Skein modules of mapping tori of the 2-torus

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Intelligence of Low Dimensional Topology
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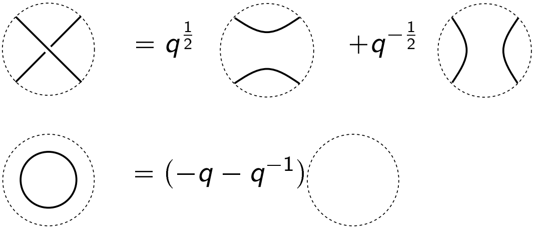
Skein modules

The skein module of a 3-manifold M is the $\mathbb{C}(q^{\frac{1}{2}})$-vector space spanned by isotopy classes of framed links in M, modulo the Kauffman bracket relations:



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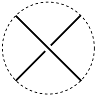
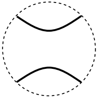
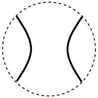
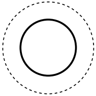
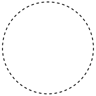
 $= q^{\frac{1}{2}}$  $+q^{-\frac{1}{2}}$ 

 $= (-q - q^{-1})$ 

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Note: We can interpret framed links locally as morphisms between tensor powers of the defining representation in $\mathrm{Rep}\, U_q\mathfrak{sl}_2$.
Considering a different Lie group G gives a more general notion of skein module $\mathrm{Sk}_G(M)$.
$\star$ = we know how to generalize to $G = \mathrm{GL}_N, \mathrm{SL}_N$.

Skein module dimensions

Let M be compact, oriented, and without boundary.

It was shown in [Gunningham-Jordan-Safronov 23] that

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Some computations:

$M = L(p,q)$	$\dim \mathrm{Sk}(M) = \lfloor \frac{p}{2} \rfloor + 1$	[Hoste-Przytycki 93]
$M = \mathbb{T}^3$	$\dim \mathrm{Sk}(M) \leqslant 9$	[Carrega 17]
	$\dim \mathrm{Sk}(M) \geqslant 9$	[Gilmer 18]
$M = \Sigma_g \times S^1$	$\dim \mathrm{Sk}(M) \leqslant 2^{2g+1} + 2g - 1$	[Gilmer-Masbaum 19]
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This talk: dimensions for mapping tori of $\mathbb{T}^2$, from [Kinnear 25].

Mapping tori of $\mathbb{T}^2$

Let $\gamma \in \mathrm{Mod}(\mathbb{T}^2)$. Consider

$$M_\gamma = (\mathbb{T}^2 \times [0,1])/((a,0) \sim (\gamma(a),1)).$$

Note $M_\gamma \cong M_\phi$ as oriented manifolds iff γ and ϕ are conjugate ($\sim$) in $\mathrm{Mod}(\mathbb{T}^2)$. Recall

$$\mathrm{Mod}(\mathbb{T}^2) \cong \mathrm{SL}_2(\mathbb{Z}) = \langle S, T | S^4 = \mathrm{Id}, (ST)^3 = S^2 \rangle.$$

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- $|\operatorname{tr}(\gamma)| = 0$: then $\gamma \sim \pm S$ for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $|\operatorname{tr}(\gamma)| = 1$: then $\gamma \sim \pm E^{\pm 1}$ for $E = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.
- $|\operatorname{tr}(\gamma)| = 2$: then $\gamma \sim \pm T^n$ for $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$.
- $|\operatorname{tr}(\gamma)| > 2$: classfication by continued fractions.

Main Theorem

Theorem

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*. Then* $\dim \mathrm{Sk}(M_\gamma)$ *is as follows:*

- $|\operatorname{tr}(\gamma)| = 0$*: then* $\dim \mathrm{Sk}(M_\gamma) = 6$*.*
- $|\operatorname{tr}(\gamma)| = 1$*: then* $\dim \mathrm{Sk}(M_\gamma) = 4$*.*
- $|\operatorname{tr}(\gamma)| = 2$*: then* $\gamma \sim \pm T^n$*, and*
$$\dim \mathrm{Sk}(M_\gamma) = \begin{cases} 9 + k & n = 2k \\ 6 + k & n = 2k + 1 \end{cases}.$$
- $|\operatorname{tr}(\gamma)| > 2$*: then*
$$\dim \mathrm{Sk}(M_\gamma) = |\operatorname{tr}(\gamma)| + 2^{c(\gamma)+1}$$
where
$$c(\gamma) = \#\{m \in \{\gcd(a-1, b, c, d-1), \operatorname{tr}(\gamma)\} : m \text{ even}\}.$$

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Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then $\dim \mathrm{Sk}(M_\gamma)$ is given by

$$\dim \mathrm{Sk}(M_\gamma) = 2^{c(\gamma)} + \frac{\prod_{i=1}^{r_+} a_i^+ + 2^{p_+}}{2} + \frac{\prod_{i=1}^{r_-} a_i^- + 2^{p_-}}{2}$$

where $r_\pm$ is the rank of

$$\mathrm{Id} \mp \gamma = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and

$$a_1^\pm = \gcd(w, x, y, z), \quad a_2^\pm = \frac{|\operatorname{tr}(\mathrm{Id} \mp \gamma)|}{a_1^\pm}$$

are the invariant factors of $\mathrm{Id} \mp \gamma$, and

$$p_\pm = \#\{a_i^\pm \text{ even} : 1 \leqslant i \leqslant r_\pm\}.$$

Elements of proof

Lemma ($\star$ [Carrega 17])

For M a 3-manifold, $\mathrm{Sk}(M)$ is graded by $\mathrm{H}_1(M;\mathbb{Z}/2\mathbb{Z})$.

$\rightsquigarrow$ For $M = M_\gamma$, skeins wrap the monodromy direction either an even or odd number of times.

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Let M be a 3-manifold, and fix $\mathbb{T}^2 \subset M$. Then $\mathrm{Sk}(M)$ is spanned by skeins which intersect $\mathbb{T}^2$ at most once.

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So

$$\mathrm{Sk}(M_\gamma) = S_0 \oplus S_1.$$

Twisted Hochschild homology

Recall $\mathrm{Sk}(\mathbb{T}^2 \times [0,1])$ is an algebra. Denote this $\mathrm{SkAlg}(\mathbb{T}^2)$.

We see that

$$S_0 = \mathrm{HH}_0^\gamma(\mathrm{SkAlg}(\mathbb{T}^2)) = \mathrm{SkAlg}(\mathbb{T}^2)/(ab - b\gamma(a)).$$

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Can give an explicit presentation of E as a 4-dimensional algebra, and calculate

$$\dim S_1 = \dim \mathrm{HH}_0^\gamma(E) = 2^{c(\gamma)}.$$

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All together gives:
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Corollary

There does not exist a symmetric monoidal functor

$$Z : \mathrm{Bord}^{\mathrm{or}}_{(4,3)} \to \mathrm{Vect}$$

sending a 3-manifold to its skein module, and giving the natural action of the mapping class groups of 3-manifolds.

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- But $\dim \mathrm{Sk}(M_\gamma)$ is not bounded above.

Further questions

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  What about $\mathrm{HH}_\bullet^\gamma$? (Bai, Kinnear)
- We considered $\mathrm{coker}(\mathrm{Id} \mp \gamma)$. In the paper [Chun-Gukov-Park-Sopenko 20] this is identified with the (almost) abelian flat $\mathrm{SL}_2(\mathbb{C})$-connections on M_γ, and is used to define the so-called $\hat{Z}$-invariant. Is there a connection with skein theory?

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