Infinitely many virtual knots whose virtual unknotting number equals one and a sequence of n-writhes

Migiwa Sakurai
(Shibaura Institute of Technology)
joint work with
Yoshiyuki Ohyama
(Tokyo Woman's Christian University)

May 27, 2025

Main theorem

In this talk, we show the following theorem.

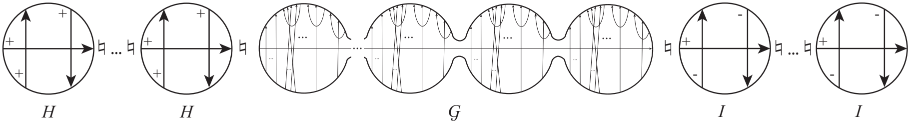
Theorem 1

Let $\{r_n\}_{n\neq 0}$ be a sequence of integers. If $\sum_{n\neq 0} nr_n = 0$, then there exist infinitely many virtual knots K_m $(m \in \mathbb{N})$ such that

$$u^v(K_m) = 1 \text{ and } J_n(K_m) = r_n$$

for any $n \neq 0$.

Construction methods:



$$\mathcal{G}_m = \underbrace{H\natural\ldots\natural H}_{m}\natural\mathcal{G}\natural\underbrace{I\natural\ldots\natural I}_{m}$$

Contents

1. Introduction

- Satoh and Taniguchi defined a virtual knot invariant J_n, called the *n*-writhe, for any non-zero integer n.
- The n-writhes give the coefficients of some polynomial invariants for virtual knots.

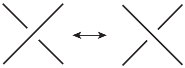
They gave a necessary and sufficient condition for a sequence of integers to be that of the n-writhes of a virtual knot as follows.

Theorem 2 ([Satoh-taniguchi, 2014])

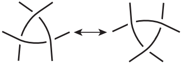
- *The sequence of n-writhes* $\{J_n(K)\}_{n\neq 0}$ *of a virtual knot K satisfies* $\sum_{n\neq 0} nJ_n(K) = 0$.
- *for any sequence of integers* $\{r_n\}_{n\neq 0}$ *with* $\sum_{n\neq 0} nr_n = 0$*, there is a virtual knot K such that* $J_n(K) = r_n$ *for any* $n \neq 0$.

It is well known that the following local moves are unknotting operations for classical and virtual knots.

For classical knots,

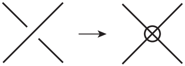


Crossing change

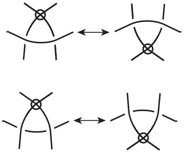


Delta-move

For virtual knots,



Virtualization



Forbidden moves

The unknotting number by the virtualization is called the virtual unknotting number and is denoted by u^v.

In our previous researches, we shew the following theorems for the virtualization.

Theorem 3 ([Ohyama-S, 2019])

For any given non-zero integer n and any given integer m, there exists a virtual knot K such that $u^v(K) = 1$ and $J_n(K) = m$.

Theorem 4 ([Ohyama-S, 2021])

Let $\{r_n\}_{n\neq 0}$ be a sequence of integers. If $\sum_{n\neq 0} nr_n = 0$, then there exists a virtual knot K such that

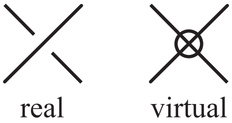
$$u^v(K) = 1 \text{ and } J_n(K) = r_n$$

for any $n \neq 0$.

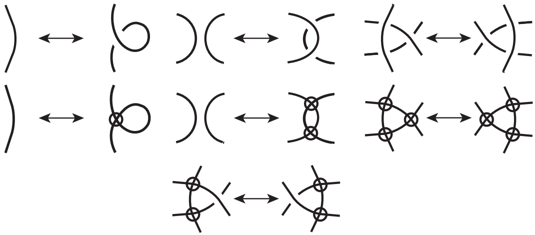
In this talk, we show that there exist infinitely many virtual knots having such properties by using the vertex connected sum on Gauss diagrams.

2. Preliminaries

A virtual knot diagram is a generalization of a knot diagram and it has real crossings and virtual crossings.

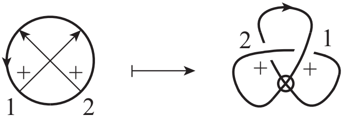


A virtual knot is an equivalence class of virtual knot diagrams under the generalized Reidemeister moves.

