On metrics for quandles

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Introduction (1/2)

- A ***quandle*** is an algebraic system, which can be regarded as a generalization of the conjugation of a group.
- Quandles are studied in many branches of mathematics;
  - as an invariant of knots in knot theory,
  - as discretization of symmetric spaces in symmetric space theory.
- Much research has focused on finite quandles;
  - for computability of knot invariants,
  - for correspondence of compact symmetric spaces.
- On the other hand, there are many interesting quandles on infinite sets.
  - knot quandles for most (classical or surface) knots,
  - discrete subquandles of non-compact symmetric spaces.

Motivation

We want to construct some tools to study countable quandles.

Introduction (2/2)

Idea

Introduce some notions and ideas in geometric group theory to quandle theory.

Geometric group theory

Discrete groups are studied from a geometric view point.

(1) (group G, generating set S) $\rightsquigarrow$ the word metric d_S (i.e. a Cayley graph).

(2) $S, T \subset G$: finite generating sets
$\Longrightarrow$ (G, d_S) and (G, d_T) are quasi-isometric.

(3) Quasi-isometry invariants (e.g. $\#$ ends, Gromov hyperbolicity, ...) of the metric space (G, d_S) become invariants of the group G.

Contents

1. Preliminary: Quandles
   - review the notion of quandles and some natural actions on them.
2. Preliminary: Schreier graphs
   - review the Schreier graph, which is a generalization of the Cayley graph.
3. Metrics for quandles
   - define two metrics for a quandle using natural actions of groups.
   - these metrics are determined up to the choice of “generating sets”.
4. Examples
   - give some examples of quandles quasi-isometric to typical metric spaces.

Quandles

Definition (Joyce (1982), Matveev (1982))

X: non-empty set with a binary operation $\triangleleft : X \times X \to X$.
$(X, \triangleleft)$: ***quandle*** if the following conditions hold:

1. $x \triangleleft x = x$ for any $x \in X$,
2. the map $s_y : X \to X$ defined by $s_y(x) := x \triangleleft y$ is bijective for any $y \in X$,
3. $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ for any $x, y, z \in X$.

- The map $s_y : X \to X$ is called the ***point symmetry*** at y.
- $x \triangleleft^{-1} y := s_y^{-1}(x)$.

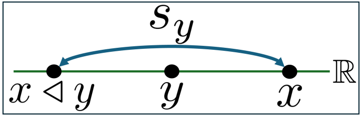
Example of quandles

Example (conjugation quandle)

G: group, $z \in G$, $z^G := \{g^{-1}zg \mid g \in G\}$.
$\rightsquigarrow \mathrm{Conj}(z^G) := (z^G, x \triangleleft y := y^{-1}xy)$: the ***conjugation quandle*** on x^G.

Example (infinite dihedral quandle)

$R_\infty := (\mathbb{Z}, x \triangleleft y := 2y - x)$: the ***infinite dihedral quandle***.



$R_\infty \cong \mathrm{Conj}(R_0^{D_\infty})$,
where D_∞: the infinite dihedral group, $R_0 \in D_\infty$: the reflection at 0.

Groups acting on quandles

- $\mathrm{Aut}(X) := \{f : X \to X \mid f$: bij. qdle. hom.$\}$: the automorphism group.
  - $fg := g \circ f$ $(f, g \in \mathrm{Aut}(X)) \rightsquigarrow X \curvearrowleft \mathrm{Aut}(X)$.
- X: ***homogeneous*** $\overset{\text{def}}{\iff} X \curvearrowleft \mathrm{Aut}(X)$: transitive.
- $\mathrm{Inn}(X) := \langle s_x \mid x \in X \rangle_{\mathrm{grp}} < \mathrm{Aut}(X)$: the ***inner automorphism group***.
- X: ***connected*** $\overset{\text{def}}{\iff} X \curvearrowleft \mathrm{Inn}(X)$: transitive.
  Each orbit of $X \curvearrowleft \mathrm{Inn}(X)$ is called a ***connected component***.
- $\mathrm{Dis}(X) := \langle s_x s_y^{-1} \mid x, y \in X \rangle_{\mathrm{grp}} < \mathrm{Aut}(X)$: the ***displacement group***.

