Knitted surfaces and their chart description : overview and the case of degree 2

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Intelligence of Low-dimensional Topology

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joint w/ Jumpei Yasuda.

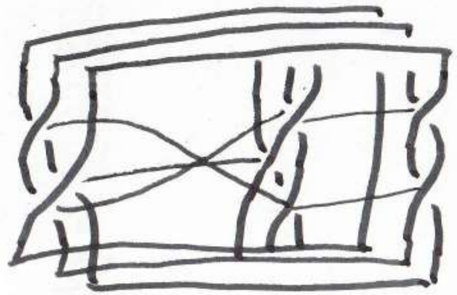
Contents

1. Intro.

braid in $D^2 \times I$

"trace of transformations of braids"

or generalization into surfaces in $D^2 \times B^2$



braided surface in $D^2 \times B^2$

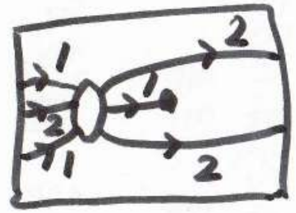
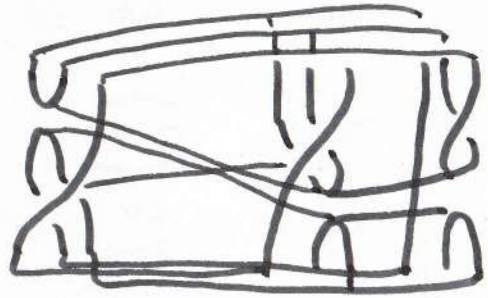


chart description

splice of crossings

knit



knitted surface

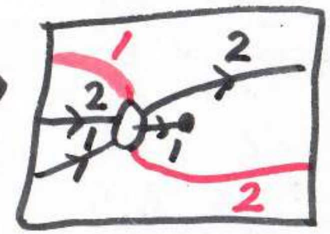


chart description

Main results

- $\forall$ Surface w/ ∂ properly embedded in $D^2 \times B^2$ is isotopic to a knitted surface : so it is presented by a chart.

maybe unorientable

- $\forall$ Trivial surface-link is presented by a chart of degree 2.
- The plat closure of $\forall$ knitted surface of degree 2 is a trivial surface-link.

2. Braids in $D^2 \times I$ & Braided surfaces in $D^2 \times B^2$

Def 1. β : a 1-mfd embedded in $D^2 \times I$

β is an m-braid

$\overset{\text{def}}{\Longleftrightarrow}$ • $p|_\beta : \beta \to I$ is a covering map of degree m

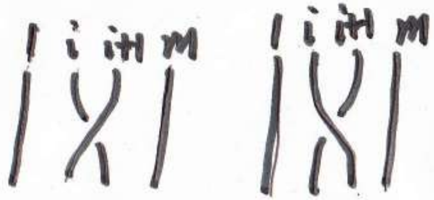
• $\partial\beta = \beta \cap \partial(D^2 \times I) = Q_m \times \partial I$

where $p : D^2 \times I \to I$ is the projection.

Fact An m-braid β is presented by an element of

$$B_m = \left\langle \sigma_1, \cdots, \sigma_{m-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j|>1) \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \ (|i-j|=1) \end{array} \right\rangle$$

the braid gp



σ_i $\qquad$ σ_i^{-1}

Def 2. S : a surface properly embedded in $D^2 \times B^2$

S is a simple braided surface of degree m

$\overset{\text{def}}{\Longleftrightarrow}$ • $p|_S : S \to B^2$ is a branched covering map of degree m

• $\#(S \cap p^{-1}(x)) = m$ or $m-1$ $(\forall x \in B^2)$

where $p : D^2 \times B^2 \to B^2$ is the projection.

Fact. S is described by "transf." of m-braids

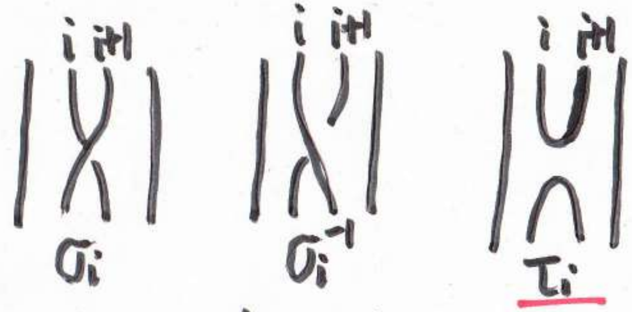
$\Updownarrow$

chart description

3. Knits in $D^2 \times I$ & Knitted surfaces in $D^2 \times B^2$

"Def" 3. β is a **knit** of degree m or a **BMW** (Birman-Murakami-Wenzl) **tangle**

$\iff$ ${}^\exists \beta_0$: an m-braid
s.t. β is obtained from β_0
by splice of some crossings



A knit β is presented by a word consisting of $\{\sigma_i, \sigma_i^{-1}, \tau_i \mid i = 1, \cdots, m-1\}$
generators of the **knit monoid**

"Def" 4 [N-Y] S : a surface properly embedded in $D^2 \times B^2$
S is a **knitted surface** of degree m or a **BMW surface**

$\iff {}^\exists h : B^2 \to [0,1]$: a surj. conti. map
s.t. $h^{-1}(t) \approx I$ $({}^\forall t \in [0,1])$: **height function**

s.t. $\{S \cap (D^2 \times h^{-1}(t))\}_{t \in [0,1]}$ is associated w/ "transformations" of knits of degree m.

