Constructing solutions of Polygon and Simplex equation

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Based on joint works with

Mochida Tomoro (Tohoku Univ.), arxiv:2510.12905

Intelligence of Low-dimensional Topology

May 20, 2026, RIMS, Kyoto University.

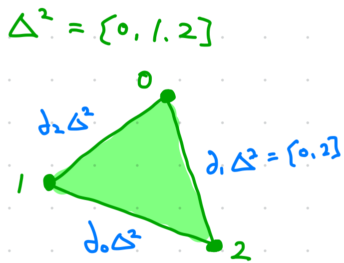
1 : Introduction and motivation

2 : Stacking solutions of polygon equation

3 : Mixed pair and simplex equation

Defintion of polygon equation

- $\Delta^n :=$ power set of $\{0, 1, \cdots, n\}$: standard n-simplicial complex
  represented by the set of ordered vertices $[0, 1, \cdots, n]$



- $\sigma \subset \Delta^n$: k-subsimplex $\rightsquigarrow$ $\sigma = [v_0, v_1, \cdots, v_k]$ s.t. $v_0 < v_1 < \cdots < v_k$

  For $0 \leq i \leq k$, $\partial_i \sigma := [v_0, \cdots, \check{v}_i, \cdots, v_k] \subset \Delta^n$: $(k-1)$-subsimplex

Definition of polygon equation

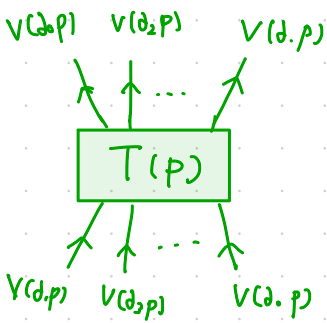
$n \geq 3$, V : vector space, $T : V^{\otimes \lfloor \frac{n-1}{2} \rfloor} \longrightarrow V^{\otimes \lfloor \frac{n}{2} \rfloor}$

- For every $s \subset \Delta^{n-1}$: $(n-3)$ - simplex :

  $V(s)$: a copy of V

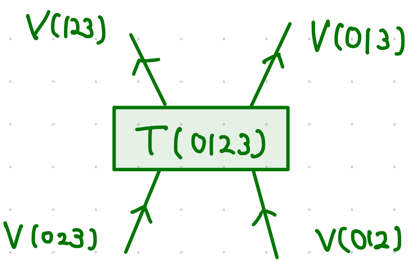
- For every $p \subset \Delta^{n-1}$: $(n-2)$ - simplex

$(T=)$ $T(p) : V(\partial_1 p) \otimes V(\partial_3 p) \otimes \cdots \otimes V(\partial_{2\lceil n/2 \rceil - 2} p) \longrightarrow V(\partial_0 p) \otimes V(\partial_2 p) \otimes \cdots \otimes V(\partial_{2\lfloor \frac{n}{2} \rfloor - 1} p)$



Example

$n = 5$ and $p = [0123]$



Definition of polygon eq.

$D_{even} := \{ \partial_i \Delta^{n-1} \subset \Delta^{n-1} \mid 0 \leq i \leq n-1,\ i \text{ is even} \}$ $\rightsquigarrow \partial \Delta^{n-1} = D_{even} \cup D_{odd}$

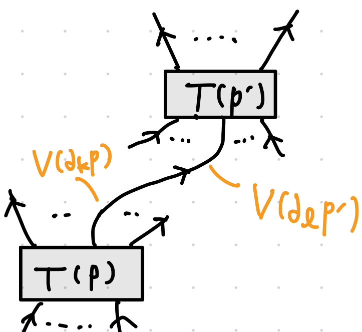
$D_{odd} := \{ \partial_i \Delta^{n-1} \subset \Delta^{n-1} \mid 0 \leq i \leq n-1,\ i \text{ is odd} \}$

LHS of polygon eq. :

Compositions of $\{T(p)\}_{p \in D_{even}}$

for $p, p' \in D_{even}$, $\exists k$: even and $\exists l$: odd such that if $\partial_k p = \partial_l p'$

$\Rrightarrow$

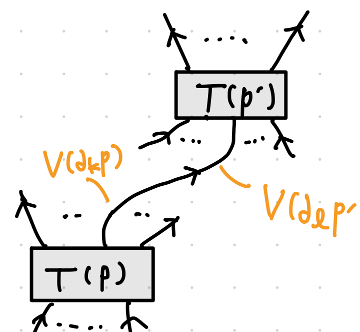


RHS of polygon eq. :

