MGR coloring invariants of Seifert surfaces

Intelligence of low-dimensional topology

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Contents

Introduction of §1

Handlebody-knot: handlebody embedded in S^3.

- H_1, H_2: handlebody-knots. $H_1 \cong H_2 \overset{\text{def}}{\Longleftrightarrow} H_1$ is ambiently isotopic to H_2.
- K: knot in $S^3 \Rightarrow N(K)$: handlebody-knot of genus 1.

Spatial surface: compact surface embedded in S^3.

- F_1, F_2: spatial surfaces. $F_1 \cong F_2 \overset{\text{def}}{\Longleftrightarrow} F_1$ is ambiently isotopic to F_2.
- Oriented spatial surface with non-empty boundary is a Seifert surface for its boundary.

$$\{\text{Knots}\}/_{\cong} \xhookrightarrow{\text{nbhd}} \{\text{Handlebody-knots}\}/_{\cong}$$
$$\xhookrightarrow{\text{boundary}} \{\text{Oriented spatial surfaces}\}/_{\cong}.$$

Handlebody-knots(1/2)

In this talk, graphs may have multiple edges and loops.
Spatial trivalent graph: finite trivalent graph embedded in S^3.
Handlebody-knot: handlebody embedded in S^3.

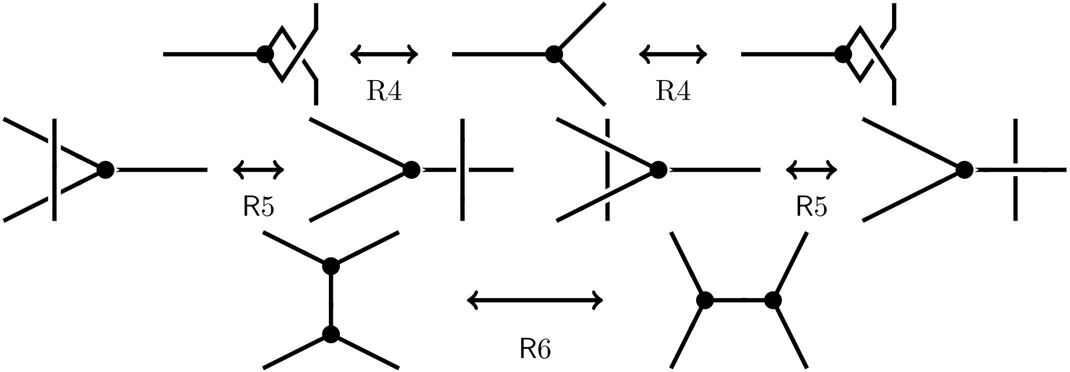
- H_1, H_2: handlebody-knots. $H_1 \cong H_2 \overset{\text{def}}{\iff} H_1$ is ambiently isotopic to H_2.
- G: spatial trivalent graph $\Rightarrow N(G)$: handlebody-knot.
- A diagram of a hdlbdy-knot H is a diagram of a spatial trivalent graph G s.t. $H \cong N(G)$.
- $H(D)$: handlebody-knot presented by a diagram D.

Handlebody-knots(2/2)

H_1, H_2: handlebody-knots, D_i: diagram of H_i $(i = 1, 2)$.

Theorem (Ishii 2008)

$H_1 \cong H_2 \overset{\text{iff}}{\Longleftrightarrow} D_1$ and D_2 are related by a finite seq. of $\mathrm{R1}$–$\mathrm{R6}$ moves and isotopies in S^2.



Spatial surfaces(1/3)

Spatial surface: compact surface embedded in S^3.

- F_1, F_2: spatial surfaces. $F_1 \cong F_2 \overset{\text{def}}{\Longleftrightarrow} F_1$ is ambiently isotopic to F_2.

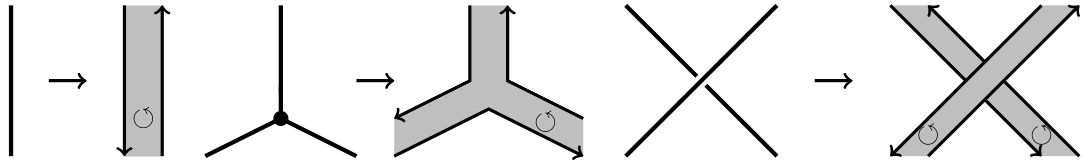
Remark

In this talk, we assume that

- spatial surfaces are connected and oriented,
- spatial surfaces are neither 2-disks nor closed surfaces.