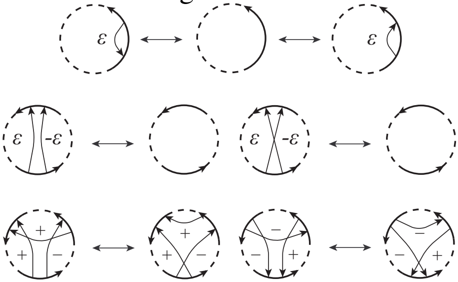


D : a virtual knot diagram

The Gauss diagram of D is the preimage of D with real crossing information.

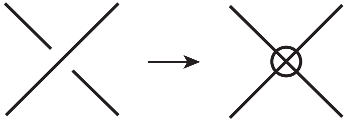


There exists a bijection from the set of virtual knots to the set of equivalence classes of their Gauss diagrams modulo the generalized Reidemeister moves of Gauss diagrams.



Generalized Reidemeister moves of Gauss diagrams

The following local move on virtual knot diagrams is called a virtualization. A virtualization is an unknotting operation.



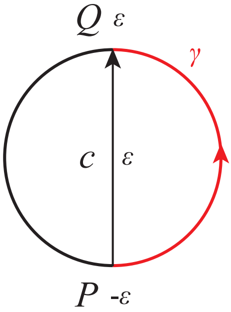
Virtualization.

The minimum number of the virtualizations needed to transform a diagram of a virtual knot K into a trivial knot diagram is called the virtual unknotting number and is denoted by $u^v(K)$.

$$K \quad \underbrace{\rightarrow \quad \rightarrow \quad \cdots \quad \rightarrow}_{\text{the min. no. of the virutalizations}} \quad O : \text{a trivial knot}$$

Invariants for virtual knots

Definition 5 ([Satoh-Taniguchi, 2014])



K : a virtual knot
G : the Gauss diagram of K
P, Q : two points on the circle $\mathbb{S}^1$ of G
$c = \overrightarrow{PQ}$: the chord from P to Q with the sign ε
We give the signs $-\varepsilon$ and ε to the endpoints P and Q, respectively.
γ : the arc on $\mathbb{S}^1$ of G from P to Q along the orientation of $\mathbb{S}^1$

The index, $i(c)$ is the sum of the signs of the points on the arc γ of the chord c. The n-writhe J_n of K is defined by

$$J_n(K) = \sum_{i(c)=n} \varepsilon(c)$$

A sequence of the n-writhes

Satoh and Taniguchi give a necessary and sufficient condition for a sequence of integers to be that of the n-writhes of a virtual knot as follows.

Theorem 6 ([Satoh-Taniguchi, 2014])

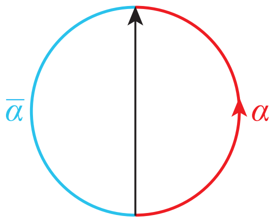
Any virtual knot K satisfies $\sum_{n\neq 0} nJ_n(K) = 0$. Conversely, for any sequence of integers $\{r_n\}_{n\neq 0}$ with

$$\sum_{n\neq 0} nr_n = 0,$$

there is a virtual knot K such that $J_n(K) = r_n$ for any $n \neq 0$.

A unmerical value $S(\alpha, \beta)$

The endpoints of a chord of a Gauss diagram G divide the circle $\mathbb{S}^1$ into two arcs α and $\overline{\alpha}$ in the following figure.



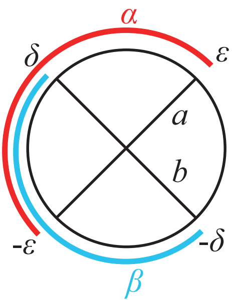
$P(\alpha)$: the set of endpoints of the chords of G in the interior of α.
$\alpha, \beta \subset \mathbb{S}^1$: arcs for distinct chords a and b of G, resp.
For $\rho \in P(\alpha)$,
- $\varepsilon(\rho)$: the sign of ρ
- $\tau(\rho)$: the other endpoint of the chord incident to ρ

$$S(\alpha, \beta) = \sum_{\rho \in P(\alpha), \tau(\rho) \in P(\beta)} \varepsilon(\rho).$$

We say that the chords a and b are linked if their endpoints appear on $\mathbb{S}^1$ alternately, and otherwise unlinked.

