Purely cosmetic surgeries and Casson–Walker–Lescop invariants

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Cosmetic surgery

M: closed oriented 3-mfd. K: knot in M

M_K: the exterior of K (i.e., $M \setminus N^\circ(K)$)

$M_K(\gamma)$: the mfd. obtained by Dehn surgery on K along slope γ

i.e. $M_K(\gamma) = (M_K \cup (D^2 \times S^1))/(\gamma \sim (\partial D^2 \times *))$

natural to ask: $\boxed{\text{Can } \textit{distinct} \text{ Dehn surgeries give the } \textit{same} \text{ manifold?}}$

Two slopes γ, γ' are *equivalent* (denoted by $\gamma \sim \gamma'$)

$$\overset{\text{def}}{\Longleftrightarrow} {}^\exists h\colon M_K \to M_K\colon \text{homeo. s.t. } h(\gamma) = \gamma'$$

γ- & γ'-surgery on K are *purely* (*chirally*) *cosmetic*

$$\overset{\text{def}}{\Longleftrightarrow} M_K(\gamma) \cong M_K(\gamma') \quad (M_K(\gamma) \cong -M_K(\gamma'))$$

Cosmetic Surgery Conjecture [Problem 1.12 (a) in K3]

$\gamma \not\sim \gamma' \Longrightarrow M_K(\gamma) \not\cong M_K(\gamma')$ (Assume: $M_K \not\cong D^2 \times S^1$)

Rem. ∃chirally cosmetic surgeries along inequivalent slopes.

In this talk "cosmetic surgeries" = purely cosmetic surgeries

Simple constraint for purely cosmetic surgeries H_1

When M is S^3 or $\mathbb{Z}$HS, or M is $\mathbb{Q}$HS and K is null-homologous,
slope γ is parametrized by $p/q \in \mathbb{Q} \cup \{1/0\}$

We denote $M_K(\gamma)$ by $M_K(p/q)$ or $K(p/q)$. (sign convention: $p \geq 0$)

Basic strategy Compare invariants of $K(p/q)$ and $K(p'/q')$

Fact

$$H_1(K(p/q)) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & (p > 0) \\ \mathbb{Z} & (p = 0) \end{cases}$$

$\Longrightarrow$ If $K(p/q) \cong K(p'/q')$, then $p' = p$. (We may assume that $p \neq 0$.)

- λ_{C}: Casson invariant defined for $\mathbb{Z}$HS
- λ_{CW}: Casson–Walker invariant defined for $\mathbb{Q}$HS
- $\lambda = \lambda_{\mathrm{CWL}}$: Casson–Walker–Lescop invariant defined for closed 3-mfd

In this talk: we use Casson–Walker–Lescop invariant λ (introduce later)

known results for a knot in S^3

Cosmetic Surgery Conjecture is true for the following families:

- genus 1 knot [Wang '06]
- cable knot, composite knot [Tao '18, '19]
- 2-bridge knot [Ichihara-J.-Mattman-Saito '21]
- 3-braid knot [Varvarezos '21]
- pretzel knot [Stipsicz-Szabó '21]
- alternating knot [Daemi-Lidman-Miller Eismeier '24] (preprint)

For non-trivial $K \subset S^3$, assume that $K(p/q) \cong K(p/q')$ $(q \neq q')$.

[Ni-Wu '15]

$q' = -q$. i.e. $p/q' = -p/q$.

[Hanselman '23] (announced in '19)

$p = 2$ & $q, q' \in \{\pm 1\}$, or $p = 1$.

[Daemi-Lidman-Miller Eismeier '24] (preprint)

$p = 2$ & $q, q' \in \{\pm 1\}$. $g(K) = 2$ & $\Delta_K(t) = 1$.

knots in HS

Remaining possibilities for purely cosmetic surgeries on a knot in S^3

$K \subset S^3$ with $g(K) = 2$ and $\Delta_K(t) = 1$ & surgery slopes $= \pm 2$

Another direction

How about for a knot in a 3-mfd $\not\cong S^3$, in particular, for a knot in HS?

In this talk: We use Casson–Walker–Lescop invariant λ (Why use it?)

- $|H_1|$ & $\lambda \longleftrightarrow$ finite type invariant of degree 0 & 1
- For $K \subset S^3$, cosmetic surgery is studied by using finite type invariant (of low degree) by [Ichihara-Wu], [Ito], [Ito-Ichihara-Saito].

[Boyer-Lines '90]

K in $\mathbb{Z}$HS w/ $\Delta''_K(1) \neq 0$ admits NO purely cosmetic surgeries.

(Surgery formulae for Casson-Walker inv. & Casson-Gordon inv.)

Rem. For $K \subset S^3$, $\Delta''_K(1) \neq 0 \Leftrightarrow a_2(K) \neq 0$ (2nd coeff. of Conway poly)

Result – $\mathbb{Q}$HS –

Dehn surgery along slope γ is integral $\overset{\text{def}}{\iff} \Delta(\gamma, \mu) = 1$
(For null-homologous knot, slope γ is integral $\iff \gamma = p/1$ for $p \in \mathbb{Z}$)
(hereafter, we change our sign convention: $q > 0$)

[Boyer-Lines '90] (implicitly)

$\forall$knot in an $\mathbb{Z}$HS admits at most two integral purely cosmetic surgery pairs.