Example

- $\mathrm{Inn}(R_\infty) = \langle s_0 : z \mapsto -z, s_1 : z \mapsto -z + 2 \rangle \cong D_\infty$.
- $\mathrm{Dis}(R_\infty) = \langle s_1 s_0 : z \mapsto z + 2 \rangle \cong 2\mathbb{Z}$.

Schreier graph

Schreier grpah

X: set, G: group, $S \subset G$: generating set, $X \curvearrowleft G$.
Then, the ***Schreier graph*** $\mathrm{Sch}(X; G, S)$ is a graph with

Vertex set $V = X$,

Edge set $E = \{(x, x \cdot g) \mid x \in X, g \in S\}$.

Remark

$G \curvearrowleft G$ given by $x \cdot g := xg$
$\rightsquigarrow \mathrm{Sch}(G; G, S)$ is the Cayley graph of G with respect to S.

Properties of Schreier graph

- $\{\text{connected components} \subset \mathrm{Sch}(X; G, S)\} \overset{1:1}{\longleftrightarrow} \{\text{orbits of } X \curvearrowleft G\}$.
- Each connected component of $\mathrm{Sch}(X; G, S)$ becomes a metric space by the path metric d_S^{Sch}.
- An orbit O is a metric space by the restriction of d_S^{Sch}.

$\rightarrow$ The metric space (O, d_S^{Sch}) depends on the choice of S.

Proposition

$S, T \subset G$: finite generating sets, O: an orbit of $X \curvearrowleft G$.
$\implies \mathrm{id} : (O, d_S^{\mathrm{Sch}}) \to (O, d_T^{\mathrm{Sch}})$: quasi-isometry.

$\rightarrow$ The quasi-isometry class $[(O, d_S^{\mathrm{Sch}})]$ is determined independently of S.

Schreier graphs associated with quandles

Definition

X: quandle.

1. $\Gamma_A^{\mathrm{Inn}}(X) := \mathrm{Sch}(X; \mathrm{Inn}(X), A)$, where $A \subset \mathrm{Inn}(X)$: generating set.
2. $\Gamma_U^{\mathrm{Dis}}(X) := \mathrm{Sch}(X; \mathrm{Dis}(X), U)$, where $U \subset \mathrm{Dis}(X)$: generating set.

- $\Gamma_A^{\mathrm{Inn}}(X)$ is a generalization of a ***diagram of a quandle*** defined by Winker (1984).

$(\because)$ $T \subset X$: generating set as quandles $\implies$ $s(T) \subset \mathrm{Inn}(X)$: generating set.

Proposition

$\{\text{conn. comp.} \subset \Gamma_A^{\mathrm{Inn}}(X)\} \overset{1:1}{\longleftrightarrow} \{\text{conn. comp.} \subset X\} \overset{1:1}{\longleftrightarrow} \{\text{conn. comp.} \subset \Gamma_U^{\mathrm{Dis}}(X)\}$.

$\to$ Each component becomes a metric space by the path metric.

- The metric d_A^{Inn} induced by $\Gamma_A^{\mathrm{Inn}}(X)$ is called the ***inner metric***.
- The metric d_U^{Dis} induced by $\Gamma_U^{\mathrm{Dis}}(X)$ is called the ***displacement metric***.

Independence of the choise of the generating set

Theorem

X: quandle, $O \subset X$: connected component.

1. $A, B \subset \mathrm{Inn}(X)$: finite generating sets $\implies (O, d_A^{\mathrm{Inn}}) \sim_{\text{q.i.}} (O, d_B^{\mathrm{Inn}})$.
2. $U, V \subset \mathrm{Dis}(X)$: finite generating sets $\implies (O, d_U^{\mathrm{Dis}}) \sim_{\text{q.i.}} (O, d_V^{\mathrm{Dis}})$.

Remark

- X: finitely generated $\implies \mathrm{Inn}(X)$: finitely generated.
- $\exists X$: finitely generated quandle s.t. $\mathrm{Dis}(X)$: **NOT** finitely generated.