Def 5

β : a tangle in $D^2 \times I$

ℓ : a simple arc embedded in $D^2 \times I$

s.t. $\ell \pitchfork \beta$.

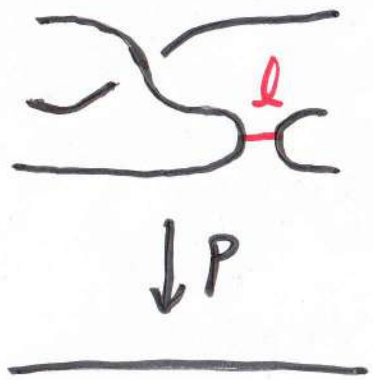
ℓ is a pairing of β

$\overset{\text{def}}{\Longleftrightarrow}$ · $\ell \cap \beta = \partial \ell \cap \mathrm{Crit}(\beta) = \partial \ell$

· $p|_\ell : \ell \to I$ has no critical pts.

· $\forall a \in \partial \ell$, $\exists$ local coordinates $(U : x)$ of a, and $(p(U) : t)$ of $p(a)$ s.t. $p|_\beta(x) = x^2$ & $p(\ell \cap U) = \{t \leqq 0\}$

where $p : D^2 \times I \to I$ is the projection.



Def 6

(β, ℓ) is an m-knit

$\overset{\text{def}}{\Longleftrightarrow}$ · β is a non-degenerate tangle w/ $\partial\beta = Q_m \times \partial I$

$p|_\beta : \beta \to I$ is a Morse function

· $\ell = \{\ell_1, \cdots, \ell_q\}$ is a set of mutually disjoint pairings of β

· $\mathrm{Crit}(\beta) \subset \ell_1 \cup \cdots \cup \ell_q$

ℓ: knit structure

Def 7

$(\beta_0, \ell_0), (\beta_1, \ell_1)$: m-knits

(β_0, ℓ_0) and (β_1, ℓ_1) are equivalent

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \{\varphi_t\}_{t\in[0,1]}$: isotopy of $D^2 \times I$ rel $\partial(D^2 \times I)$

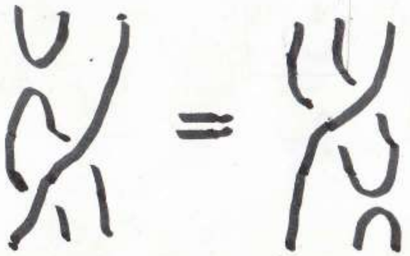
s.t. $\varphi_1(\beta_0) = \beta_1$, $\varphi_1(\ell_0) = \ell_1$, $\varphi_t(\beta_0 ; \ell_0)$ is an m-knit $(\forall t \in [0,1])$

$D_m := \{\, [\beta] \mid \beta \text{ is an } m\text{-knit} \,\}$

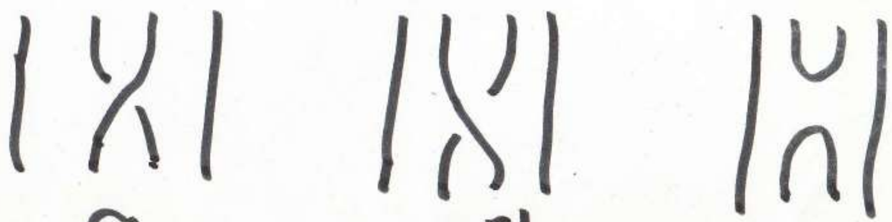
Prop 1.

$$D_m = \left\langle \begin{matrix} \sigma_1, \cdots, \sigma_{m-1}, \\ \sigma_1^{-1}, \cdots, \sigma_{m-1}^{-1}, \\ \tau_1, \cdots, \tau_{m-1} \end{matrix} \;\middle|\; \begin{matrix} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = e, \ \sigma_i\tau_i = \tau_i\sigma_i = \tau_i \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \ \tau_i\sigma_j\sigma_i = \sigma_j\sigma_i\tau_j \ (|i-j|=1) \\ \sigma_i\sigma_k = \sigma_k\sigma_i, \ \sigma_i\tau_k = \tau_k\sigma_i, \ \tau_i\tau_k = \tau_k\tau_i \ (|i-k|>1) \end{matrix} \right\rangle$$

monoid.



$\tau_i\sigma_{i+1}\sigma_i \qquad \sigma_{i+1}\sigma_i\tau_{i+1}$



$\sigma_i \qquad \sigma_i^{-1} \qquad \tau_i$

Remark $D_m \supset B_m$ submonoid

Def 8. S : a surface embedded in $D^2 \times B^2$

L : an orientable surface embedded in $D^2 \times B^2$

s.t. $L \pitchfork S$.