Compositions of $\{T(p)\}_{p \in D_{odd}}$

for $p, p' \in D_{odd}$, $\exists k$: even and $\exists l$: odd such that if $\partial_k p = \partial_l p'$

$\Rrightarrow$



Definition of Polygon equation

$$\begin{cases} n=2k+1 \rightsquigarrow T: V^{\otimes k} \longrightarrow V^{\otimes k} \text{ solution of } n\text{-gon eq.} \\ n=2k \rightsquigarrow T: V^{\otimes k-1} \longrightarrow V^{\otimes k} \text{ solution of } n\text{-gon eq.} \end{cases}$$

if $\partial \Delta^{n-1} = D_{even} \cup D_{odd} \rightsquigarrow \text{LHS} = \text{RHS}$ for $T(p) := T$ for $\forall p \in \Delta^{n-1}$

For $p \in \Delta^{n-1}$: $(n-2)$-simplex $\rightsquigarrow S(\varphi): V(\partial_0 p) \otimes V(\partial_2 p) \otimes \cdots \longrightarrow V(\partial_1 p) \otimes V(\partial_3 p) \otimes \cdots$

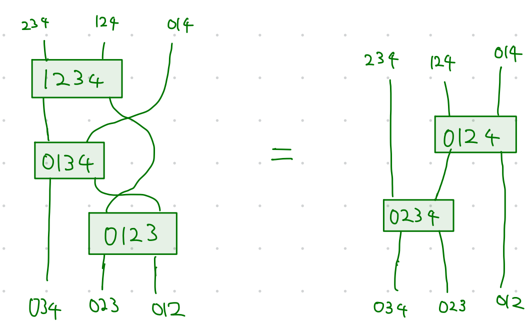
$\longrightarrow$ dual n-gon eq.

For $n=5$:

$T: V \otimes V \longrightarrow V \otimes V$: solution of 5-gon eq.

$$\partial \Delta^4 = \underbrace{\{[1234], [0134], [0123]\}}_{D_{even}} \cup \underbrace{\{[0234], [0124]\}}_{D_{odd}}$$

$T_{12} T_{13} T_{23} = T_{23} T_{12}$ over $V^{\otimes 3}$



Introduction

Theorem) [Pachner, '91]

Any two triangulation of closed PL $(n-2)$-manifold are related by a finite sequence of Pachner $(k, n-k)$-moves $(1 \le k \le n-1)$.

$\longrightarrow$ n-gon eq. are algebraic realization of a Pachner move.

Polygon equation was introduced by Dimakis and Müller-Hoissen in 2015

$\longrightarrow$ Algebraic realization of "chains" of higher Tamari order

Introduction

- $n=4$:

Finite dimensional semi-simple coalgebra

$\longrightarrow$ Solution of dual 4-gon eq.

$\longrightarrow$ Invariant of closed oriented surface   Fukuma-Hosono-Kawai '92

- $n=5$:

6j-symbol, a solution of 5-gon (pentagon eq.)

$\longrightarrow$ Invariant of closed (oriented) 3-manifold.

Finite dimensional (semi-simple) Hopf algebra

$\longrightarrow$ Solution of 5-gon eq.

$\longrightarrow$ Invariant of closed oriented 3-manifold. [M.-Suzuki-Terashima]

Main result 1:

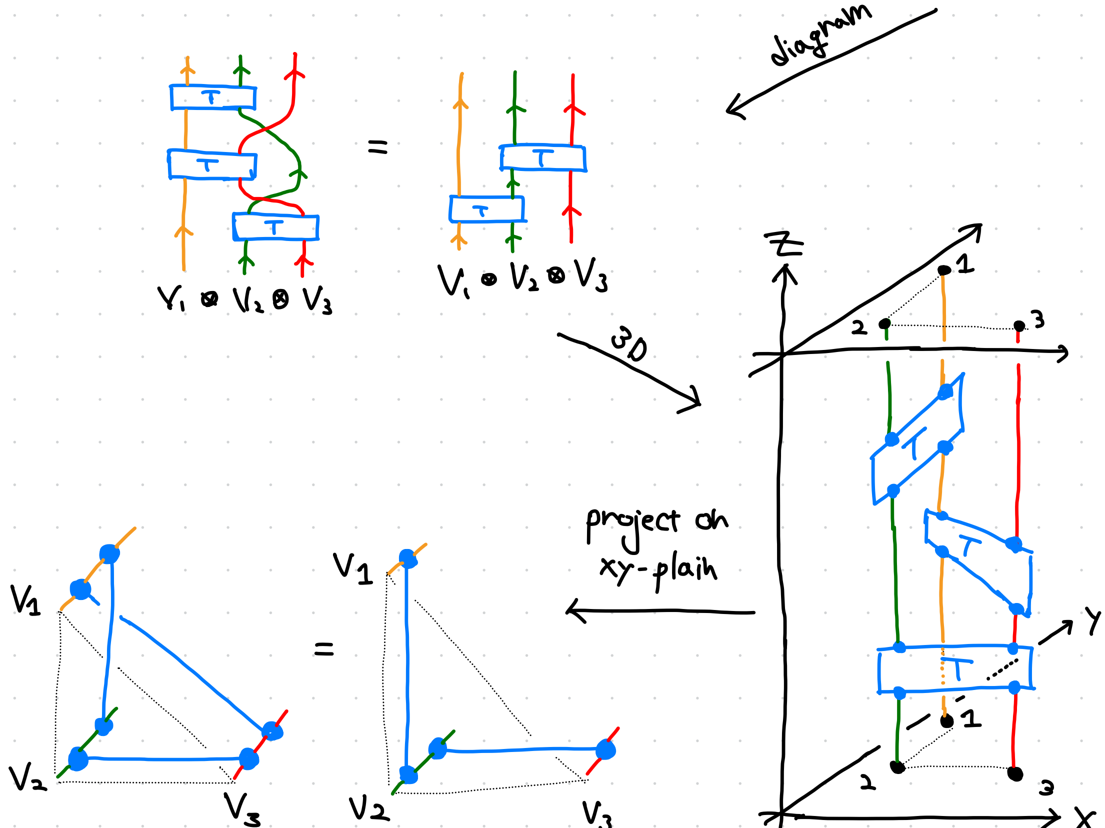
Constructing solution of polygon equation

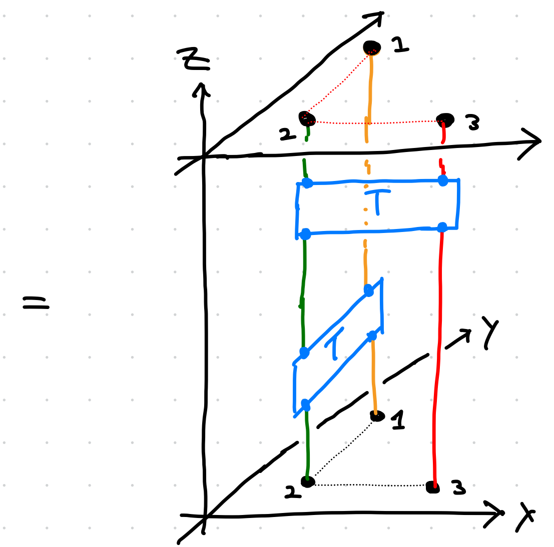
Rewriting the definition

5-gon (pentagon equation):

$$T_{12}\,T_{13}\,T_{23} = T_{23}\,T_{12}$$

over $V \otimes V \otimes V$

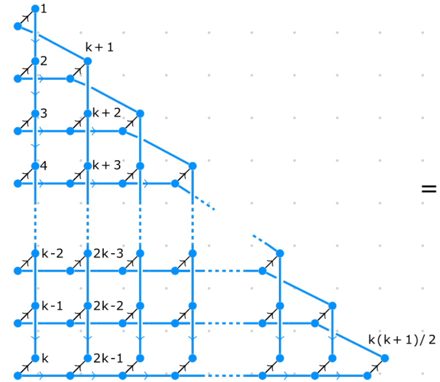




Rewriting the definition

$(2k+1)$ - gon equation :