Then the intersection number $\alpha \cdot \beta$ of the arcs α and β is calculated as follows.

1. If a and b are unlinked, then $\alpha \cdot \beta = S(\alpha, \beta)$ and
2. Assume that a and b are linked as shown in the following figure, where $\varepsilon, \delta \in \pm$. Then it holds that $\alpha \cdot \beta = S(\alpha, \beta) + \frac{1}{2}(\varepsilon + \delta)$ and $\beta \cdot \alpha = S(\beta, \alpha) - \frac{1}{2}(\varepsilon + \delta)$.



Linked chords a and b.

The first intersection polynomial

Definition 7 ([Higa, Nakamura, Nakanishi, and Satoh, 2023])

K : a virtual knot
D : a diagram of K with m crossings c_i for $1 \leq i \leq m$
G : the Gauss diagram of D
γ_i : the arc oriented from the tail of c_i to its head along the orientation of $\mathbb{S}^1$

$$W_D(t) = \sum_i \varepsilon_i (t^{\gamma_i \cdot \overline{\gamma_i}} - 1),$$

$$f_{01}(D) = \sum_{i,j} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \overline{\gamma_j}} - 1) \text{ and}$$

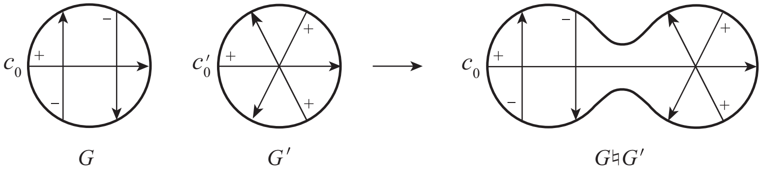
$$I_D(t) = f_{01}(D) - w_D W_D(t),$$

where ε_i is the sign of c_i and $w_D = \sum_{i=1}^{m} \varepsilon_i$ is the writhe of D. The polynomial $I_D(t)$ is an invariant of K called the first intersection polynomial and denoted by $I_K(t)$.

Vertex connected sum

Notation (vertex connected sum)

Let G and G' be Gauss diagrams with chords c_0 and c_0' respectively. If c_0 and c_0' have the same sign and $i(c_0) = i(c_0') = 0$, then the vertex connected sum $G\natural G'$ with respect to c_0 and c_0' is the Gauss diagram obtained by removing the interiors of regular neighborhoods of the head of c_0 and the tail of c_0' from the diagrams and connecting them as shown in the following figure.



The vertex connected sum $G\natural G'$.

3. Proof of the main result

Notation ($M_{k\ell}$ for $k, \ell = 0, 1$)

G : a Gauss diagram with the chords c_0 and c_i for $1 \leq i \leq s$

γ_i : the arc associated with the chord c_i

We define the sets of the numbers $M_{k\ell}$ for $k, \ell = 0, 1$ on the Gauss diagram G by

$$M_{11}(G) = \{\, i \mid \gamma_i \text{ has both the head of } c_0 \text{ and the tail of it} \,\},$$

$$M_{10}(G) = \{\, i \mid \gamma_i \text{ has the head of } c_0 \text{ but doesn't have the tail of it} \,\},$$

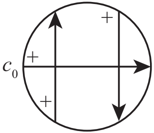
$$M_{01}(G) = \{\, i \mid \gamma_i \text{ doesn't have the head of } c_0 \text{ but has the tail of it} \,\} \text{ and}$$

$$M_{00}(G) = \{\, i \mid \gamma_i \text{ has neither the head of } c_0 \text{ nor the tail of it} \,\}.$$

Lemma 8

L : a virtual knot with the Gauss diagram H as shown in the following fig.
K : a virtual knot
G : a Gauss diagram of K with the chords c_i' for $0 \le i \le s$
The chords c_i' has the sign ε_i'.
γ_i' : the arc of G associated with c_i'
Then, for the virtual knot $L\natural K$ with the vertex connected sum $H\natural G$, we have

$$\begin{aligned} I_{L\natural K}(t) &= I_L(t) + I_K(t) \\ &+ \sum_{i\in M_{11}(G)} \varepsilon_i'(t^{\gamma_i'\cdot\overline{\gamma_i'}}-1)(t^{-1}-1) + \sum_{i\in M_{10}(G)} \varepsilon_i'(t^{\gamma_i'\cdot\overline{\gamma_i'}}-1)(t^{-1}-1) \\ &+ \sum_{i\in M_{01}(G)} \varepsilon_i'(t^{\gamma_i'\cdot\overline{\gamma_i'}}-1)(t-1) + \sum_{i\in M_{00}(G)} \varepsilon_i'(t^{\gamma_i'\cdot\overline{\gamma_i'}}-1)(t-1). \end{aligned}$$



H

Outline of the proof of Lemma 8.

