Simplicity of the kernel of the flux homomorphism for non-orientable open surfaces

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Simplicity of diffeomorphism groups

A group G is *simple* if $\nexists$ non-trivial normal subgroup.

($\Longleftrightarrow \forall$ homomorphism to another group is trivial or injective.)

S: a surface

$\mathrm{Diff}_c(S)$: the group of compactly supported diffeomorphism group

$$\mathrm{Diff}_{c,0}(S) \hookrightarrow \mathrm{Diff}_c(S) \twoheadrightarrow \mathrm{Mod}(S)$$

$\rightsquigarrow \mathrm{Diff}_c(S)$ is not simple.

Theorem (Thurston 1974)

For every manifold M, the group $\mathrm{Diff}_{c,0}(M)$ is simple.

Digression

Theorem (Thurston 1974)

For every manifold M, the group $\mathrm{Diff}^{\infty}_{c,0}(M)$ is simple.

Theorem (Mather 1974, 1975)

For every manifold M, the group $\mathrm{Diff}^{r}_{c,0}(M)$ is simple unless $r = \dim(M) + 1$.

Even the simplicity of $\mathrm{Diff}^{2}_{0}(S^1)$ is still open.

Area-preserving diffeomorphism group

S: orientable open surface (i.e. non-compact without boundary)
$\omega \in \Omega^2(S)$: area form

$\mathrm{Diff}_c(S;\omega) := \{f \in \mathrm{Diff}_c(S) \colon f^*\omega = \omega\}$

- Is $\mathrm{Diff}_c(S;\omega)$ simple?

Area-preserving diffeomorphism group

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- Is $\mathrm{Diff}_c(S;\omega)$ simple?
  $\rightsquigarrow$ No! There is a homomorphism $\mathrm{Diff}_c(S;\omega) \to \mathrm{Mod}(S)$.
  The kernel is the identity component $\mathrm{Diff}_{c,0}(S;\omega)$.

Flux homomorphism

$\mathrm{Diff}_{c,0}(S;\omega) \trianglelefteq \mathrm{Diff}_c(S;\omega)$

- Is $\mathrm{Diff}_{c,0}(S;\omega)$ simple?

Flux homomorphism

$\mathrm{Diff}_{c,0}(S;\omega) \trianglelefteq \mathrm{Diff}_c(S;\omega)$

- Is $\mathrm{Diff}_{c,0}(S;\omega)$ simple?
  $\rightsquigarrow$ No! There is the flux homomorphism

$$\mathrm{Flux}_S \colon \mathrm{Diff}_{c,0}(S;\omega) \twoheadrightarrow \mathrm{H}^1_c(S;\mathbb{R}).$$

(The kernel is the group of Hamiltonian diffeomorphisms.)

Definition of Flux_S

For $\eta \in \Omega^1(S)$ such that $d\eta = \omega$ and for $g \in \mathrm{Diff}_{c,0}(S;\omega)$, the difference $\eta - g^*\eta$ is closed since $d\eta - dg^*\eta = \omega - g^*\omega = 0$.
Since g is compactly supported, so is $\eta - g^*\eta$.

$\rightsquigarrow \mathrm{Flux}_S(g) := [\eta - g^*\eta] \in \mathrm{H}^1_c(S;\mathbb{R})$

Calabi homomorphism

$\mathrm{KerFlux}_S \trianglelefteq \mathrm{Diff}_{c,0}(S;\omega) \trianglelefteq \mathrm{Diff}_c(S;\omega)$

- Is $\mathrm{KerFlux}_S$ simple?

Calabi homomorphism

$\mathrm{KerFlux}_S \trianglelefteq \mathrm{Diff}_{c,0}(S;\omega) \trianglelefteq \mathrm{Diff}_c(S;\omega)$

- Is $\mathrm{KerFlux}_S$ simple?
  $\rightsquigarrow$ No! There is the Calabi homomorphism

$$\mathrm{Cal}_S\colon \mathrm{KerFlux}_S \twoheadrightarrow \mathbb{R}.$$

Definition of Cal_S

1. Take $g \in \mathrm{KerFlux}_S \rightsquigarrow \eta - g^*\eta$ is exact.
2. $\exists! \varphi_g \in \Omega^0_c(S)$ such that $d\varphi_g = \eta - g^*\eta$.
3. $\mathrm{Cal}_S(g) := \int_S \varphi_g \omega$

Theorem (Banyaga 1978)

KerCal_S *is simple.*

Orientable and area preserving world

$\downarrow$ Is this simple?

$$\mathrm{Diff}_{c,0}(S;\omega) \to \mathrm{Diff}_{c}(S;\omega) \to \mathrm{Mod}(S)$$

$$\mathrm{KerFlux}_S \to \mathrm{Diff}_{c,0}(S;\omega) \xrightarrow{\mathrm{Flux}_S} \mathrm{H}^1_c(S;\mathbb{R})$$

$$\mathrm{KerCal}_S \to \mathrm{KerFlux}_S \xrightarrow{\mathrm{Cal}_S} \mathbb{R}$$

KerCal_S is simple (Banyaga 1978).

