mathbf{x}) = M(\mathbf{x}) - 2(k_1 + \cdots + k_n),$$
$$A(U_1^{k_1} \cdots U_n^{k_n} \mathbf{x}) = A(\mathbf{x}) - (k_1 + \cdots + k_n).$$

- The differential ∂ drops M by 1 and preserve A.
- $GC^-(g) = \bigoplus_{d,s \in \mathbb{Z}} GC_d^-(g, s)$ is a bigraded chain complex,

where $GC_d^-(g, s)$ is the subspace with $(M, A) = (d, s)$.

10 The definition of $GH^-(g)$

- $GC^-(g)$ has two combinatorial gradings, Maslov and Alexander gradings. These gradings are absolute gradings.

- $GC^-(g) = \bigoplus_{d,s\in\mathbb{Z}} GC_d^-(g,s)$ is a bigraded chain complex.

Definition. Let $GH^-(g)$ be the homology of $GC^-(g)$ regarded as bigraded $\mathbb{F}[U]$-module, where the action of U is induced by multiplication by U_i.

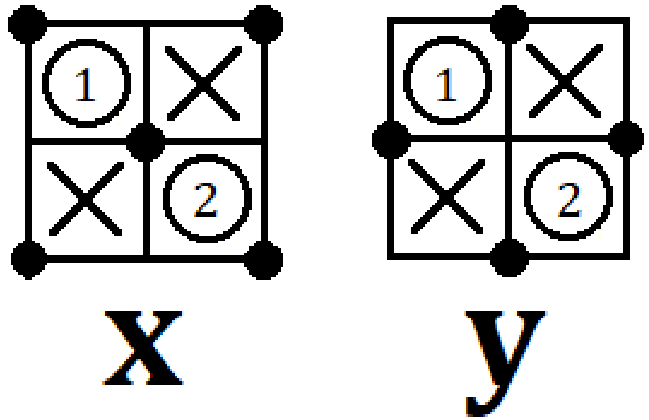
Fact. multiplication by U_i is chain homotopic to multiplication by U_j.

Theorem (Manolescu, et.al.,'07)**.** If g and g' represent the same knot, then $GH^-(g) \cong GH^-(g')$.

The idea of the proof. g and g' are connected by grid moves.
We can construct a combinatorial quasi-isomorphism corresponding to each grid move. □

11 Example

g : 2×2 grid diagram representing the unknot.