L is a <u>pairing</u> of S

$\overset{\text{def}}{\Longleftrightarrow}$

- $L \cap S = \partial L \cap \mathrm{Fold}(S)$ is a non-empty set and $\partial L \setminus \mathrm{Fold}(S)$ consists of simple open arcs in $D^2 \times \partial B^2$.
- $p|_L : L \to B^2$ has no critical pts.
- For $\forall a \in \mathrm{Fold}(S)$, $\exists$ local coordinates $(U : x, y)$ of a and $(p(U) : s, t)$ of $p(a)$ s.t. $p|_S(x, y) = (x, y^2)$ and

$$p(L \cap U) = \begin{cases} \{(s,t) \mid s \in \mathbb{R},\ t \leqq 0\} & \text{if } p(a) \notin \partial B^2 \\ \{(s,t) \mid s \geqq 0,\ t \leqq 0\} & \text{if } p(a) \in \partial B^2 \end{cases}$$

Def 9

$(S, \mathcal{L})$ is a knitted surface of degree m

$\overset{\text{def}}{\Longleftrightarrow}$

- S is a surface in $D^2 \times B^2$
- $\mathcal{L} = \{L_1, \cdots, L_q\}$ is a set of mutually disjoint pairings of S
- $S \cap (D^2 \times \{y_0\}) = Q_m \times \{y_0\}$, $(L_1 \cup \cdots \cup L_q) \cap (D^2 \times \{y_0\}) = \phi$
- $\mathrm{Fold}(S) \subset L_1 \cup \cdots \cup L_q$

$\mathcal{L}$: knit structure

Def 10

$(S_0, \mathcal{L}_0), (S_1, \mathcal{L}_1)$: knitted surfaces of degree m

$(S_0, \mathcal{L}_0)$ and $(S_1, \mathcal{L}_1)$ are equivalent

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \{\Phi_t\}_{t \in [0,1]}$: isotopy of $D^2 \times B^2$ rel $D^2 \times \{y_0\}$

s.t. $\Phi_1(S_0) = S_1$, $\Phi_1(\mathcal{L}_0) = \mathcal{L}_1$, $(\Phi_t(S_0), \Phi_t(\mathcal{L}_0))$ is a knitted surf.

$(\forall t \in [0,1])$

4. Chart description of braided surfaces & knitted surfaces

Chart of a simple braided surface

a simple braided surface $S \subset D^2 \times B^2$

$$D^2 \times B^2 = I \times I \times B^2 \xrightarrow{p_1 : \text{proj.}}$$

$$S \longrightarrow p_1(S) \subset I \times B^2$$

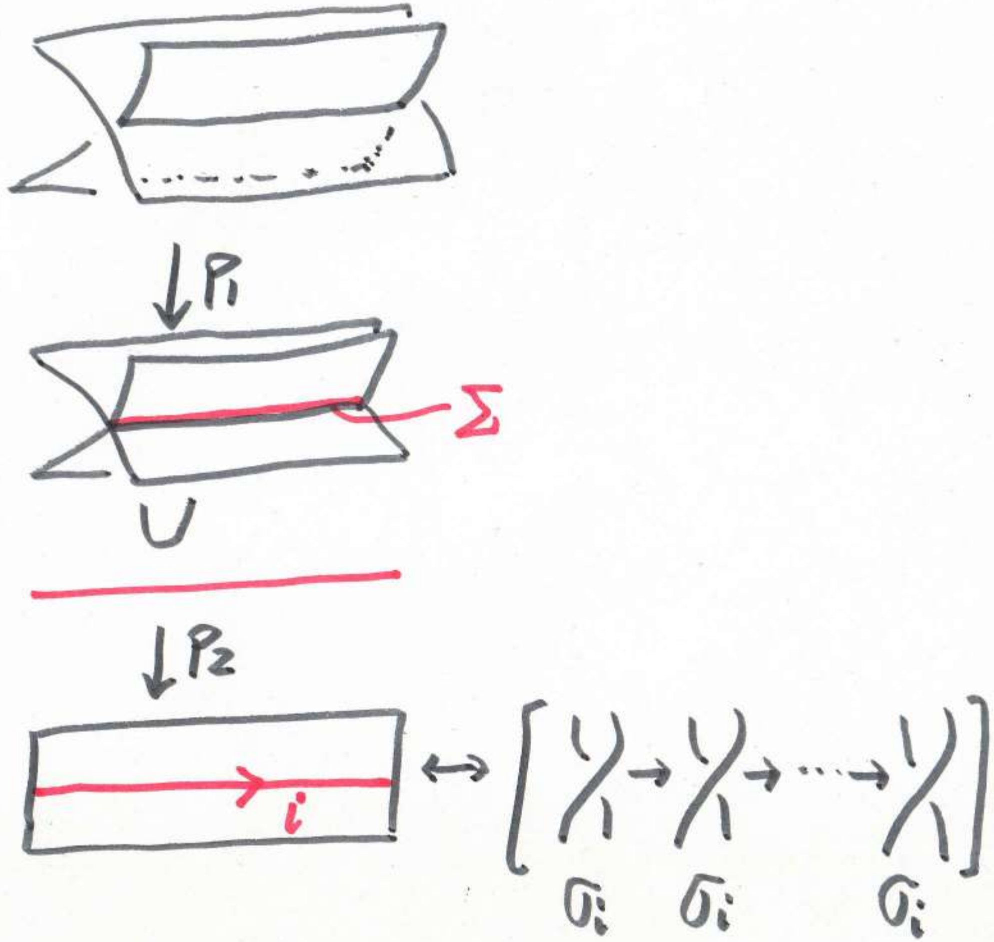
$$p_1(S) \supset \Sigma$$ singularity set