D : a virtual knot diagram of K

E : a virtual knot diagram of L

The Gauss diagram H has the chords c_0, c_1 and c_2 as shown in the following fig.

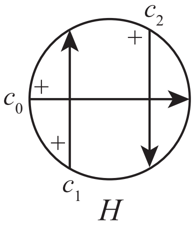
ε_k : the sign of c_k for $k = 0, 1, 2$

γ_k : the arc of c_k

The Gauss diagram $H \natural G$ has the chords c_0, c_1, c_2 and c'_i.

Γ_k : the arc associated with the chord c_k of the Gauss diagram $H \natural G$

Γ'_i : the arc associated with the chord c'_i



We consider the intersection numbers of any two chords of $H\natural G$ from the following cases:

- Case 1: any two chords of H except for c_0.
- Case 2: any two chords of G except for c_0.
- Case 3: the chord c_0 and the other chords.
- Case 4: a chord of H and a chord of G.

Calculating $I_{L\natural K}(t)$ from Cases 1, 2, 3 and 4,

$$\begin{aligned}
I_{L\natural K}(t) &= \sum_{k,\ell=1,2} \varepsilon_k\varepsilon_\ell(t^{\Gamma_k\cdot\overline{\Gamma_\ell}} - 1) + \sum_{1\le p,q\le s} \varepsilon'_p\varepsilon'_q(t^{\gamma'_p\cdot\overline{\gamma'_q}} - 1) \\
&+ \varepsilon_0\varepsilon_0(t^{\Gamma_0\cdot\overline{\Gamma_0}} - 1) + \sum_{k=1,2} \varepsilon_0\varepsilon_k(t^{\Gamma_0\cdot\overline{\Gamma_k}} - 1) + \sum_{\ell=1,2} \varepsilon_\ell\varepsilon_0(t^{\Gamma_\ell\cdot\overline{\Gamma_0}} - 1) \\
&+ \sum_{1\le i\le s} \varepsilon'_0\varepsilon'_i(t^{\gamma'_0\cdot\overline{\gamma'_i}} - 1) + \sum_{1\le i\le s} \varepsilon'_i\varepsilon'_0(t^{\gamma'_i\cdot\overline{\gamma'_0}} - 1) \\
&+ (w_D - 1)W_L(t) + (w_E - 1)W_K(t)
\end{aligned}$$

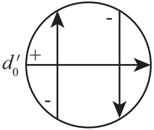
$$
\begin{aligned}
&+ \sum_{i \in M_{11}(G)} \varepsilon_i'(t^{-1} - 1)(t^{\gamma_i' \cdot \overline{\gamma_i'}} - 1) + \sum_{i \in M_{00}(G)} \varepsilon_i'(t - 1)(t^{\gamma_i' \cdot \overline{\gamma_i'}} - 1) \\
&+ \sum_{i \in M_{10}(G)} \varepsilon_i'(t^{-1} - 1)(t^{\gamma_i' \cdot \overline{\gamma_i'}} - 1) + \sum_{i \in M_{01}(G)} \varepsilon_i'(t - 1)(t^{\gamma_i' \cdot \overline{\gamma_i'}} - 1) \\
&- (w_E + w_D - 1)\{W_L(t) + W_K(t)\}.
\end{aligned}
$$

Rearranging this formula, we obtain the formula of Lemma 8. □

Lemma 9

M : a virtual knot with the Gauss diagram I as shown in the following fig.
K : a virtual knot
G : a Gauss diagram of K with the chords d_i for $0 \leq i \leq s$
The chord d_i has the sign δ_i.
ζ_i : the arc of G associated with d_i
Then, for the virtual knot $K\natural M$ with the vertex connected sum $G\natural I$, we have

$$I_{K\natural M}(t) = I_K(t) + I_M(t)$$
$$- \sum_{i \in M_{11}(G)} \delta_i(t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t-1) - \sum_{i \in M_{10}(G)} \delta_i(t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t^{-1}-1)$$
$$- \sum_{i \in M_{01}(G)} \delta_i(t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t-1) - \sum_{i \in M_{00}(G)} \delta_i(t^{\zeta_i \cdot \overline{\zeta_i}} - 1)(t^{-1}-1).$$