Difference between the two metrics

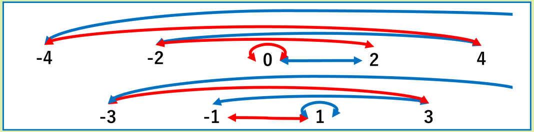
In general, d^{Inn} and d^{Dis} are **NOT** quasi-isometric.

example: R_∞

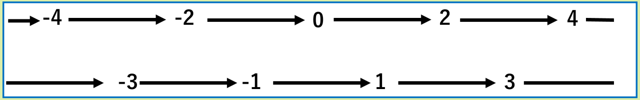
$X := R_\infty = (\mathbb{Z}, x \lhd y := 2y - x)$.

- $S := \{s_0 : z \mapsto -z, s_1 : z \mapsto -z + 2\} \subset \mathrm{Inn}(X)$: generating set.
- $T := \{s_0 s_1 : z \mapsto z + 2\} \subset \mathrm{Dis}(X)$: generating set.

$\Gamma_S^{\mathrm{Inn}}(R_\infty)$:



$\Gamma_T^{\mathrm{Dis}}(R_\infty)$:



$\rightsquigarrow$ They are not quasi-isometric since #ends are different.

Special cases

Proposition

X: homogeneous quandle, $O, O' \subset X$: connected component.

- $A \subset \mathrm{Inn}(X)$: finite generating set $\Longrightarrow$ $(O, d_A^{\mathrm{Inn}}) \sim_{\mathrm{q.i.}} (O', d_A^{\mathrm{Inn}})$.
- $U \subset \mathrm{Dis}(X)$: finite generating set $\Longrightarrow$ $(O, d_U^{\mathrm{Dis}}) \sim_{\mathrm{q.i.}} (O', d_U^{\mathrm{Dis}})$.

Proposition

X: quandle, $U \subset \mathrm{Dis}(X)$: finite generating set.
$O \subset X$: connected component, $O \curvearrowleft \mathrm{Dis}(X)$: free
$\Longrightarrow$ $(\mathrm{Dis}(X), d_U) \sim_{\mathrm{q.i.}} (O, d_U^{\mathrm{Dis}})$.

- X: non-trivial quandle $\Longrightarrow$ $X \curvearrowleft \mathrm{Inn}(X)$: **NOT** free.

$(\because)$ $s_x(x) = x \triangleleft x = x$ for any $x \in X$.

Free quandle

A: finite set with $\#A \geq 2$, $F[A]$: free group on A.
$FQ[A] := (A \times F[A])/(a,u) \sim (a, a^n u)$.
$\rightsquigarrow (FQ[A], [(a,u)] \triangleleft [(b,v)] := [a, uv^{-1}bv])$: ***free quandle***.

Proposition

- $\bar{A} := \{s_a \mid a \in A\} \subset \mathrm{Inn}(FQ[A])$: generating set.
- $\forall O \subset FQ[A]$: connected component, $(O, d_{\bar{A}}^{\mathrm{Inn}}) \sim_{\mathrm{q.i.}}$ tree.

Remark

$\mathrm{Dis}(FQ[A])$ is NOT finitely generated.

Generalized Alexander quandles

G: group, $\sigma : G \to G$: group automorphism.

$\rightsquigarrow \mathrm{GAlex}(G,\sigma) := (G, g \triangleleft h := \sigma(gh^{-1})h)$: ***generalized Alexander quandle***.

- $O \curvearrowleft \mathrm{Dis}(X)$: free $\iff O \cong \mathrm{GAlex}$.
- GAlex: homogeneous quandle.

Theorem

$X := \mathrm{GAlex}(G,\sigma)$, $S \subset \mathrm{Dis}(X)$: finite generating set

$\implies \forall O \subset X$: connected component, $(O, d_S^{\mathrm{Dis}}) \sim_{\mathrm{q.i.}} (\mathrm{Dis}(X), d_S)$

Lemma (cf. Higashitani–Kurihara (2024))

Let $X := \mathrm{GAlex}(G,\sigma)$.