$$I \times B^2 \xrightarrow{p_2 : \text{proj.}} B^2$$

$$\Sigma \longrightarrow p_2(\Sigma) \subset B^2$$

$$p_2(\Sigma) = \Gamma$$

Γ equip a label & orientation to each edge $\rightsquigarrow$ the chart Γ of S

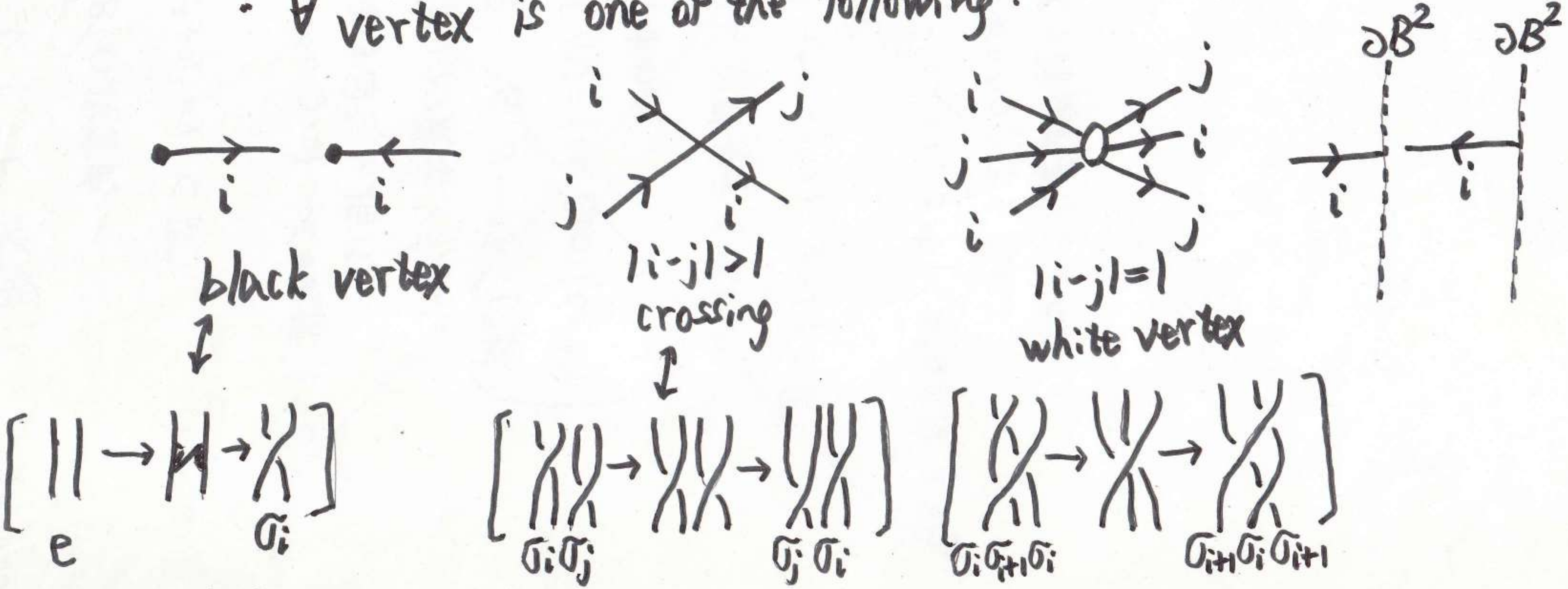


Def 11. Γ: a finite graph in B^2

Γ is a chart of degree m

$\overset{\text{def}}{\Longleftrightarrow}$ · $\forall$ edge has an ori. and a label $\in \{1, 2, \cdots, m-1\}$.

· $\forall$ vertex is one of the following:



Thm [Kamada]

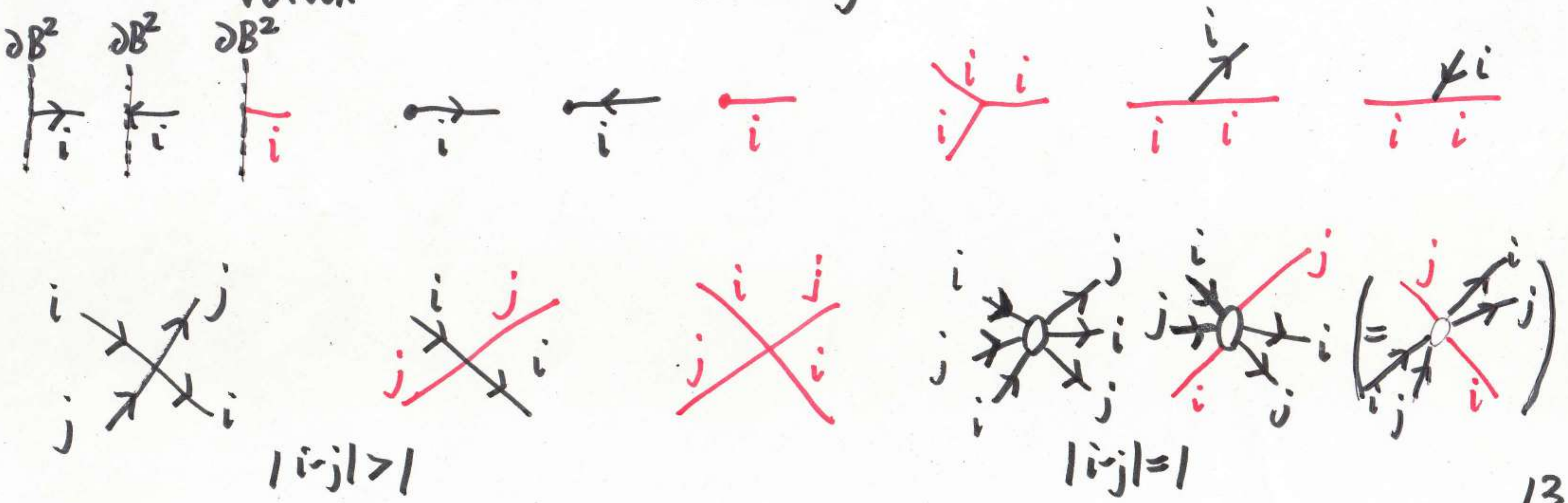
$\forall$ simple braided surface is described by a chart.