Today: What happens if the surface is *non-orientable*?

Area density for non-orietable surfaces

N: non-orientable open surface

$L_N \to N$: orientation line bundle of N

$\Omega^2(N; L_N) = \mathrm{Sect}(N, \wedge^2 T^*N \otimes L_N)$: the space of twisted 2-forms

$\omega \in \Omega^2(N; L_N)$: area-density (i.e., nowhere-zero section)

$\mathrm{Diff}_c(N; \omega) := \{g \in \mathrm{Diff}_c(N) \colon g^*\omega = \omega\}$
: the group of area density preserving diffeomorphisms

$\mathrm{Diff}_{c,0}(N; \omega) \hookrightarrow \mathrm{Diff}_c(N; \omega) \to \mathrm{Mod}(N)$

Flux homomorphism for non-orietable surface

$\mathrm{Diff}_{c,0}(N;\omega) \trianglelefteq \mathrm{Diff}_c(N;\omega)$

- Is $\mathrm{Diff}_{c,0}(N;\omega)$ simple?

Flux homomorphism for non-orietable surface

$\mathrm{Diff}_{c,0}(N;\omega) \trianglelefteq \mathrm{Diff}_c(N;\omega)$

- Is $\mathrm{Diff}_{c,0}(N;\omega)$ simple?
  $\rightsquigarrow$ No! There is the flux homomorphism

$$\mathrm{Flux}_N \colon \mathrm{Diff}_{c,0}(N;\omega) \twoheadrightarrow \mathrm{H}^1_c(N;L_N).$$

Definition of Flux_N

For $\eta \in \Omega^1(N;L_N)$ such that $d\eta = \omega$ and for $g \in \mathrm{Diff}_{c,0}(N;\omega)$,
the difference $\eta - g^*\eta \in \Omega^1(N;L_N)$ is closed since
$d\eta - dg^*\eta = \omega - g^*\omega = 0$.
Since g is compactly supported, so is $\eta - g^*\eta$.

$\rightsquigarrow \mathrm{Flux}_N(g) := [\eta - g^*\eta] \in \mathrm{H}^1_c(N;L_N)$

Main result: simplicity

$\mathrm{KerFlux}_N \trianglelefteq \mathrm{Diff}_{c,0}(N;\omega) \trianglelefteq \mathrm{Diff}_c(N;\omega)$

- Is $\mathrm{KerFlux}_N$ simple?

Main result: simplicity

$\mathrm{KerFlux}_N \trianglelefteq \mathrm{Diff}_{c,0}(N;\omega) \trianglelefteq \mathrm{Diff}_c(N;\omega)$

- Is $\mathrm{KerFlux}_N$ simple?

Theorem (Kim–M.)

$\mathrm{KerFlux}_N$ *is simple.*

Comparison: orientable/non-orientable world

$\downarrow$ Is this simple?

$$\mathrm{Diff}_{c,0}(S;\omega) \to \mathrm{Diff}_c(S;\omega) \to \mathrm{Mod}(S)$$

$$\mathrm{KerFlux}_S \to \mathrm{Diff}_{c,0}(S;\omega) \xrightarrow{\mathrm{Flux}_S} \mathrm{H}^1_c(S;\mathbb{R})$$

$$\mathrm{KerCal}_S \to \mathrm{KerFlux}_S \xrightarrow{\mathrm{Cal}_S} \mathbb{R}$$

KerCal_S is simple (Banyaga 1978).

$\downarrow$ Is this simple?

$$\mathrm{Diff}_{c,0}(N;\omega) \to \mathrm{Diff}_c(N;\omega) \to \mathrm{Mod}(N)$$

$$\mathrm{KerFlux}_N \to \mathrm{Diff}_{c,0}(N;\omega) \xrightarrow{\mathrm{Flux}_N} \mathrm{H}^1_c(N;\mathbb{R})$$

$\mathrm{KerFlux}_N$ is simple (Kim–M.).

Simplicity trick

M : manifold, $G < \mathrm{Diff}_c(M)$

Definition

G has the *displacing property (DP)* if for two distinct points $x, y \in M$, there exist $g \in G$ and a point $z \in M \setminus \{x, y\}$ such that $g(y) = z$ and $x \notin \mathrm{supp}(g)$.