- $\mathrm{Dis}(X) \cong P := e \cdot \mathrm{Inn}(X) < G$.
- $\sigma : x \mapsto g^{-1}xg \ (g \in G) \implies P = [\langle\langle g \rangle\rangle_G, \langle\langle g \rangle\rangle_G]$.

($\langle\langle g \rangle\rangle_G$: the normal closure of $\{g\}$ in G.)

Ex: Euclidean space and hyperbolic plane

Proposition

$\forall O \subset \mathrm{GAlex}(\mathbb{Z}^n, t)$: conn. comp., $(O, d^{\mathrm{Dis}}) \sim_{\mathrm{q.i.}} \mathbb{E}^k$ $(k = \mathrm{rank}(t-1))$

Proposition

$\Delta^+(p,q,r) := \langle a,b,c \mid a^p = b^q = c^r = abc = 1 \rangle$, $\sigma : x \mapsto a^{-1}xa$.
$X := \mathrm{GAlex}(\Delta^+(p,q,r), \sigma)$.
$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \implies \forall O \subset X$: connected component, $(O, d^{\mathrm{Dis}}) \sim_{\mathrm{q.i.}} \mathbb{H}^2$.

Remark

$Y := \mathrm{Conj}(\{a\}^{\Delta^+(p,q,r)}) < \mathrm{Conj}(\Delta^+(p,q,r))$
$\implies \exists X \twoheadrightarrow Y$: quandle homomorphism & $Y \subset \mathbb{H}^2$: discrete subquandle.

Ex: 3-dimensional geometry (1/2)

$\mathcal{O}(K, n)$: 3-orbifold with

basespace S^3,

singular locus a knot $K \subset S^3$ with the cone-angle $2\pi/n$.

$G_n(K) := \pi_1^{\mathrm{orb}}(\mathcal{O}(K, n)) \cong G(K)/\langle\langle \mu^n \rangle\rangle_{G(K)}$ ($\mu \in G(K)$: meridian).

$\sigma \in \mathrm{Aut}(G_n(K)) : x \mapsto \mu^{-1} x \mu$, $X_n(K) := \mathrm{GAlex}(G(K), \sigma)$.

Proposition

- $\mathrm{Dis}(X) \cong \pi_1(M_n(K))$, where $M_n(K)$: n-branched covering space of S^3 along K.
- $\mathcal{O}(K, n)$: geometric

  $\Longrightarrow$ $\forall O \subset X_n(K)$: connected component, $(O, d^{\mathrm{Dis}}) \sim_{\mathrm{q.i.}} \widetilde{\mathcal{O}(K, n)}$.

- $\exists X_n(K) \twoheadrightarrow Q_n(K)$: quandle homomorphism.

Ex: 3-dimensional geometry (2/2)

We can construct many quandles quasi-isometric to 3-dim. Geometries:

$$S^2 \times \mathbb{R}, S^3, \mathbb{E}^3, \mathrm{Nil}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}}_2, \mathrm{Sol}, \mathbb{H}^3.$$

Example

- K: hyperbolic knot, n: "enough large" $\Longrightarrow$ $X_n(K) \sim_{\text{q.i.}} \mathbb{H}^3$ (cf. hyperbolic Dehn surgery theorem).
- $X_3(4_1) \sim_{\text{q.i.}} \mathbb{E}^3$, $X_6(3_1) \sim_{\text{q.i.}} \mathrm{Nil}$ (cf. Dunbar(1988)).
- $X_2(\text{Pretzel}(-2, 3, 7)) \sim_{\text{q.i.}} \widetilde{\mathrm{SL}}_2$ (cf. Montesinos trick).

We can distinguish many countable quandles !

Summary

- We have defined two metrics for each connected component of a quandle by natural actions.
- Their quasi-isometry classes are determined up to the choice of "generating sets", but they are different.
- We gave examples of quandles quasi-isometric to typical metric spaces.

On going work

- Construct quandles whose components with d^{Inn} are quasi-isometric to typical metric spaces.
- Study the case $X \curvearrowleft \mathrm{Dis}(X)$ is not free.
- Study quasi-isometric invariants for quandles.
  (e.g. properties of Gromov hyperbolic quandles.)