Spatial surfaces(2/3)

D: diagram of a spatial trivalent graph.
The spatial surface $F(D)$ is obtained from the following.



We remark that full twists are presented by kinks in diagrams.

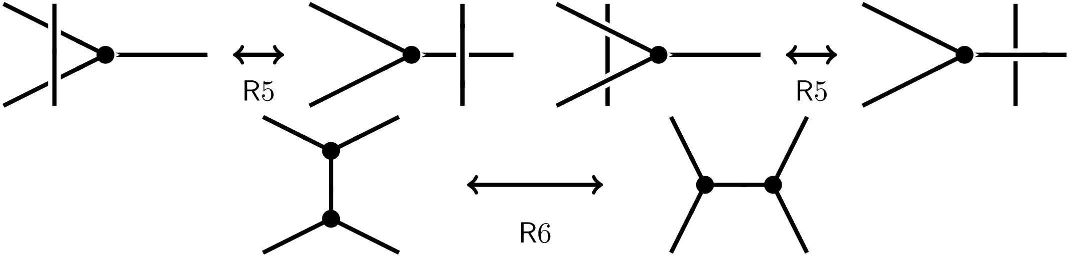
- $\forall F$: spatial surface, $\exists D$: diagram of a spatial trivalent graph s.t. $F \cong F(D)$.
- A <u>diagram</u> of a spatial surface F is a diagram D s.t. $F \cong F(D)$.

Spatial surfaces(3/3)

F_1, F_2: spatial surfaces, D_i: diagram of F_i $(i = 1, 2)$.

Theorem (Matsuzaki 2021)

$F_1 \cong F_2 \overset{\text{iff}}{\Longleftrightarrow} D_1$ and D_2 are related by a fin. seq. of $\mathrm{R2}$, $\mathrm{R3}$, $\mathrm{R5}$, $\mathrm{R6}$ moves and iso. in S^2.



Remark

$\mathrm{R1}$ and $\mathrm{R4}$ moves change framings of spatial surfaces.

Introduction of §2

The number of colorings by an algebraic system yields an invariant of "Object"
$\Rightarrow$ the algebraic system needs to have "Universal structure."

Object	Universal structure
Oriented knot	Quandle (Joyce 1982, Matveev 1982)
Knot	Symmetric quandle (Kamada 2007, Kamada-Oshiro 2010)
Handlebody-knot	Multiple conjugation quandle (Ishii 2015)
Spatial surface	?

- Multiple group rack (Ishii-Matsuzaki-Murao 2020), Heap rack (Saito-Zappala 2024).

Main result 1 (A. 2025)

? is a groupoid rack.

- $\{$Groupoid racks$\} \supset \{$Multiple group racks$\} \cup \{$Heap racks$\}$.

Racks and quandles

Definition (Joyce 1982, Matveev 1982, Fenn-Rourke 1992)

X: set, $\triangleleft : X \times X \to X$: binary operation on X.

$X = (X, \triangleleft)$: quandle $\overset{\text{def}}{\iff}$ (i)–(iii):

(i) $\forall x \in X$, $x \triangleleft x = x$.

(ii) $\forall y \in X$, the map $S_y : X \ni x \mapsto x \triangleleft y \in X$ is bijective.

(iii) $\forall x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

$X = (X, \triangleleft)$: rack $\overset{\text{def}}{\iff}$ (ii) and (iii).

Example

- $n \in \mathbb{Z}_{>0}$. $R_n = (\mathbb{Z}_n, x \triangleleft y = 2y - x)$: dihedral quandle.
- $n \in \mathbb{Z}_{>0}$. $C_n = (\mathbb{Z}_n, x \triangleleft y = x + 1)$: cyclic rack.
- $(X_1, \triangleleft_1), (X_2, \triangleleft_2)$: racks. $X_1 \times X_2$ is a rack w/ $(x_1, x_2) \triangleleft (y_1, y_2) = (x_1 \triangleleft_1 y_1, x_2 \triangleleft_2 y_2)$.

Groupoid racks (1/3)

Definition (Ishii 2015, Ishii-Matsuzaki-Murao 2020)

$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$: disjoint union of groups, $\triangleleft : X \times X \to X$: binary operation on X.