Theorem 1 [Ichihara-J.-Tsutsumi]

$\forall$null-homologous knot in $\mathbb{Q}$HS admits at most two integral purely cosmetic surgery pairs.

[Daemi-Lidman '25]

$\forall$null-homotopic knot K in $\mathbb{Q}$HS Y, $\pi_1 K(p/q) \cong \pi_1 Y \implies p/q = \pm 1$.

Rem. [Ichihara] $\forall$knot in any closed oriented 3-mfd. admits only finitely many purely cosmetic surgery pairs.

Results – 1st Betti number 1 or 2 (not $\mathbb{Q}$HS) –

A link $L = \cup_{i=1}^{n} K_i$ in S^3 is algebraically split $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{lk}(K_i, K_j) = 0$ $(i \neq j)$

Theorem 2 [Ichihara-J.-Tsutsumi]

$L = K \cup K_0$: algebraically split 2-comp. link, $M = K_0(0)$
$\hat{a}_1(L) \neq 0 \Longrightarrow K$ (in M) admits NO purely cosmetic surgeries.

Rem. $\hat{a}_1(L) \neq 0 \iff a_3(L) \neq 0$ for a 2-comp. link L

Theorem 3 [Ichihara-J.-Tsutsumi]

$L = K \cup K' \cup K_0$: alg. split 3-comp. link, $M = K' \cup K_0(p_1/q_1, 0)$
$p_1 \hat{a}_1(K \cup K_0) + q_1 \hat{a}_1(L) \neq 0 \Longrightarrow K$ (in M) admits NO purely cosmetic surgeries.

Theorem 4 [Ichihara-J.-Tsutsumi]

$L = K \cup K_0 \cup K'_0$: alg. split 3-comp. link, $M = K_0 \cup K'_0(0, 0)$
$\hat{a}_1(K \cup K_0 \cup K'_0) \neq 0 \Longrightarrow K$ (in M) admits NO purely cosmetic surgeries.

Rem. $\hat{a}_1(L') = 0$ for $L' \subset L$ with $\sharp L' \geq 4$ if L is alg. split [Hoste '86].

Rational surgery formula of CWL inv. λ

$$L = K_1 \cup \cdots \cup K_n \subset S^3, \qquad M = L(p_1/q_1, \ldots, p_n/q_n) \ (q_i > 0)$$

Surgery formula [Lescop '96]

$$\frac{\lambda(M)}{\prod_{i=1}^n q_i} = (-1)^{b_-} \sum_{\emptyset \neq J \subset N} \left(\det B_{N\setminus J}\ \hat{a}_1(L_J) + \frac{(-1)^{\sharp J} \det A_{N\setminus J}\ \theta(A_J)}{24} \right) + |\det A| \left(\frac{\sigma(A)}{8} + \sum_{i=1}^n \frac{s(p_i, q_i)}{2} \right).$$

- $J \subset N = \{1, \ldots, n\}$, $L_J = \cup_{j \in J} K_j$ (sublink of L)
- $\hat{\nabla}_L(z) = z^{n-1} \left(\hat{a}_0(L) + \hat{a}_1(L) z^2 + \cdots \right)$: Conway polynomial
- $A = (a_{ij})$: linking matrix, b_-: # of negative eigenvalues of A
- $A_{N\setminus J} = (a_{ij})_{i,j \in N\setminus J}$
- $B_{N\setminus J} = (b_{ij})_{i,j \in N\setminus J}$ with $b_{ij} = \begin{cases} a_{ij} & (i \neq j) \\ a_{ii} + \sum_{k \in J} a_{ki} & (i = j) \end{cases}$
- $\theta(A_J) = \begin{cases} (p_j^2 + q_j^2 + 1)/q_j^2 & (J = \{j\}) \\ 0 & (\sharp J \geq 2) \end{cases}$ if $\exists K_i$ s.t. $\mathrm{lk}(K_i, K_j) = 0$
- $s(p, q)$: Dedekind sum

Surgery description of a knot in $\mathbb{Q}$HS (WANT: simple desc.)

Fact

$\forall\,\mathbb{Z}$HS is obtained by Dehn surgery on an alg. split link along ± 1 slopes.

Proposition

$\exists\,\mathbb{Q}$HS cannot be obtained by Dehn surgery on an alg. split link (along rational coefficients).

Proposition [cf. Ichihara-J. '25]

K: *null-homologous* knot in $\mathbb{Q}$HS M
$\exists$surgery desc. of M by a link $\cup_{i=1}^{n} K_i \subset S^3$ w/ $\mathrm{lk}(K, K_i) = 0$ for $\forall i$.