$$\mathbf{x} : (M, A) = (0, 0),$$

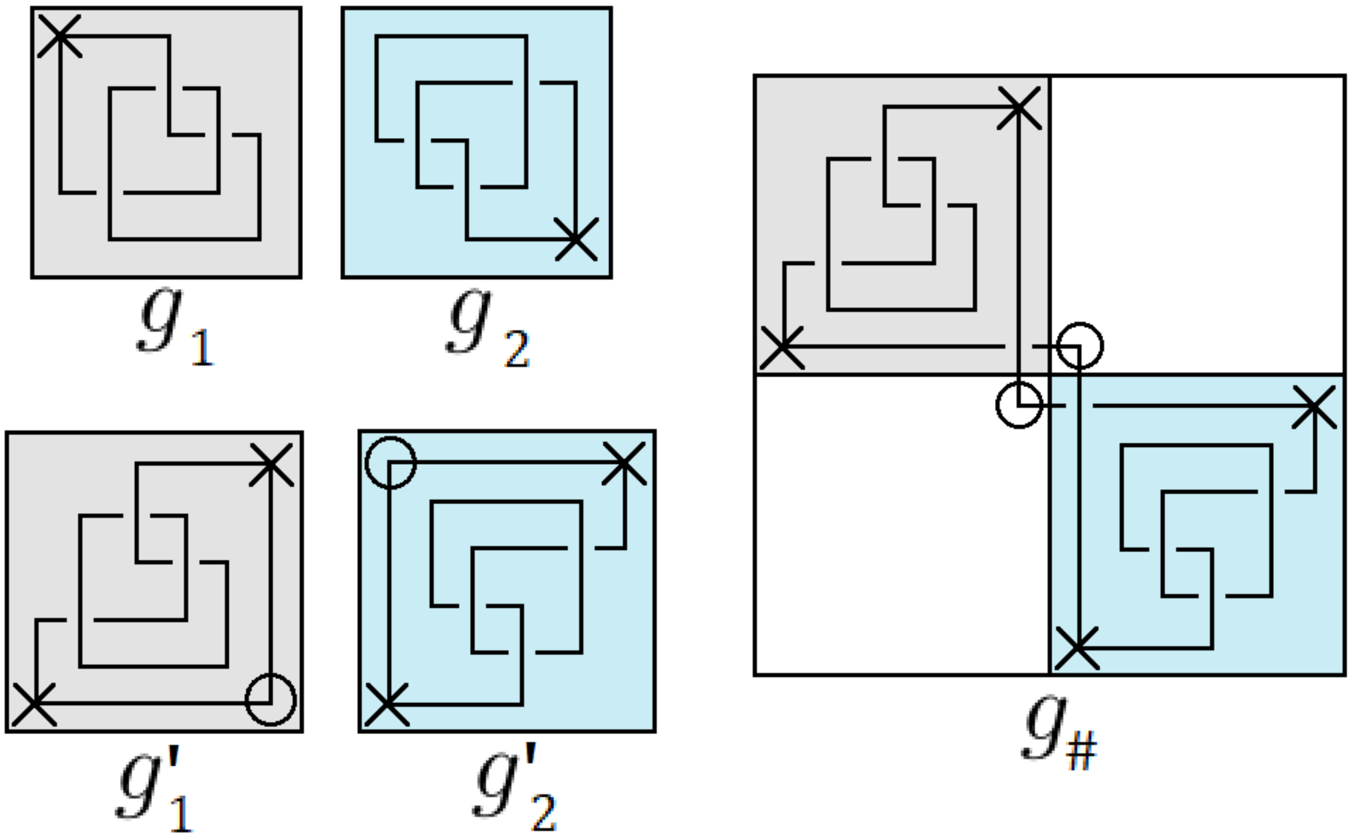
$$\mathbf{y} : (M, A) = (1, 1),$$

$$U_1, U_2 : (M, A) = (-2, -1).$$

$$\partial(\mathbf{x}) = 0,\ \partial(\mathbf{y}) = U_1\mathbf{x} + U_2\mathbf{x}.$$

$$GH^-(g) \cong \mathbb{F}[U].$$

12 Main results : a grid diagram representing $K_1 \# K_2$



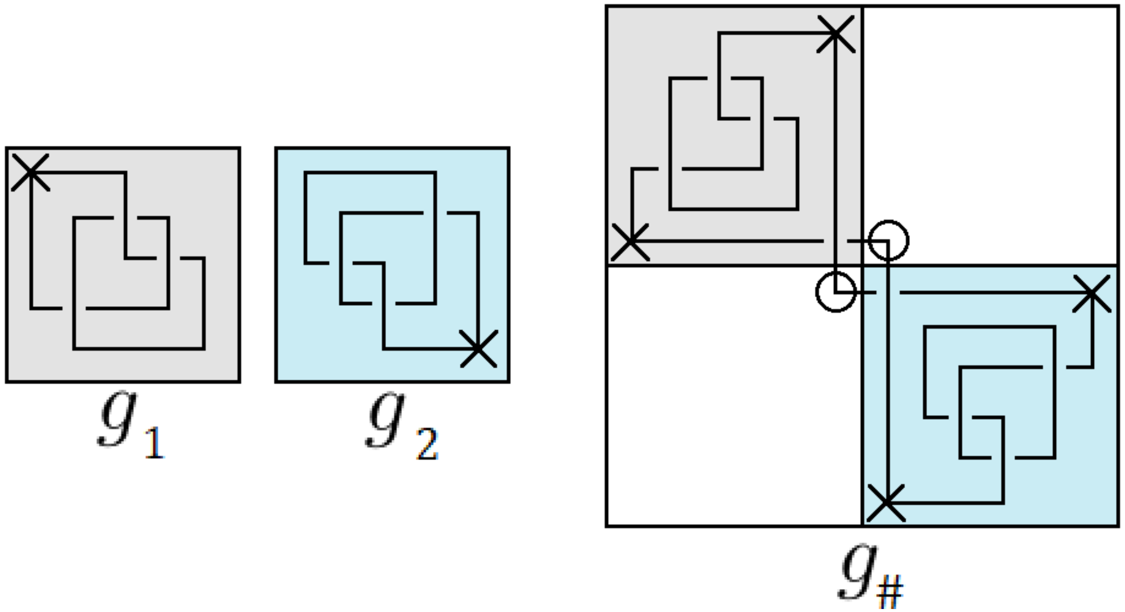
Let g_1, g_2 be diagrams representing K_1, K_2.
g'_1, g'_2 are obtained from g_1, g_2 by a stabilization.
$g_\#$ is obtained from g'_1, g'_2. $g_\#$ represents $K_1 \# K_2$.