$X = (X, \triangleleft)$: <u>multiple conjugation quandle</u> (MCQ) $\overset{\text{def}}{\Longleftrightarrow}$ (i)–(iv):

(i) $\forall \lambda \in \Lambda$, $\forall a, b \in G_\lambda$, $a \triangleleft b = b^{-1}ab$.

(ii) $\forall x \in X$, $\forall \lambda \in \Lambda$, $\forall a, b \in G_\lambda$, $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$ and $x \triangleleft e_\lambda = x$,
where e_λ is the identity element of G_λ.

(iii) $\forall x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

(iv) $\forall x \in X$, $\forall \lambda \in \Lambda$, $\exists \mu \in \Lambda$ s.t. $\forall a, b \in G_\lambda$, $a \triangleleft x, b \triangleleft x \in G_\mu$ and $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$.

$X = (X, \triangleleft)$: <u>multiple group rack</u> (MGR) $\overset{\text{def}}{\Longleftrightarrow}$ (ii)–(iv).

Groupoid racks (2/3)

$\mathcal{C}$: groupoid $\overset{\text{def}}{\Longleftrightarrow}$ category in which all morphisms are invertible.

Definition (A. 2025)

$\mathcal{C}$: groupoid, $X = \mathrm{Mor}(\mathcal{C})$: the set of all morphisms, $\triangleleft : X \times X \to X$: binary operation on X.

$X = (X, \triangleleft)$: groupoid rack associated with $\mathcal{C}$ $\overset{\text{def}}{\Longleftrightarrow}$ (i)–(iii):

(i) $\forall x, f, g \in X$ w/ $\mathrm{cod}(f) = \mathrm{dom}(g)$, $x \triangleleft (fg) = (x \triangleleft f) \triangleleft g$ and $x \triangleleft \mathrm{id}_\lambda = x$, where id_λ is the identity of $\lambda \in \mathrm{Ob}(\mathcal{C})$.

(ii) $\forall x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

(iii) $\forall x, f, g \in X$ w/ $\mathrm{cod}(f) = \mathrm{dom}(g)$, $\mathrm{cod}(f \triangleleft x) = \mathrm{dom}(g \triangleleft x)$ and $(fg) \triangleleft x = (f \triangleleft x)(g \triangleleft x)$.

Groupoid racks (3/3)

An MGR $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ can be regarded as the grpd. rack ass. with the following grpd. $\mathcal{C}$.

- $\mathrm{Ob}(\mathcal{C}) = \Lambda$, $\mathrm{Mor}(\lambda,\mu) = \begin{cases} G_\lambda & (\lambda=\mu), \\ \emptyset & (\lambda\neq\mu). \end{cases}$, Composition: $G_\lambda \times G_\lambda \to G_\lambda$; $(a,b)\mapsto ab$.
- The identity morphism of $\lambda\in\Lambda$ is identity element of the group G_λ.
- The inverse morphism of a morphism $x \in G_\lambda$ is $x^{-1}\in G_\lambda$.

Proposition

X: groupoid rack associated with a groupoid $\mathcal{C}$.
If $\mathcal{C}$ satisfies that $\forall\lambda,\mu\in\mathrm{Ob}(\mathcal{C})$ with $\lambda\neq\mu$, $\mathrm{Mor}(\lambda,\mu)=\emptyset$, then X is an MGR.

Proposition

$(X,\triangleleft)$: grpd. rack associated with a grpd. $\mathcal{C}$. If $\mathcal{C}$ satisfies the following, then X is an MCQ.

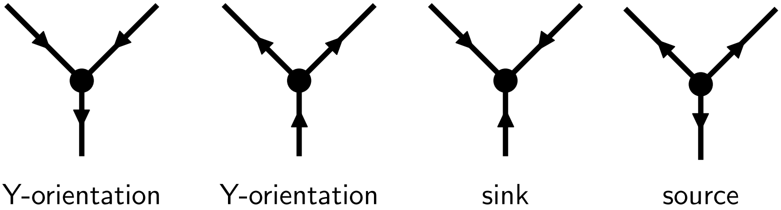
(i) $\forall\lambda,\mu\in\mathrm{Ob}(\mathcal{C})$ with $\lambda\neq\mu$, $\mathrm{Mor}(\lambda,\mu)=\emptyset$.