I

Lemma 10

K : a virtual knot
G : a Gauss diagram of K with the chords d_i for $0 \leq i \leq s$
The chord d_i has the sign δ_i.
ζ_i : the arc of G associated with d_i.
Then for a natural number m, we have

$$I_{\underbrace{L\natural\ldots\natural L}_{m}\natural K\natural\underbrace{M\natural\ldots\natural M}_{m}}(t) = I_K(t) + m\sum_{i\in M_{11}(G)}\delta_i(t^{\zeta_i\cdot\overline{\zeta_i}}-1)(t^{-1}-t)$$
$$+\, m\sum_{i\in M_{00}(G)}\delta_i(t^{\zeta_i\cdot\overline{\zeta_i}}-1)(t-t^{-1}).$$

Theorem 1 (Restated) Let $\{r_n\}_{n\neq 0}$ be a sequence of integers. If $\sum_{n\neq 0} nr_n = 0$, then there exist infinitely many virtual knots K_m ($m \in \mathbb{N}$) such that

$$u^v(K_m) = 1 \text{ and } J_n(K_m) = r_n$$

for any $n \neq 0$.

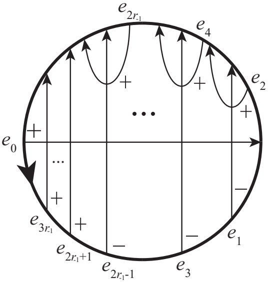
Outline of the proof of Theorem 1.

If $\{r_n\} = \{0\}$, it is proven in [Y. Ohyama and M. Sakurai, 2023].

Assume that $\{r_n\} \neq \{0\}$ and $\{r_n\} = \{r_{-1}, r_1\}$.

We consider the case where $r_{-1} > 0$.

$\mathcal{G}$: the Gauss diagram as shown in the following fig.



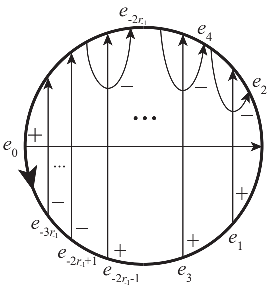
$\mathcal{G}_m := \underbrace{H\natural\ldots\natural H}_{m}\natural\mathcal{G}\natural\underbrace{I\natural\ldots\natural I}_{m}.$

$\mathcal{K}_m$: the virtual knot with the Gauss diagram $\mathcal{G}_m$

- from the construction, it holds that $u^v(K_m) = 1$ and $J_n(K_m) = r_n$.
- From Lemma 10, we obtain

$$\begin{aligned} I_{\mathcal{K}_m}(t) &= I_{\mathcal{K}}(t) \\ &+ m\sum_{u\in M_{11}(\mathcal{G})}\eta_u(t^{\iota_u\cdot\overline{\iota_u}}-1)(t^{-1}-t) + m\sum_{u\in M_{00}(\mathcal{G})}\eta_u(t^{\iota_u\cdot\overline{\iota_u}}-1)(t-t^{-1}) \\ &= I_{\mathcal{K}}(t) + mr_{-1}(t^{-1}-1). \end{aligned}$$

Similarly for the case where $r_{-1} < 0$, it also holds that $I_{\mathcal{K}_m}(t) \neq I_{\mathcal{K}_s}(t)$ for $m \neq s$ if $\mathcal{G}$ is the Gauss diagram as shown in the following fig.

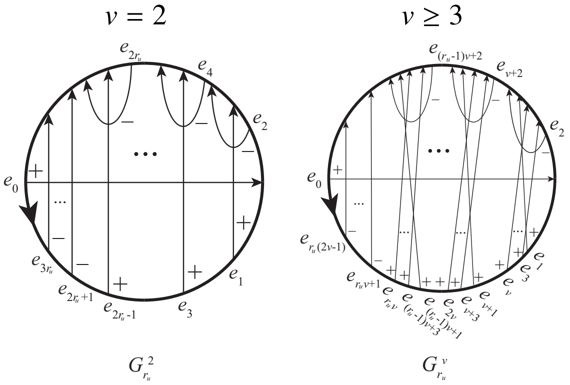


$\mathcal{G}$

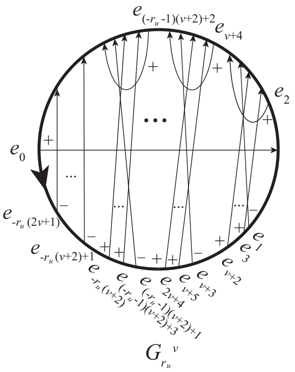
Assume that $\{r_n\} \neq \{r_{-1}, r_1\}, \{0\}$.