$GC^-(g'_1)$ is a complex over $\mathbb{F}[U_1, \ldots, U_n]$.
$GC^-(g'_2)$ is a complex over $\mathbb{F}[U_{n+1}, \ldots, U_{2n}]$.
$GC^-(g_\#)$ is a complex over $\mathbb{F}[U_1, \ldots, U_{2n}]$.

13 Main results : a Künneth formula

Theorem (K,'24)**.** There are a subcomplex $C \subset GC^-(g_\#)$ and two quasi-isomorphisms

$$C \to GC^-(g_\#) \quad \text{and} \quad C \to \frac{GC^-(g_1) \otimes_{\mathbb{F}} GC^-(g_2)}{U_1 = U_{2n}}.$$

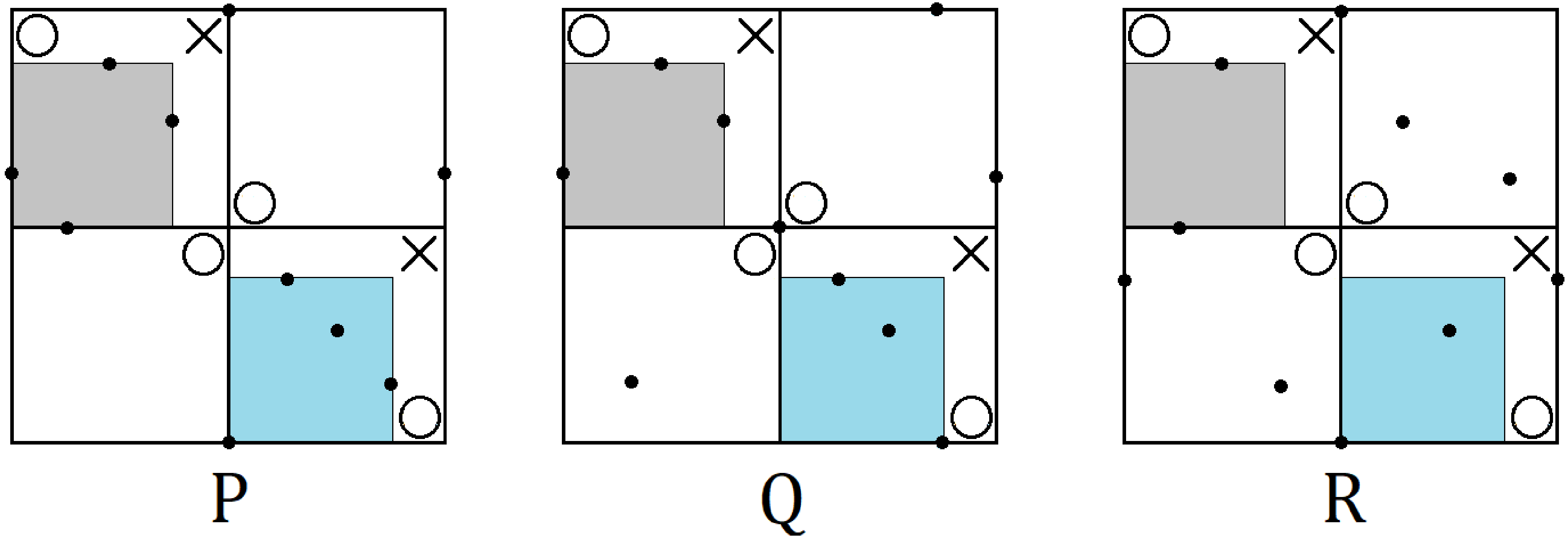


Corollary. $GH^-(K_1 \# K_2) \cong GH^-(K_1) \otimes_{\mathbb{F}[U]} GH^-(K_2)$.

Definition. $\tau(K) = -i$, where $GH^-(K) \cong \mathbb{F}[U]_{(2i,i)} \oplus \mathrm{Tors}$.

Corollary. $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$.

14 The construction of C



Classify the states $\mathbf{S}(g_\#) = \mathbf{P} \sqcup \mathbf{Q} \sqcup \mathbf{R}$, where

$\mathbf{P}$: n points each in the gray and light blue areas,

$\mathbf{Q}$: two points at $\alpha_1 \cap \beta_1$ and $\alpha_{n+1} \cap \beta_{n+1}$, and $n-1$ points each in the gray and light blue area,