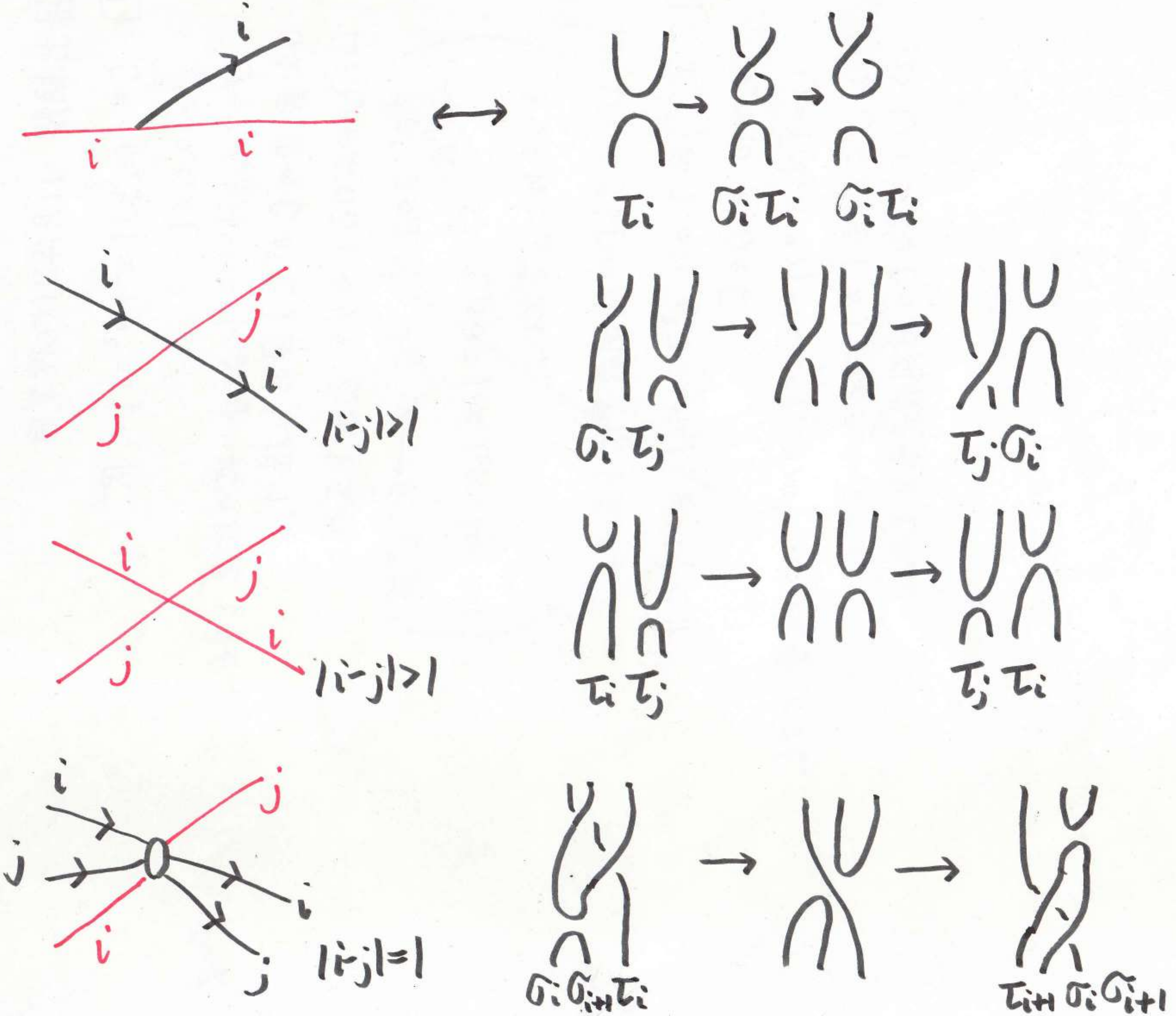
Charts of knitted surfaces

<u>Def 12.</u> Γ: a finite graph in B^2

Γ is a <u>chart</u> for a knitted surface of degree m (or a <u>BMW chart</u>)

$\overset{\text{def}}{\iff}$
- $\Gamma \cap \partial B^2$ consists of vertices of degree 1.
- $\forall$ edge has a label $\in \{1, 2, \cdots, m-1\}$.
- $\forall$ edge is either oriented or non-oriented.
- $\forall$ vertex is one of the following:





i
i
i
$\longleftrightarrow$
τ_i
$\sigma_i\tau_i$
$\sigma_i\tau_i$
i
j
j
i
$|i-j|>1$
$\sigma_i\tau_j$
$\tau_j\sigma_i$
i
j
j
i
$|i-j|>1$
$\tau_i\tau_j$
$\tau_j\tau_i$
i
j
j
i
i
j
$|i-j|=1$
$\sigma_i\sigma_{i+1}\tau_i$
$\tau_{i+1}\sigma_i\sigma_{i+1}$

Thm 1 [NY] Any knitted surface is presented by a chart.

i.e. For $\forall$ knitted surface S, $\exists$ chart Γ of S,

and for $\forall$ chart Γ, $\underbrace{\text{the surface } S \text{ presented by } \Gamma}_{\text{locally constructed surf.}}$,

is a knitted surface.