$T: V^{\otimes k} \longrightarrow V^{\otimes k}$



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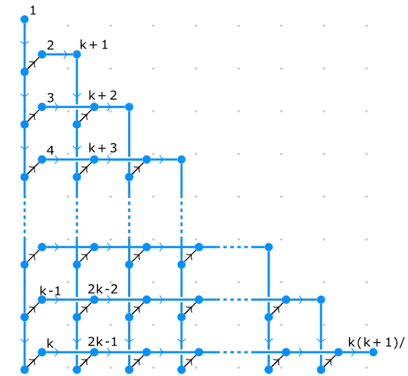
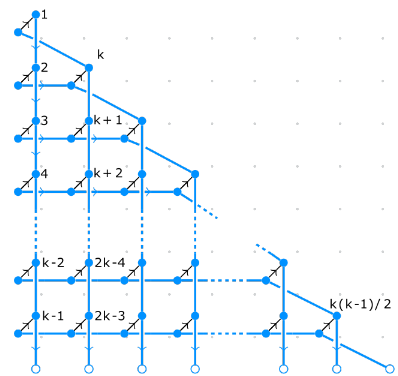


$2k$ - gon equation :

$T: V^{\otimes (k-1)} \longrightarrow V^{\otimes k}$



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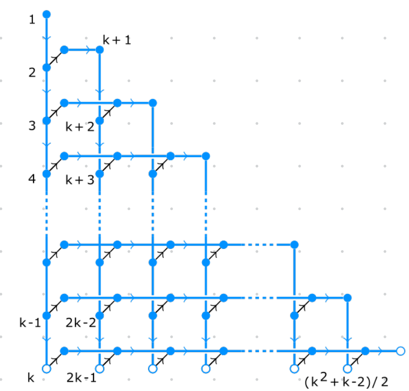
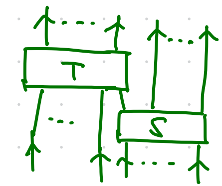


Stacking solutions

For $T : V^{\otimes k} \longrightarrow V^{\otimes k'}$ & $S : V^{\otimes \ell} \longrightarrow V^{\otimes \ell'}$

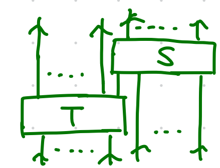
$$T \circ_{\ell} S : V^{\otimes k+\ell-1} \longrightarrow V^{\otimes k'+\ell'-1}$$

$$:= (T \otimes \mathrm{id}_V^{\otimes \ell'-1}) \circ (\mathrm{id}_V^{\otimes k-1} \otimes S)$$



$$T \circ_{r} S : V^{\otimes k+\ell-1} \longrightarrow V^{\otimes k'+\ell'-1}$$

$$:= (\mathrm{id}_V^{\otimes k'-1} \otimes S) \circ (T \otimes \mathrm{id}^{\ell-1})$$



$$T \leftrightarrow S \overset{\text{def}}{\iff} S^{\otimes k'} \circ T^{[\ell]} = T^{[\ell']} \otimes S^{\otimes k}$$

where $T^{[n]} : (V^{\otimes n})^{\otimes k} \longrightarrow (V^{\otimes n})^{\otimes k}$ is defined by

$$(v_{1,1} \otimes \cdots \otimes v_{1,n}) \otimes \cdots \otimes (v_{k,1} \otimes \cdots \otimes v_{k,n}) \xmapsto{\text{permute}} (v_{1,1} \otimes \cdots \otimes v_{k,1}) \otimes \cdots \otimes (v_{1,n} \otimes \cdots \otimes v_{k,n})$$

$$\xmapsto{T^{\otimes n}} (w_{1,1} \otimes \cdots \otimes w_{k,1}) \otimes \cdots \otimes (w_{n,1} \otimes \cdots \otimes w_{n,k}) \xmapsto{\text{permute}} (w_{1,1} \otimes \cdots \otimes w_{1,n}) \otimes \cdots \otimes (w_{k,1} \otimes \cdots \otimes w_{k,n})$$

Stacking solutions

$T^{(n)}, T'^{(n)}$: solution of n-gon eq. $S^{(n)}, S'^{(n)}$: solution of dual n-gon eq.

Proposition) [Mochida - M.]