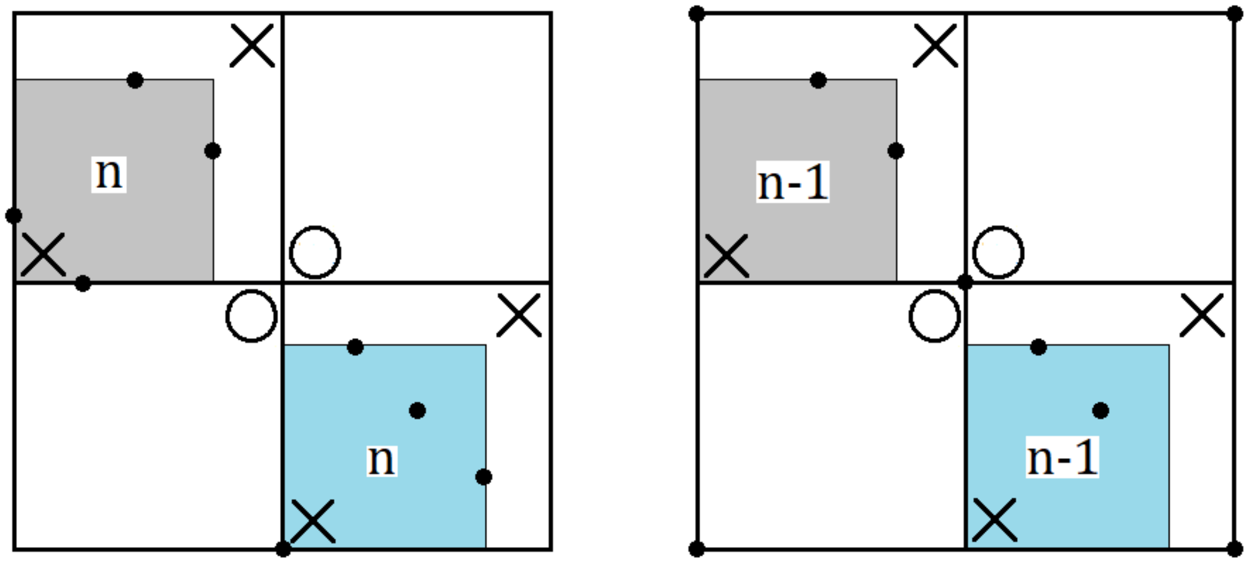
$\mathbf{R}$: the others.

Let C be the free $\mathbb{F}[U_1, \ldots, U_{2n}]$-module generated by $\mathbf{P} \cup \mathbf{Q}$.

15 The construction of C

C : the free $\mathbb{F}[U_1, \ldots, U_{2n}]$-module generated by $\mathbf{P} \cup \mathbf{Q}$.

Proposition. *C is the subcomplex of $GC^-(g_\#)$*



P **Q**

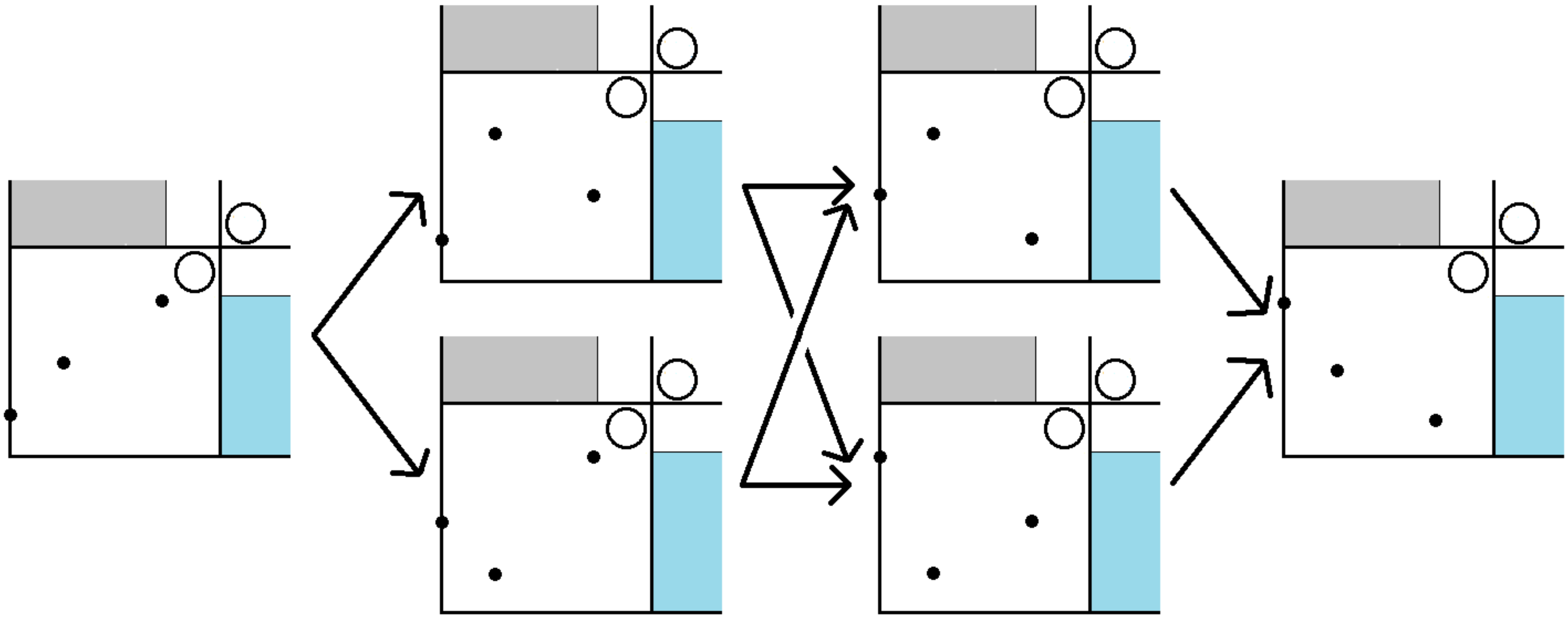
Proof. The differential of C can not count a rectangle satisfying $r \cap \{\text{the gray area}\} \neq \emptyset$ and $r \cap \{\text{the light blue area}\} \neq \emptyset$.