$L = K \cup K_1 \cup \cdots \cup K_n$: $(n+1)$-comp. link in S^3 w/ $\mathrm{lk}(K, K_j) = 0$

$$M = L(p/q, p_1/q_1, \ldots, p_n/q_n) \Longrightarrow A = \begin{pmatrix} p/q & {}^t\mathbf{0} \\ \mathbf{0} & * \end{pmatrix}$$

Proof of Theorem 1

Surgery formula [Lescop '96]

$$\frac{\lambda(M)}{\prod_{i=1}^{n} q_i} = \left\{ (-1)^{b_-} \sum_{\emptyset \neq J \subset N} \left(\det B_{N \setminus J}\ \hat{a}_1(L_J) + \frac{(-1)^{\sharp J} \det A_{N \setminus J}\ \theta(A_J)}{24} \right) + |\det A| \left(\frac{\sigma}{8} + \sum_{i=1}^{n} \frac{s(p_i, q_i)}{2} \right) \right\}.$$

Lemma

$L = K_0 \cup K_1 \cup \cdots \cup K_n$: $(n+1)$-comp. link in S^3 w/ $\mathrm{lk}(K_0, K_j) = 0$
$M = L(p_0/q_0, p_1/q_1, \ldots, p_n/q_n)$

$$\frac{\lambda(M)}{\prod_{i=0}^{n} q_i} = (-1)^{b_-} \Bigg(\sum_{\emptyset \neq J \subset N} \frac{p_0}{q_0} \det B_{N \setminus J} \hat{a}_1(L_J) + \sum_{J' \subset N, J = J' \cup \{0\}} \det B_{N \setminus J'} \hat{a}_1(L_J) + \sum_{\emptyset \neq J \subset N} \frac{p_0}{24 q_0} (-1)^{\sharp J} \det A_{N \setminus J} \theta(\mathbb{L}_J) - \det A_N \frac{p_0^2 + q_0^2 + 1}{24 q_0^2} - \sum_{i=1}^{n} \frac{p_0}{24 q_0} \det A_{N \setminus \{i\}} \frac{p_i^2 + q_i^2 + 1}{24 q_i^2} \Bigg) + |\det A_N|\, |p_0| \left(\frac{\sigma}{8} + \frac{s(p_0, q_0)}{2} + \sum_{i=1}^{n} \frac{s(p_i, q_i)}{2} \right)$$

Proof of Theorem 1

Assume: null-homologous knot K in $\mathbb{Q}$HS M admits integral purely cosmetic surgeries. Considering H_1, $p' = -p$ holds.

$\rightsquigarrow M = L(p, p_1/q_1, \ldots, p_n/q_n) \cong M' = L(-p, p_1/q_1, \ldots, p_n/q_n) \quad (p > 0)$

$$(-1)^{b_-} = -(-1)^{b'_-} \quad \text{and} \quad \sigma - 2 = \sigma'$$

$$\begin{aligned}\frac{\lambda(M)}{\prod_{i=1}^n q_i} =& (-1)^{b_-}\Bigg(\sum_{\emptyset \neq J \subset N} p \det B_{N\setminus J}\hat{a}_1(L_J) + \sum_{J' \subset N, J = J' \cup \{0\}} \det B_{N\setminus J'}\hat{a}_1(L_J) \\ &+ \sum_{\emptyset \neq J \subset N} \frac{p}{24}(-1)^{\sharp J}\det A_{N\setminus J}\theta(\mathbb{L}_J) - \det A_N \frac{p^2+2}{24} \\ &- \sum_{i=1}^n \frac{p}{24}\det A_{N\setminus\{i\}}\frac{p_i^2+q_i^2+1}{24 q_i^2}\Bigg) + |\det A_N|\,|p|\left(\frac{\sigma}{8} + \frac{0}{2} + \sum_{i=1}^n \frac{s(p_i,q_i)}{2}\right)\end{aligned}$$

$$\begin{aligned}\rightsquigarrow \frac{\lambda(M) - \lambda(M')}{\prod_{i=1}^n q_i} =& \, 2(-1)^{b_-}\sum_{J' \subset N, J = J' \cup \{0\}} \det B_{N-J'}\hat{a}_1(L_J) \\ &- \frac{(-1)^{b_-}\det A_N}{12}(p^2+2) + \frac{|\det A_N|}{4}p.\end{aligned}$$

Proof of Theorem 1

Since $\lambda(M) - \lambda(M') = 0$, we have

$$0 = 2(-1)^{b_-} \sum_{J' \subset N, J = J' \cup \{0\}} \det B_{N-J'} \hat{a}_1(L_J) - \frac{(-1)^{b_-} \det A_N}{12}(p^2+2) + \frac{|\det A_N|}{4} p$$

Then we have quadratic equation $c_2 p^2 + c_1 p + c_0 = 0$ with

$$\begin{aligned} c_2 &= \frac{(-1)^{b_-} \det A_N}{12} \\ c_1 &= -\frac{|\det A_N|}{4} \\ c_0 &= -2(-1)^{b_-} \sum_{J' \subset N, J = J' \cup \{0\}} \det B_{N-J'} \hat{a}_1(L_J) + 2c_2 \end{aligned}$$

Here $c_2 \neq 0$ since M is $\mathbb{Q}$HS. $\square$