(1) If $T^{(2k+1)} \leftrightarrow T'^{(n)}$ $\Rightarrow$ $\begin{cases} T^{(2k+1)} \underset{l}{\circ} T'^{(n)} : \text{solution of } (n+2k-2)\text{-gon eq.} \\ T^{(2k+1)} \otimes T'^{(n)} : \text{solution of } (n+2k)\text{-gon eq.} \end{cases}$

(2) If $S^{(2k+1)} \leftrightarrow S'^{(n)}$ $\Rightarrow$ $\begin{cases} S^{(2k+1)} \underset{r}{\circ} S'^{(n)} : \text{solution of dual } (n+2k-2)\text{-gon eq.} \\ S^{(2k+1)} \otimes S'^{(n)} : \text{solution of dual } (n+2k)\text{-gon eq.} \end{cases}$

(3) If $S^{(2k)} \leftrightarrow T^{(n)}$ $\Rightarrow$ $\begin{cases} S^{(2k)} \underset{l}{\circ} T^{(n)} : \text{solution of dual } (n+2k-3)\text{-gon eq.} \\ S^{(2k)} \otimes T^{(n)} : \text{solution of dual } (n+2k-1)\text{-gon eq.} \end{cases}$

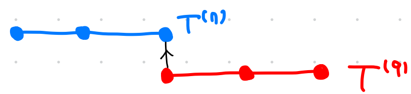
(4) If $T^{(2k)} \leftrightarrow S^{(n)}$ $\Rightarrow$ $\begin{cases} T^{(2k)} \underset{r}{\circ} S^{(n)} : \text{solution of } (n+2k-3)\text{-gon eq.} \\ T^{(2k)} \otimes S^{(n)} : \text{solution of } (n+2k-1)\text{-gon eq.} \end{cases}$

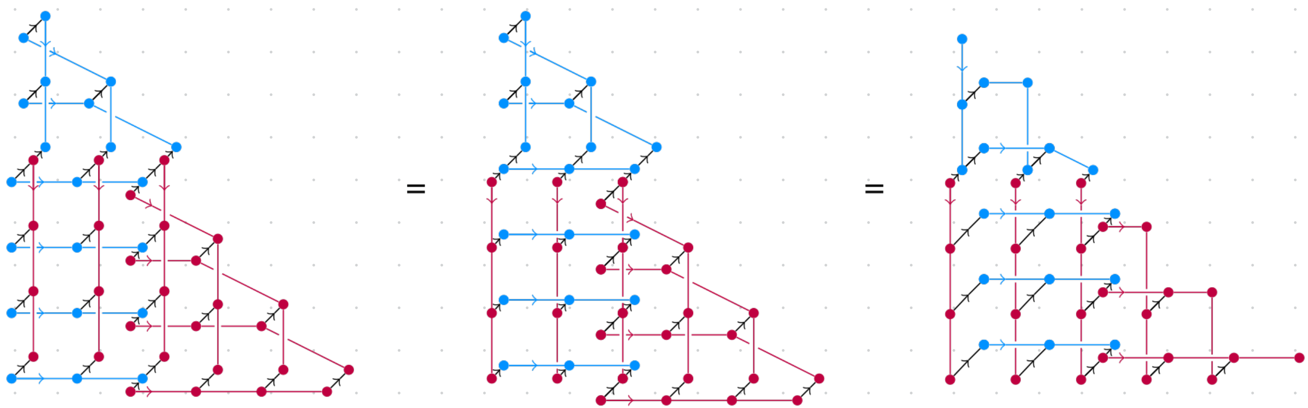
Stacking solutions

(1) If $T^{(2k+1)} \longleftrightarrow T'^{(n)} \Rightarrow T^{(2k+1)} \underset{l}{\circ} T'^{(n)}$: solution of $(n+2k-2)$-gon eq.

proof)

Consider the case, for example, $2k+1 = 7$, $n = 9$.

$T^{(7)} \underset{l}{\circ} T^{(9)} =$ 



Set theoretic solution

In [Müller '24], he proposed the following conjectures:

(1) If $S^{(2k)}$ is a dual $2k$-gon map

$\Rightarrow T^{(2k+1)}(a_1, \cdots, a_k) := (a_1, S^{(2k)}(a_1, \cdots, a_k))$

is a $(2k+1)$-gon map.

---

(2) Let $S^{(2k)}$ be a dual $2k$-gon map, $u \in A$

$T^{(2k+1)}(a_1, \cdots, a_k) := (u, S^{(2k)}(a_1, \cdots, a_k))$

is a $(2k+1)$-gon map.