Let P and Q be the free modules generated by $\mathbf{P}$ and $\mathbf{Q}$.

We have $\partial(P) \subset P$ and $\partial(Q) \subset C$, i.e., $C = \mathrm{Cone}(Q \to P)$. □

16 Key observation 1

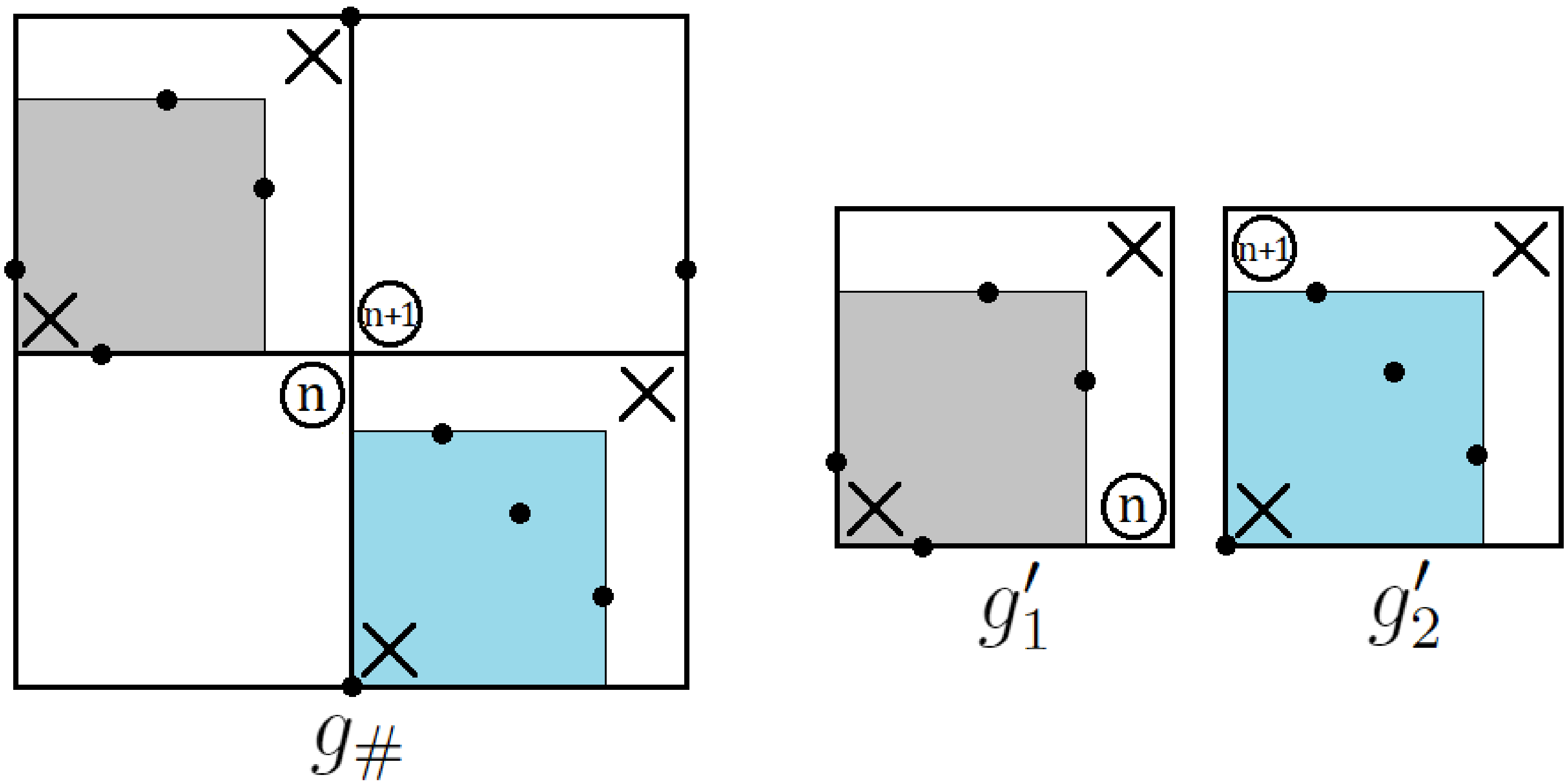
- $GC^{-}(g_{\#})/C$ is "the sum of many acyclic complexes".