Thm 2 [NY]

Any surface w/ ∂ properly embedded in $D^2 \times B^2$

is isotopic to a knitted surface.

5. Charts of degree 2 and trivial surface-knots

__Def.__ F : a surface-knot in $\mathbb{R}^4$

F is __trivial__ $\overset{def}{\Longleftrightarrow}$ $F \cong S^2$ or $(\# T)\#(\# P_+)\#(\# P_-)$

standard 2-sphere

 $\subset \mathbb{R}^3\times\{0\} \subset \mathbb{R}^4$

standard torus

 $\subset \mathbb{R}^3\times\{0\} \subset \mathbb{R}^4$

standard positive projective plane

standard negative proj. plane

__Def.__ Γ : a 2-chart (chart of degree 2) s.t.

$S(\Gamma)$: knitted surf. presented by Γ

The __plat closure__ of $S(\Gamma)$ is the surface-link obtained from $S(\Gamma)\subset D^2\times B^2\subset\mathbb{R}^4$ by pasting  to  $=\partial S(\Gamma)\subset D^2\times\partial B^2$.

etc.

Main Thm [NY]

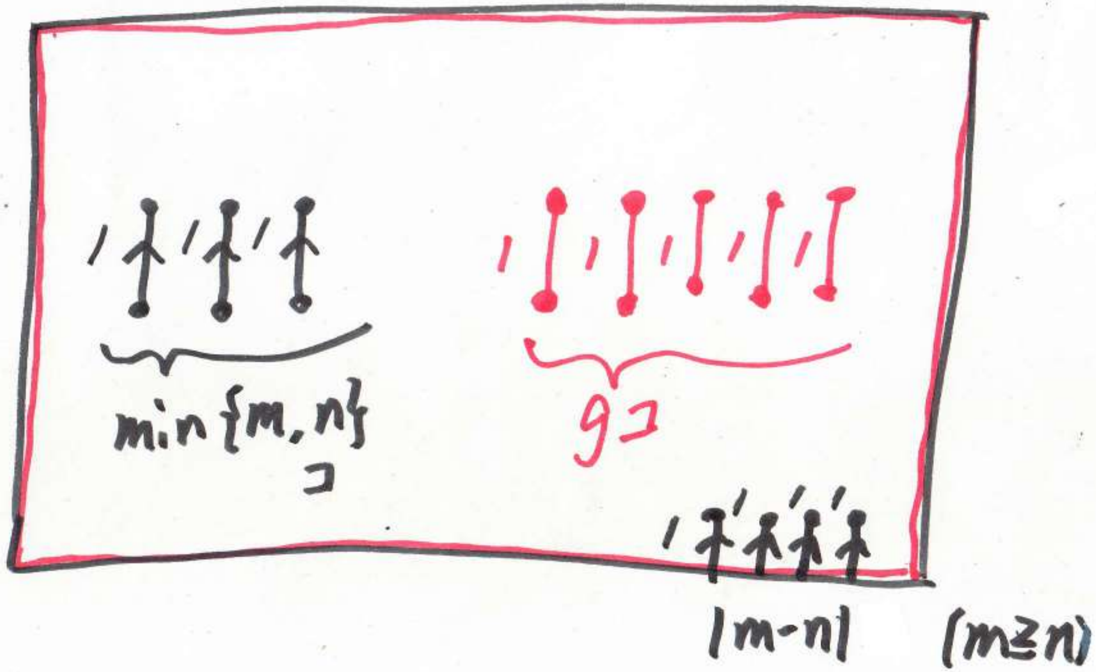
(1) $\forall$ Trivial surface-link is ambiently isotopic to the plat closure of some knitted surface of degree 2.

(2) The plat closure of $\forall$ knitted surface of degree 2 is a trivial surface-link.

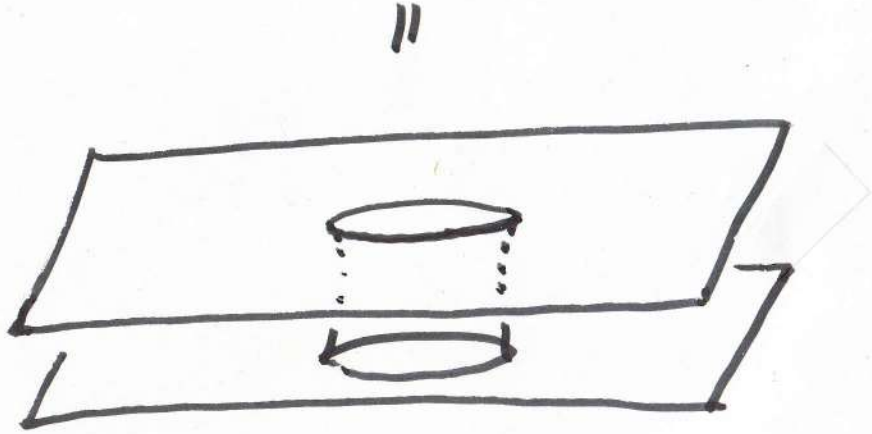
Thm 3 [NY] F : a trivial surface-knot s.t.

$$F \cong (\underset{g}{\#} T) \# (\underset{m}{\#} P_+) \# (\underset{n}{\#} P_-).$$