For $U \subset M$ open, $G_U := \{g \in G \colon \mathrm{supp}(g) \subset U\}_0$

Definition

G has the *fragmentation property (FP)* if for every $g \in G$ and for every open cover $\mathcal{U}$ of M by open balls, there exist $U_1, \ldots, U_n \in \mathcal{U}$ and $g_1, \ldots, g_n \in G$ such that

1. $g = g_1 \cdots g_n$, and
2. $g_i \in G_{U_i}$ for every i.

Strong fragmentation property

Definition

G has the *fragmentation property (FP)* if for every $g \in G$ and for every open cover $\mathcal{U}$ of M by open balls, there exist $U_1, \ldots, U_n \in \mathcal{U}$ and $g_1, \ldots, g_n \in G$ such that

1. $g = g_1 \cdots g_n$, and
2. $g_i \in G_{U_i}$ for every i.

Definition

G has the *strong fragmentation property (SFP)* if for every $g \in G$ and for every open cover $\mathcal{U}$ of M by open balls, there exist $U_1, \ldots, U_n \in \mathcal{U}$ and $g_1, \ldots, g_n \in G$ such that

1. $g = g_1 \cdots g_n$, and

2'. $g_i \in [G_{U_i}, G_{U_i}]$ for every i.

If G_{U_i} is perfect, then FP $\Longleftrightarrow$ SFP.

Simplicity trick

Theorem (Higman, Epstein, Thurston)

If G acts on M transitively and has DP and SFP, then G is simple.

- If G is $\mathrm{Diff}_c(M)$ (resp. KerCal_S), then G acts on M (resp. S) transitively and has DP and *FP*.
- Moreover, G_U is perfect for every open ball U (Thurston (resp. Banyaga)). In particular, G has SFP.

For non-orientable N, the group $G = \ker \mathrm{Flux}_N$ acts on N transitively and has DP and FP (easy).

For an open disk U, the group G_U is not perfect.

$G_U \cong \mathrm{Diff}_c(U, \omega)_0 \xrightarrow{\mathrm{Cal}_U} \mathbb{R}$: surjective homomorphism

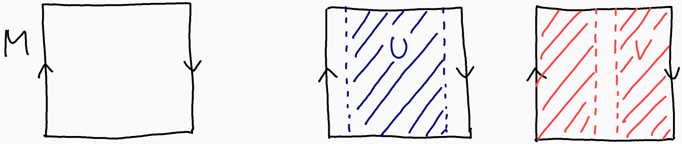
Improving FP to SFP

$G_U \cong \mathrm{Diff}_c(U,\omega)_0 \xrightarrow{\mathrm{Cal}_U} \mathbb{R}$: surjective homomorphism

$[G_U, G_U] \trianglelefteq \mathrm{KerCal}_U$ + the simpliticy of KerCal_U

$\rightsquigarrow [G_U, G_U] = \mathrm{KerCal}_U$

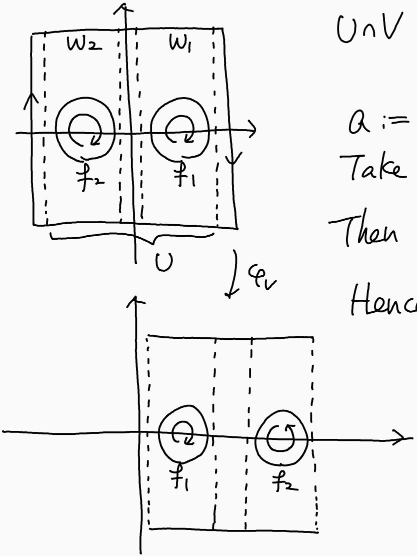
For a Möbius band M, Take the open cover $\{U, V\}$ as follows.



Lemma (Key Lemma)

For every $g \in G_U$, there exist $h_1 \in G_U$ and $h_2 \in G_V$ such that

- $g = h_1 h_2$
- $\mathrm{Cal}_U(h_1) = \mathrm{Cal}_V(h_2) = 0$



$U \cap V = W_1 \sqcup W_2$

$a := Cal_U(g)$

Take $f_i \in G_{W_i}$ s.t. $Cal_U(f_i) = \frac{a}{2}$

Then $Cal_V(f_1) = \frac{a}{2}$ and $Cal_V(f_2) = -\frac{a}{2}$

Hence $g = \underbrace{(g f_2^{-1} f_1^{-1})}_{h_1} \cdot \underbrace{(f_1 f_2)}_{h_2}$

and

$Cal_U(h_1) = Cal_V(h_2) = 0$