Each subcomplex of n-dimensional cube is acyclic.
$\Longrightarrow GC^{-}(g_{\#})/C$ is acyclic.
$\Longrightarrow$ the inclusion $C \to GC^{-}(g_{\#})$ is a quasi-isomorphism.

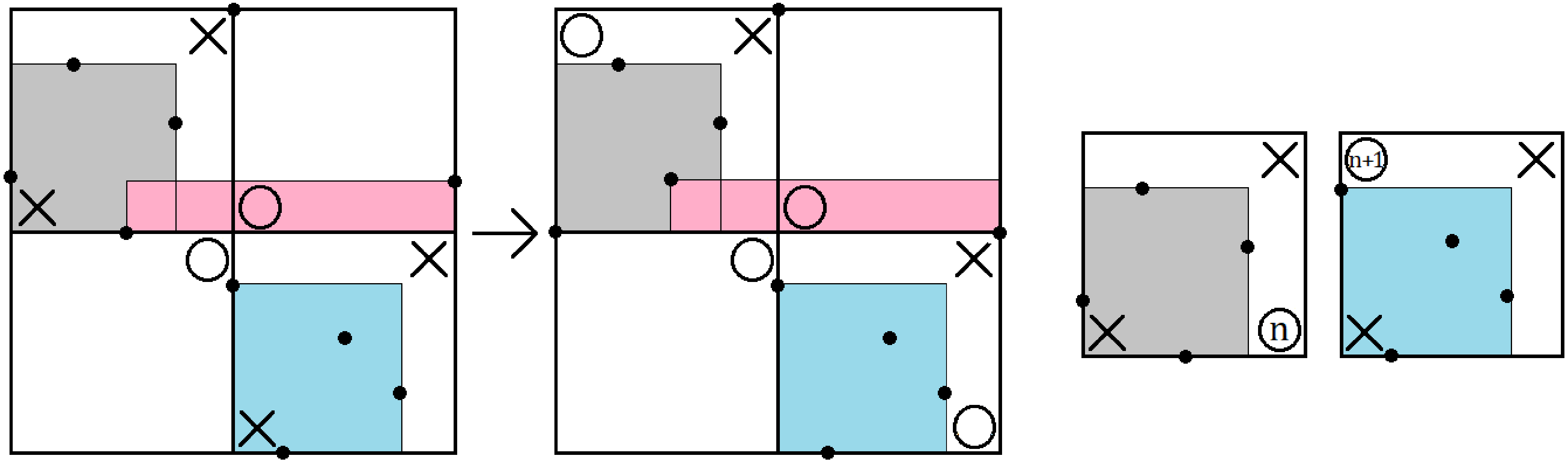
17 Key observation 2 : P v.s. $GC^-(g'_1) \otimes_{\mathbb{F}} GC^-(g'_2)$

C : the free $\mathbb{F}[U_1, \ldots, U_{2n}]$-module generated by $\mathbf{P} \cup \mathbf{Q}$.
P : the free $\mathbb{F}[U_1, \ldots, U_{2n}]$-module generated by $\mathbf{P}$.



There is a natural identification of states $\mathbf{P} \to \mathbf{S}(g'_1) \times \mathbf{S}(g'_2)$.
The differential of P moves two points either in the gray or light blue area.