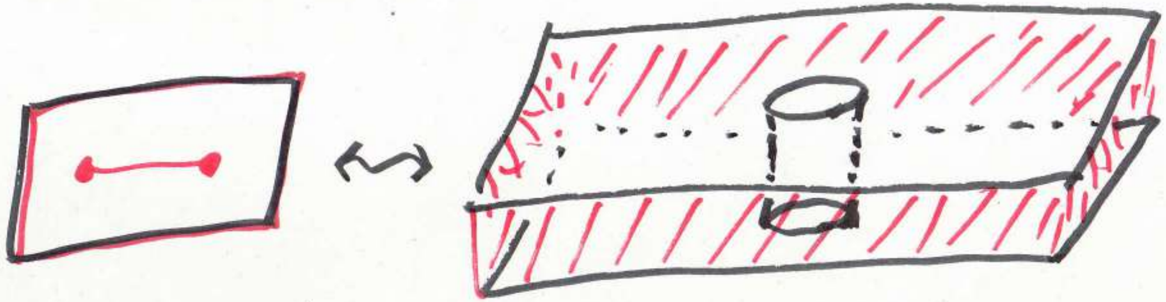
Then, F is the plat closure of the knitted surface presented by a 2-chart



$\because$)

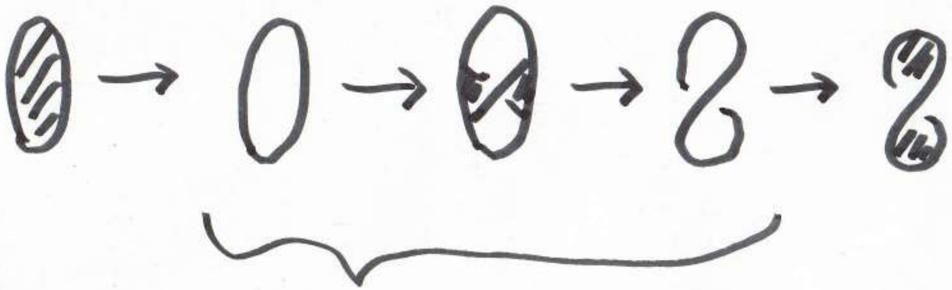


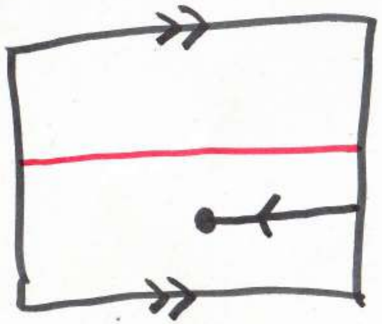
plat closure



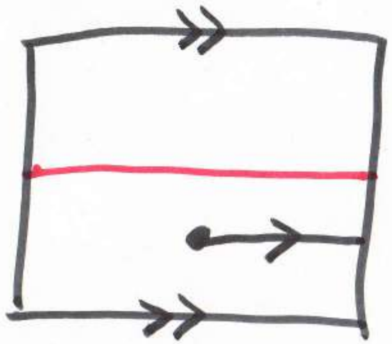
$\therefore$  presents $\#_g T$.

P_+ : 

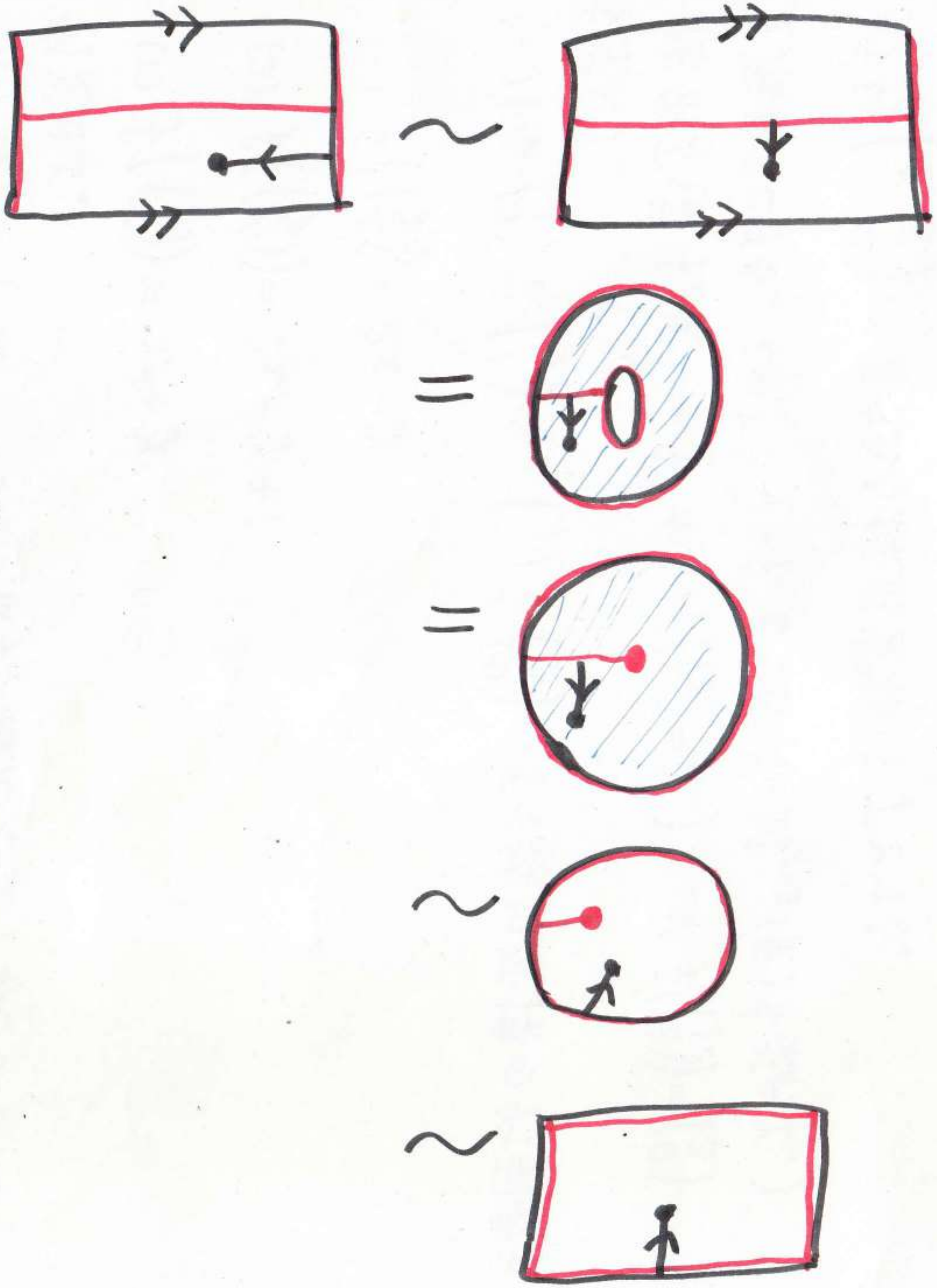
P_- : 