$\Leftrightarrow S^{(2k)}(u, \cdots, u) = (u, \cdots, u)$

---

(3) If $T^{(2k+1)}$ is a $(2k+1)$-gon map

$\Rightarrow T^{(2k+2)}(a_1, \cdots, a_k) := (T^{(2k+1)}(a_1, \cdots, a_k), a_k)$

is a $(2k+2)$-gon map.

(4) Let $T^{(2k+1)}$ be a $(2k+1)$-gon map, $u \in A$.

$T^{(2k+2)}(a_1, \cdots, a_k) := (T^{(2k+1)}(a_1, \cdots, a_k), u)$

is a $(2k+1)$-gon map.

$\Leftrightarrow T^{(2k+1)}(u, \cdots, u) = (u, \cdots, u)$

---

(5) If $S^{(2k+1)}$ is a dual $(2k+1)$-gon map

$\Rightarrow T^{(2k+2)}(a_1, \cdots, a_k) := (a_1, S^{(2k+1)}(a_1, \cdots, a_k))$

is a $(2k+2)$-gon map.

---

(6) Let $S^{(2k+1)}$ be a dual $(2k+1)$-gon map, $u \in A$.

$T^{(2k+2)}(a_1, \cdots, a_k) := (u, S^{(2k+1)}(a_1, \cdots, a_k))$

is a $(2k+2)$-gon map

$\Leftrightarrow S^{(2k+1)}(u, \cdots, u) = (u, \cdots, u)$

Set theoretic solution

proof)

(1) $T^{(2k+1)}(x_1,\cdots;x_k) := (x_1, S^{(2k)}(x_1,\cdots;x_k))$ is just $T^{(2k+1)} = \Delta \mathbin{\overset{\circ}{r}} S^{(2k)}$,

where $\Delta : X \to X \times X$.

- Δ is a solution of 4-gon eq.

Since $\Delta \leftrightarrow S^{(2k)}$, $T^{(2k+1)}$ is a solution of $(2k+1)$-gon eq.

(2) $T^{(2k+1)}(x_1,\cdots;x_k) := (u, S^{(2k)}(x_1,\cdots;x_k))$ has a form $T^{(2k+1)} = \Delta_u \mathbin{\overset{\circ}{r}} S^{(2k)}$

where $\Delta_u : X \longrightarrow X \times X$, $x \longmapsto (u, x)$.

- Δ_u is a sol. of 4-gon eq.

$S^{(2k)}(u;\cdots u) = (u;\cdots;u) \Rightarrow \Delta_u \longleftrightarrow S^{(2k)}$

$\blacksquare$

Examples of solution ①

H : commutative and cocommutative bialgebra

$\hookrightarrow \begin{cases} M \cdot H \otimes H \longrightarrow H & \text{such that commutative} \\ \Delta : H \longrightarrow H \otimes H & \text{such that cocommutative} \end{cases}$

for $n \geq 3$, the map $T^{(n)}$: $T^{(3)} = id_H$.

$$\boxed{T^{(2k+1)} := \underbrace{\Delta \underset{n}{\circ} M \underset{\ell}{\circ} \Delta \underset{n}{\circ} \cdots \underset{n}{\circ} M}_{2k-2}, \quad T^{(2k)} := \underbrace{\Delta \underset{n}{\circ} M \underset{\ell}{\circ} \Delta \underset{n}{\circ} \cdots \underset{\ell}{\circ} \Delta}_{2k-3},}$$

$\longrightarrow$ $T^{(n)}$ is a solution of n-gon eq.

"proof"

(1) $T^{(5)} \longleftrightarrow T^{(5)}$ & $T^{(5)} \longleftrightarrow \Delta$

by (co)commutativity of M and Δ

Examples of solution ②

$(H_i, 1_i, M_i, \varepsilon_i, \Delta_i)_{1 \le i \le n}$: family of bialgebras with (co)unit

- $\hat{H} := \bigotimes_{i=1}^{n} H_i$,

- $\hat{M}_i : \hat{H} \otimes \hat{H} \longrightarrow \hat{H} \quad (0 \le i \le n-1)$

$$\hat{M}_0 (x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n) := M_1(x_1 \otimes y_1) \otimes M_2(x_2 \otimes y_2) \otimes \cdots \otimes M_n(x_n \otimes y_n)$$

$$\hat{M}_i (x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n) := \prod_{j=1}^{i} \varepsilon_j(y_j) \prod_{j=i+1}^{n} \varepsilon_j(x_j) \cdot (x_1 \otimes \cdots \otimes x_i \otimes y_{i+1} \otimes \cdots \otimes y_n)$$