(ii) $\forall\lambda\in\mathrm{Ob}(\mathcal{C})$, $\forall a,b\in\mathrm{Mor}(\lambda,\lambda)$, $a\triangleleft b = b^{-1}ab$.

Groupoid rack colorings (1/4)

D: diagram of a spatial trivalent graph.

Y-orientation of D: orientation of D without sinks and sources.



Remark (Ishii 2015, Lebed 2015)

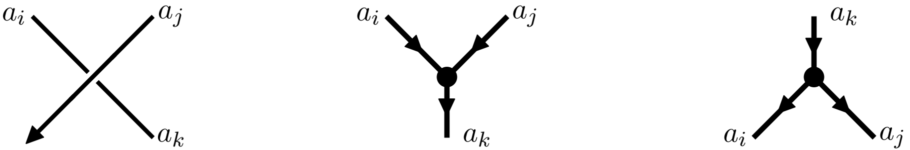
Every diagram admits a Y-orientation.

Y-oriented diagram: diagram with a Y-orientation.

Groupoid rack colorings (2/4)

D: Y-oriented diagram, X: set, $\triangleleft : X \times X \to X$, $P \subset X \times X$, $\mu : P \to X$.

$C : \mathcal{A}(D) \to X$: $\underline{X\text{-coloring}}$ of $D \overset{\text{def}}{\Longleftrightarrow} C$ satisfies the following conditions.



$C(a_i) \triangleleft C(a_j) = C(a_k)$ $\quad$ $\mu(C(a_i), C(a_j)) = C(a_k)$ $\quad$ $\mu(C(a_i), C(a_j)) = C(a_k)$

$\mathrm{Col}_X(D)$: the set of all X-colorings of D.

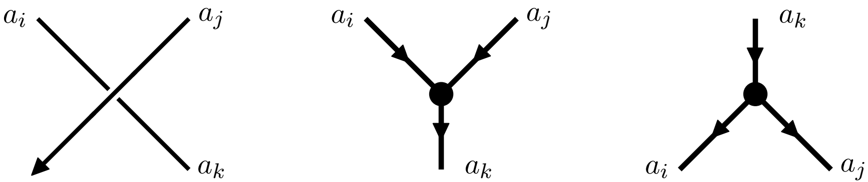
Main result 1 (A. 2025)

- $(X, \triangleleft)$: groupoid rack $\Rightarrow |\mathrm{Col}_X(D)|$: invariant of the spatial surface $F(D)$.
- $|\mathrm{Col}_X(D)|$ is an invariant of the spatial surface $F(D) \Rightarrow \bigcup_{(x,y)\in P} \{x, y\}$: groupoid rack.

Groupoid rack colorings (3/4)

D: Y-oriented diagram, $\mathcal{A}(D)$: the set of all arcs of D, X: groupoid rack.

$C : \mathcal{A}(D) \to X$: <u>X-coloring</u> of D $\overset{\text{def}}{\Longleftrightarrow}$ C satisfies the following conditions.



$C(a_i) \triangleleft C(a_j) = C(a_k)$ $\quad$ $C(a_i)C(a_j) = C(a_k)$ $\quad$ $C(a_i)C(a_j) = C(a_k)$

$\mathrm{Col}_X(D)$: the set of all X-colorings of D.

Theorem (Ishii 2015)

- X: MCQ $\Rightarrow$ $|\mathrm{Col}_X(D)|$ is an invariant of the handlebody-knot $H(D)$.
- $|\mathrm{Col}_X(D)|$ is an invariant of the handlebody-knot $H(D)$ $\Rightarrow$ X: MCQ.

Groupoid rack colorings (4/4)

Definition (Saito-Zappala 2024)

$R = (R, *)$: rack.
$R \times R$ is a rack with $(x, y) \triangleleft (z, w) = ((x *^{-1} z) * w, (y *^{-1} z) * w)$, where $x *^{-1} z := S_z^{-1}(x)$.
The heap rack $R \times R$ is the rack $(R \times R, \triangleleft)$ with the partial operation $(x, y)(y, z) = (x, z)$.

A heap rack X can be regarded as a groupoid rack.
D: Y-oriented diagram.

Theorem (Saito-Zappala 2024)

$|\mathrm{Col}_X(D)|$ is an invariant of the spatial surface $F(D)$.