$\mathcal{G}$: the Gauss diagram in Theorem 3

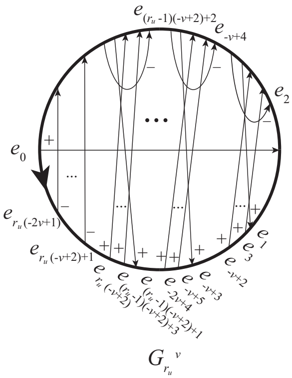
The Gauss diagram $\mathcal{G}$ is obtained from the Gauss diagram $G(u, r_u)$ for any integer u by the vertex connected sum with respect to e_0's. The Gauss diagram $G(u, r_u)$ is the Gauss diagram obtained from the Gauss diagram $G^v_{r_u}$ by the vertex connected sum with respect to e_0 as shown in the following figs.



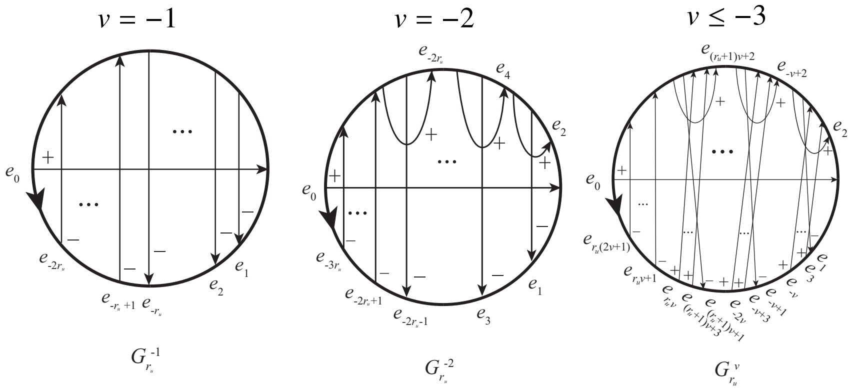
The Gauss diagram $G^v_{r_u}$ for $u \geq 2$ and $r_u > 0$.



The Gauss diagram $G^{v}_{r_u}$ for $u \geq 2$ and $r_u < 0$.



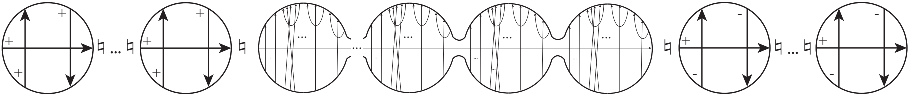
The Gauss diagram $G^{v}_{r_u}$ for $u \leq -1$ and $r_u > 0$.



The Gauss diagram $G^{v}_{r_u}$ for $u \leq -1$ and $r_u < 0$.

$\mathcal{G}_m := \underbrace{H\natural\ldots\natural H}_{m}\natural\mathcal{G}\natural\underbrace{I\natural\ldots\natural I}_{m}.$

$\mathcal{K}_m$: the virtual knot with the Gauss diagram $\mathcal{G}_m$



- from the construction, it holds that $u^v(K_m) = 1$ and $J_n(K_m) = r_n$.
- From Lemma 10, we have

$$
\begin{aligned}
&I_{\mathcal{K}_m}(t) = I_{\mathcal{K}}(t) \\
&+ m(t^{-1} - t)\sum_{u=A}^{-2}\sum_{v=u}^{-2} -r_u\left(t^{v+1} - 1\right) + m(t - t^{-1})\sum_{u=2}^{B}\sum_{v=2}^{u} -r_u\left(t^{v-1} - 1\right) \\
&= I_{\mathcal{K}}(t) + m(-r_A)t^A + \cdots + m(-r_B)t^B. \qquad \square
\end{aligned}
$$

Thank you for your attention