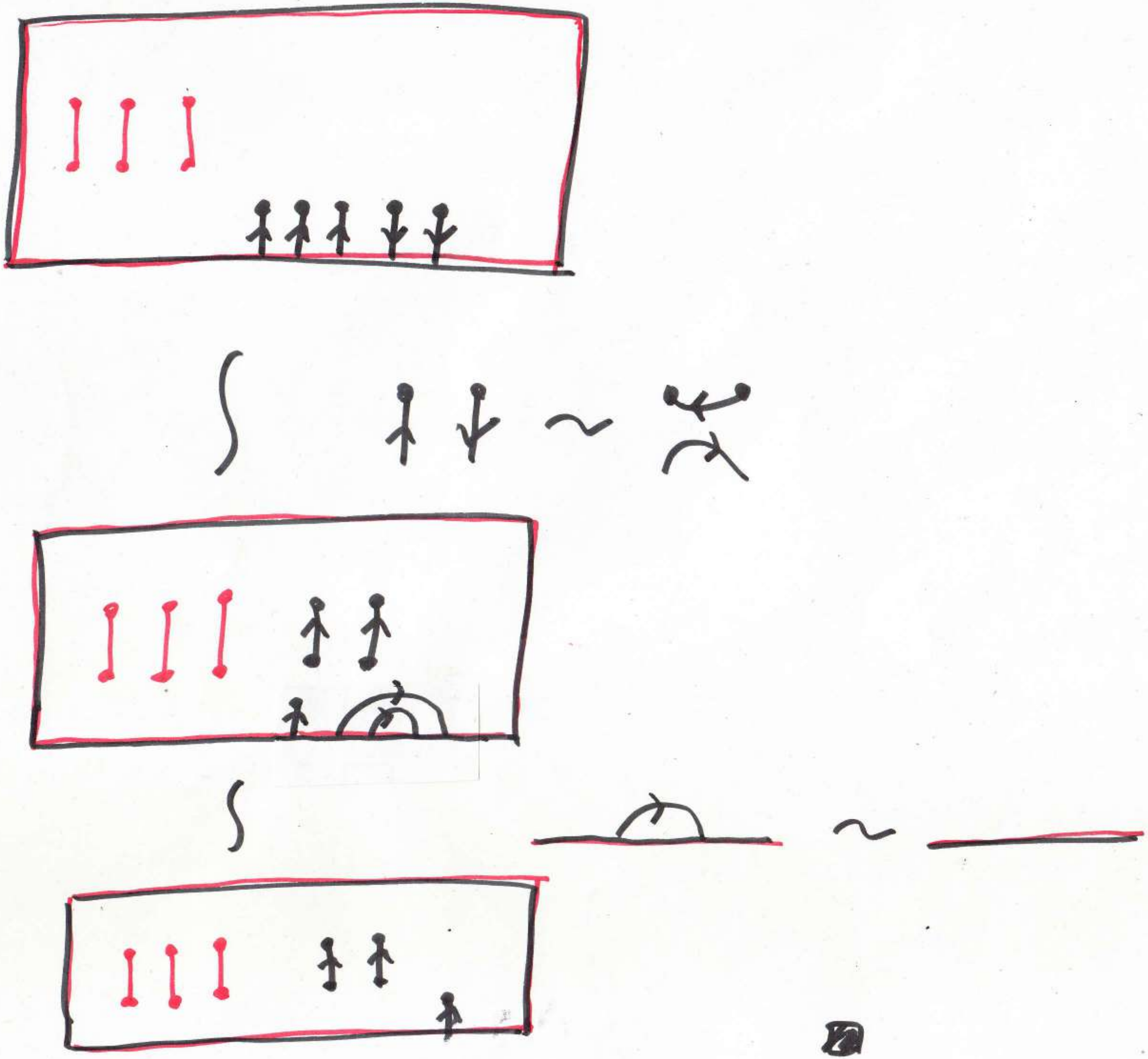


P_+



P_-





Thank you for your attention.