Introduction of §3

Main result 2

$\forall L$: oriented link, $\exists \{(F_n, F'_n)\}_{n\in\mathbb{N}}$: pairs of Seifert surfaces for L s.t.:

(i) $\forall n \in \mathbb{N}$, $N(F_n) \cong N(F'_n)$ as handlebody-knots.

(ii) $\forall n \in \mathbb{N}$, Seifert matrices of F_n and F'_n are unimodular-congruent.

(iii) $\forall n \in \mathbb{N}$, $F_n \ncong F'_n$ as spatial surfaces.

Outline of proof

(Step 1) Construct such a family $\{(F_n, F'_n)\}_{n\in\mathbb{N}}$.
(Step 2) Check that $\forall m, n \in \mathbb{N}$, $F_m \ncong F_n$.
(Step 3) Verify the claims (i)–(iii).

(Step 1) Construct $\{(F_n, F'_n)\}_{n\in\mathbb{N}}$ (1/3)

L: oriented link.
F: Seifert surface for L with $F \ncong$ (2-disk).
D: diagram of F.

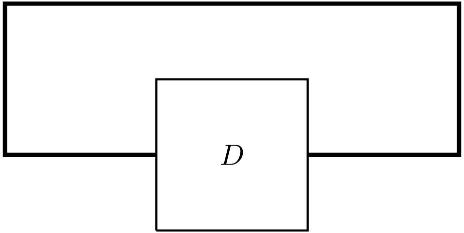
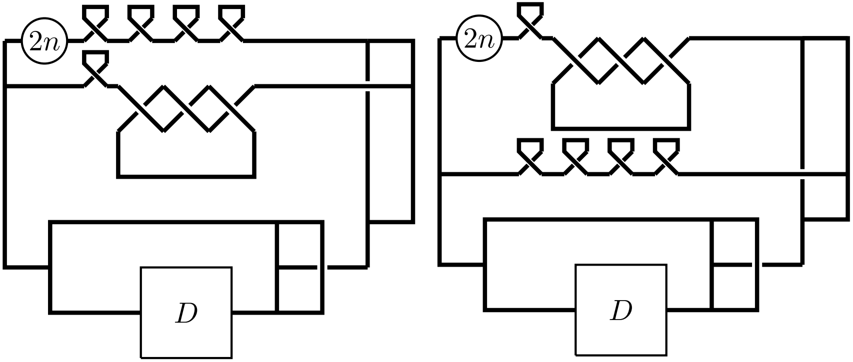


Figure: A diagram D of the Seifert surface F

(Step 1) Construct $\{(F_n, F'_n)\}_{n\in\mathbb{N}}$ (2/3)

For each $n \in \mathbb{N}$, take the following diagrams D_n and D'_n.

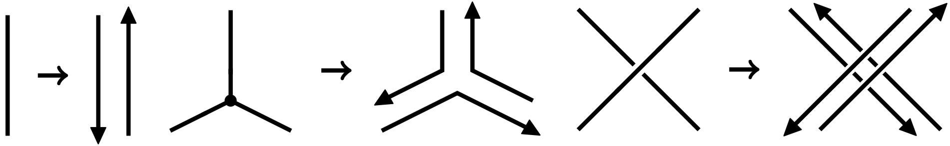


D_n D'_n


: positive n kinks.

(Step 1) Construct $\{(F_n, F'_n)\}_{n\in\mathbb{N}}$ (3/3)

- $\partial F(D_n) \cong \partial F(D'_n) \cong \partial F(D) \cong L$.



- For each $n \in \mathbb{N}$, define $F_n := F(D_n)$ and $F'_n := F(D'_n)$.
- $\{(F_n, F'_n)\}_{n\in\mathbb{N}}$: pairs of Seifert surfaces for L.

(Step 2) Check that $\forall m, n \in \mathbb{N}$, $F_m \not\cong F_n$

V: Seifert matrix of the Seifert surface $F(D)$ for L.

For each $k \in \mathbb{N}$, F_k has the Seifert matrix $V_k = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ -1 & 0 & 4 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 4+2k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & V \end{pmatrix}$.

$s = \max\{i \in \mathbb{N} \mid \exists$ non-zero $(i \times i)$-minor of $V\}$ (If $\nexists\, s$, we set $s = 0$).