18 Key observation 2 : P v.s. $GC^-(g_1') \otimes_{\mathbb{F}} GC^-(g_2')$



The rectangles counted by ∂ may contain O_{n+1} but can not contain O_n.
The identification $\mathbf{P} \to \mathbf{S}(g_1') \times \mathbf{S}(g_2')$ induces the isomorphism

$$P \cong \frac{GC^-(g_1') \otimes_{\mathbb{F}} GC^-(g_2')}{U_n = U_{n+1}}[U],$$

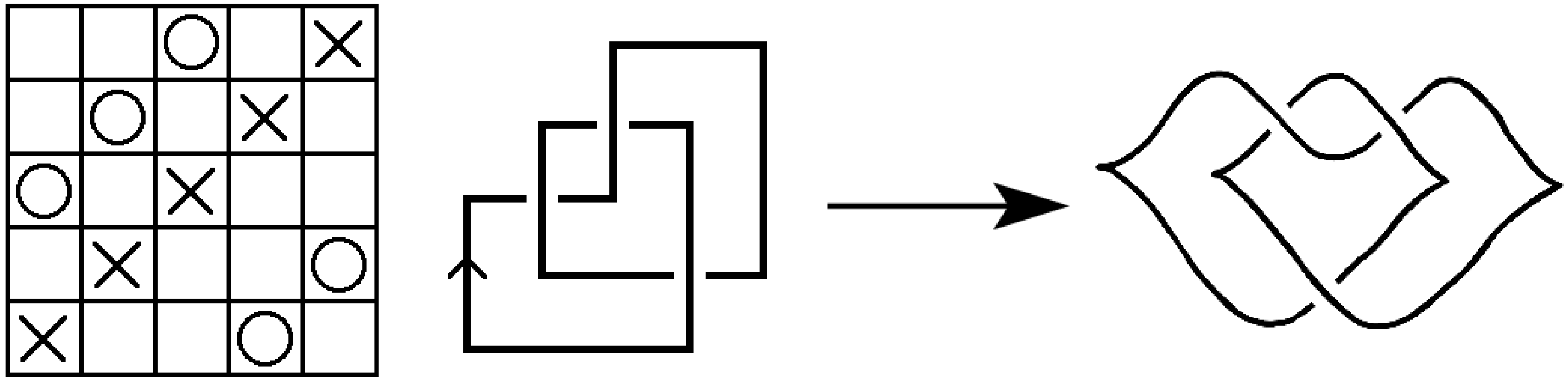
where $U_i \mapsto U_i$ for $i \neq n$ and $U_n \mapsto U$.

There are quasi-isomorphisms $GC^-(g_i') \to GC^-(g_i)$ for $i = 1, 2$.

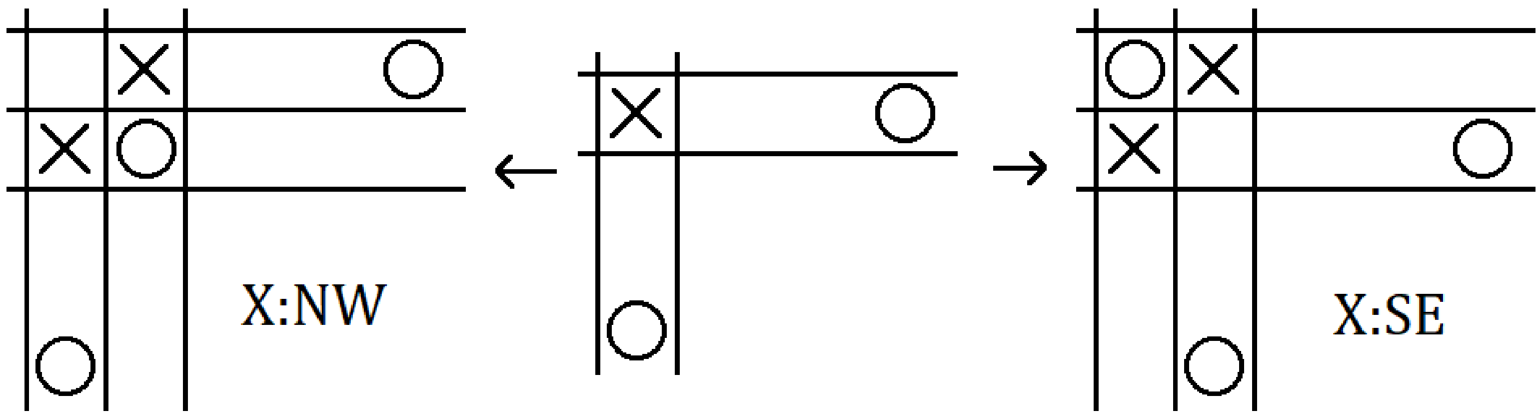
In fact, $C = \mathrm{Cone}(Q \to P) \cong \dfrac{GC^-(g_1) \otimes_{\mathbb{F}} GC^-(g_2)}{U_n = U_{n+1}}$

19 grid homology and Legendrian knots

Any Legendrian knot is represented by grid diagrams.