$E_{k,3+s} := \gcd\{(3+s) \times (3+s)\text{-minors of } V_k\}$.

Definition and Proposition

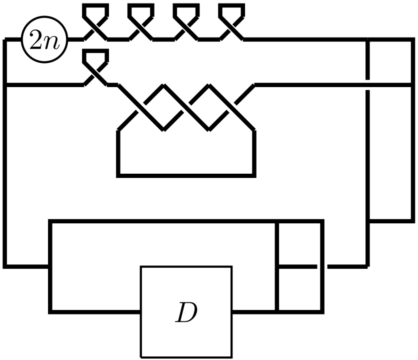
F_1, F_2: spatial surfaces, V_i: Seifert matrix of F_i $(i = 1, 2)$.

(i) $F_1 \cong F_2 \Rightarrow V_1$ and V_2 are unimodular-congruent, i.e., $\exists P$: unimodular s.t. $V_1 = P^T V_2 P$.

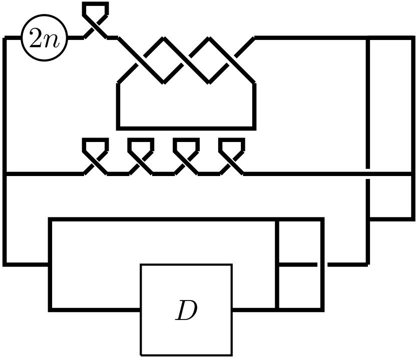
(ii) $\forall k \in \mathbb{N}$, $\gcd\{k \times k\text{-minors of } V_i\}$ is an invariant of F_i.

For any $m, n \in \mathbb{N}$, $E_{m,3+s} \neq E_{n,3+s}$. By the above Prop., $F_m \not\cong F_n$.

(Step 3) (i) $N(F_n) \cong N(F'_n)$



D_n



D'_n

- $N(F_n) \cong H(D_n)$ and $N(F'_n) \cong H(D'_n)$.

(Step 3) (ii) Seifert matrices of F_n and F'_n are unimodular-congruent

V: Seifert matrix of the spatial surface $F(D)$.

- $\forall n \in \mathbb{N}$, F_n and F'_n have the same Seifert matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & \mathbf{0} \\ -1 & 0 & 4 & 0 & \mathbf{0} \\ -1 & 0 & 0 & 4+2n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & V \end{pmatrix}$.

(Step 3) (iii) $F_n \not\cong F'_n$ (1/5)

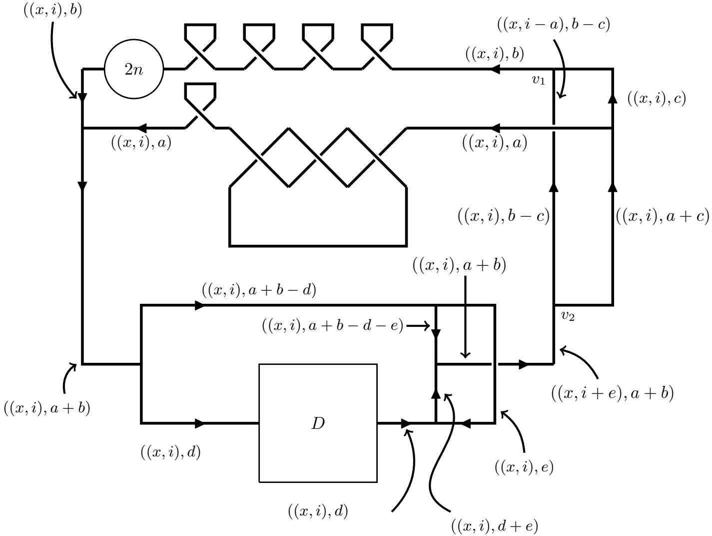
Example (MGR)

$X = (R_3 \times C_2) \times \mathbb{Z}_2 = \bigsqcup_{(x,i)\in R_3\times C_2} \{(x,i)\} \times \mathbb{Z}_2$ is an MGR with

$$((x,i),a) \triangleleft ((y,j),b) = \begin{cases} ((x,i),a) & \text{if } b = 0, \\ ((2y-x, i+1), a) & \text{if } b = 1 \end{cases}, \quad ((x,i),a)((x,i),b) = ((x,i),a+b).$$

This construction is motivated by (Ishii 2015, Ishii-Matsuzaki-Murao 2020).

(Step 3) (iii) $F_n \not\cong F'_n$ (2/5): X-coloring of D_n



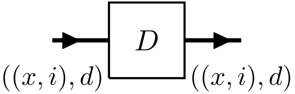
Coloring conditions at vertices v_1 and v_2.

$$\begin{cases} i - a = i \pmod 2, \\ i + e = i \pmod 2. \end{cases}$$

Thus,

$$a = 0, \ e = 0.$$

(Step 3) (iii) $F_n \not\cong F'_n$ (3/5)

$\mathrm{Col}_X(D;((x,i),d)) := \mathrm{Col}_X\Big($ 

$\Big).$

Then,

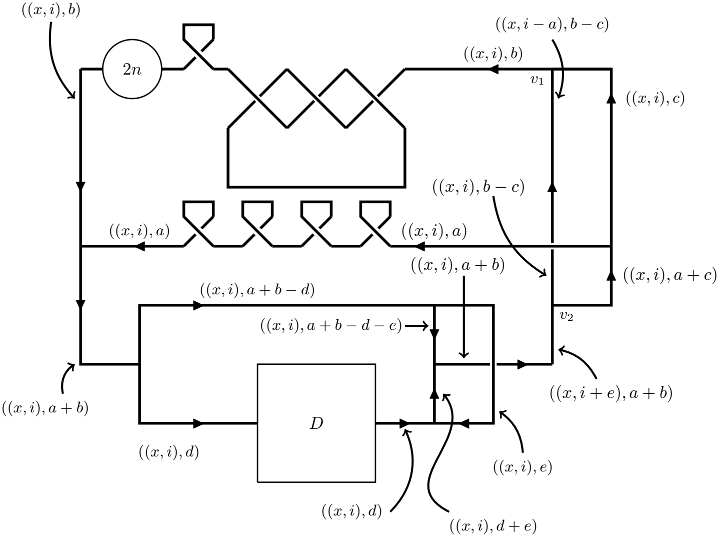
$$|\mathrm{Col}_X(D_n)| = \left| \bigsqcup_{((x,i),(b,c,d))\in(R_3\times C_2)\times\mathbb{Z}_2^3} |\mathrm{Col}_X(D;((x,i),d))| \right.$$

$$= 4\left(\bigsqcup_{((x,i),d)\in X} |\mathrm{Col}_X(D;((x,i),d))| \right).$$

Remark

$|\mathrm{Col}_X(D_n)| > 0$ because $|\mathrm{Col}_X(D;((x,i),0))| > 0$ for any $(x,i) \in R_3 \times C_2$.

(Step 3) (iii) $F_n \not\cong F'_n$ (4/5): X-coloring of D'_n



Coloring conditions at vertices v_1 and v_2.

$$\begin{cases} i - a = i \pmod 2, \\ i + e = i \pmod 2. \end{cases}$$

Thus,

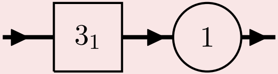
$$a = 0, \;\; e = 0.$$

(Step 3) (iii) $F_n \not\cong F'_n$ (5/5)

Remark

For any $(x,i) \in R_3 \times C_2$,

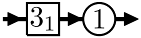
$$\bigsqcup_{b\in C_2} \left| \mathrm{Col}_X(\right.$$



$$\left. ; (x,i), b) \right| = \left|\mathrm{Col}_X(\textcircled{4}; (x,i), 0)\right| + 3\left|\mathrm{Col}_X(\textcircled{4}; (x,i), 1)\right|$$

$$= 4.$$

$$\left|\mathrm{Col}_X(D'_n)\right| = \bigsqcup_{((x,i),(b,c,d))\in(R_3\times C_2)\times\mathbb{Z}_2^3} \Bigg(\left|\mathrm{Col}_X(D, ((x,i),d))\right| \cdot \Bigg|\mathrm{Col}_X($$



$$, ((x,i),b))\Bigg|\Bigg)$$

$$\overset{\text{Rem.}}{=} 8 \left(\bigsqcup_{((x,i),d)\in X} \left|\mathrm{Col}_X(D, ((x,i),d))\right| \right) > \left|\mathrm{Col}_X(D_n)\right|.$$

Therefore, $F_n \not\cong F'_n$.

Thank you for your attention!!