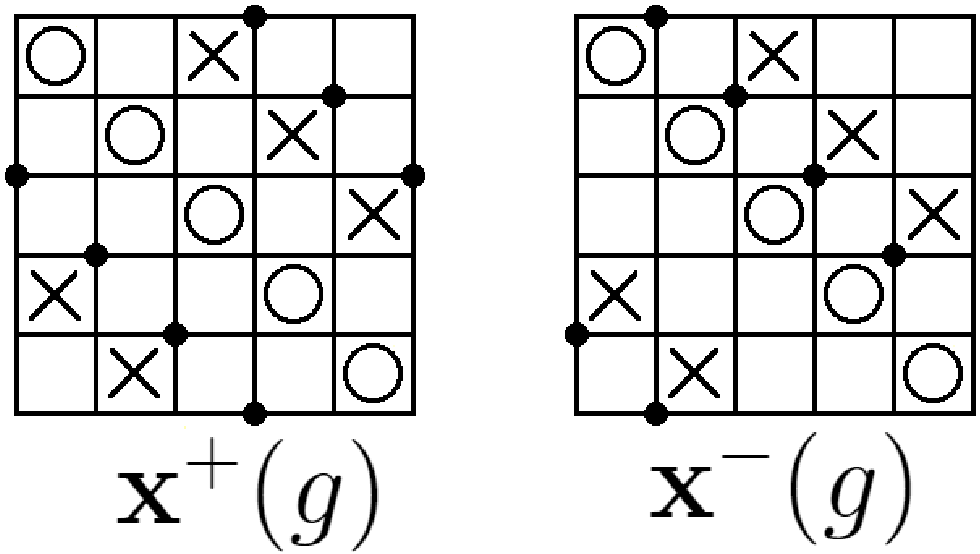


Fact. Two grid diagrams representing the same Legendrian knot are connected by a finite sequence of commutations and (de-)stabilizations of types $X : NW$ and $X : SE$.



20 The Legendrian grid invariant

For a grid diagram g, let $\mathbf{x}^+(g), \mathbf{x}^-(g)$ be the canonical states.



The homology classes $\lambda^{\pm}(g) := [\mathbf{x}^{\pm}(g)] \in GH^-(g)$ are called the Legendrian grid invariant.

Theorem (Ozsváth, et.al., '08)**.** Suppose that g and g' represent the Legendrian isotopic knots. Then there is a bigraded isomorphism $\phi\colon GH^-(g) \to GH^-(g')$ such that $\phi(\lambda^{\pm}(g)) = \lambda^{\pm}(g')$.

21 The additivity of the Legendrian grid invariant

The additivity of the Legendrian grid invariant using HFK (Vértesi,'08).

Theorem (K,'24)**.** Let g_1, g_2, and $g_\#$ be grid diagrams representing Legendrian knots $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_1 \# \mathcal{K}_2$ respectively. Then there is an isomorphism

$$\Phi\colon GH^-(g_1) \otimes GH^-(g_2) \to GH^-(g_\#),$$

such that $\Phi(\lambda^\pm(g_1) \otimes \lambda^\pm(g_2)) = \lambda^\pm(g_\#)$.

Proof. Our quasi-isomorphisms $f\colon C \to GC^-(g_\#)$ and $g\colon C \to \frac{GC^-(g_1)\otimes_{\mathbb{F}} GC^-(g_2)}{U_1=U_{2n}}$ satisfy $f(\mathbf{x}^\pm(g_\#)) = \mathbf{x}^\pm(g_\#)$ and $g(\mathbf{x}^\pm(g_\#)) = \mathbf{x}^\pm(g_1) \otimes \mathbf{x}^\pm(g_2)$.

$$\begin{array}{ccccc}
GC^-(g_\#) & \xleftarrow{\quad f \quad} & C & \xrightarrow{\quad g \quad} & \frac{GC^-(g_1)\otimes_{\mathbb{F}} GC^-(g_2)}{U_1=U_{2n}} \\
\Cup & & \Cup & & \Cup \\
\mathbf{x}^\pm(g_\#) & \longleftarrow\!\!\!\dashv & \mathbf{x}^\pm(g_\#) & \longmapsto & \mathbf{x}^\pm(g_1) \otimes \mathbf{x}^\pm(g_2)
\end{array}$$

□