- $\hat{\Delta}_i : \hat{H} \longrightarrow \hat{H} \otimes \hat{H} \quad (0 \le i \le n-1)$

$$\hat{\Delta}_0 (x_1 \otimes \cdots \otimes x_n) := (x_{1(1)_1} \otimes x_{2(1)_2} \otimes \cdots \otimes x_{n(1)_n}) \otimes (x_{1(2)_1} \otimes x_{2(2)_2} \otimes \cdots \otimes x_{n(2)_n})$$

$$\hat{\Delta}_i (x_1 \otimes \cdots \otimes x_n) := (x_1 \otimes \cdots \otimes x_i \otimes 1_{i+1} \otimes \cdots \otimes 1_n) \otimes (1_1 \otimes \cdots \otimes 1_i \otimes x_{i+1} \otimes \cdots \otimes x_n)$$

$$\boxed{T^{(2k+1)} := \hat{\Delta}_0 \circ_r \hat{M}_0 \circ_l \hat{\Delta}_1 \circ_r \hat{M}_1 \circ_l \cdots \circ_r \hat{M}_{k-1} \ , \quad T^{(2k)} := \hat{\Delta}_0 \circ_r \hat{M}_0 \circ_l \hat{\Delta}_1 \circ_r \hat{M}_1 \circ_l \cdots \circ_l \hat{\Delta}_{k-1}}$$

Example of solution ③

$\mathcal{G}$: strict n-groupoid , $\mathcal{G}_n$: set of n-morphisms of $\mathcal{G}$

$$H := \mathbb{k}[\mathcal{G}_n] = \left\{ \sum_i a_i g_i \;\middle|\; a_i \in \mathbb{k},\ g_i \in \mathcal{G}_n \right\}$$

for $0 \leq i \leq n-1$:

$$\Delta_i : H \longrightarrow H \otimes H, \quad g \longmapsto g \otimes g$$

$$M_i : H \otimes H \longrightarrow H, \quad f \otimes g \longmapsto \delta^{s_{n;i}(f)}_{t_{n;i}(g)}\, f \circ_{n;i} g$$

$$T^{(2k+1)} := \Delta_0 \circ_r M_0 \circ_l \Delta_1 \circ_r \cdots\cdots \circ_r M_{k-2}, \quad T^{(2k)} := \Delta_0 \circ_r M_0 \circ_l \Delta_1 \circ_r \cdots\cdots \circ_l \Delta_{k-2}$$

$$S^{(2k+1)} := M_{n-1} \circ_l \Delta_{n-1} \circ_r \cdots\cdots \circ_l \Delta_{n-k+1}, \quad S^{(2k)} := M_{n-1} \circ_l \Delta_{n-1} \circ_r \cdots\cdots \circ_r M_{n-k+1}$$

Conclusion (so far...)

- n-gon eq. was "algebraic realization" of middle part of Pachner move.
- The "commutative" pair of sol. of polygon eq. gave a sol. of higher polygon eq.
- Solutions can be constructed systematically from (malltiple) bialgebra(s).

Main result 2

Constructing solution of simplex equation
from solutions of polygon equation.

Polygon and simplex eq.

The simplex equation is a family of equation parametrized by $n \geq 1$
$\longrightarrow$ for each n, the equation is called n-simplex equation

Introduced by Bazhanov - Stroganov '82
as higher dimensional analogue of Yang-Baxter.

- 2-simplex equation

$R: V \otimes V \longrightarrow V \otimes V, \quad R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ (a.k.a. Yang-Baxter eq.)

- 3-simplex equation

$R: V \otimes V \otimes V \longrightarrow V \otimes V \otimes V, \quad R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}$

(a.k.a. Tetrahedron. eq)

Polygon and simplex eq.

- In Kashaev - Sergeev '96, they showed that the pair

  T: solution of 5-gon eq., S: solution of dual 5-gon eq.

satisfing the ten-term relation:

$$S_{12} T_{13} S_{14} T_{24} S_{34} = T_{24} S_{34} T_{14} S_{12} T_{13}$$

$$R^3 := S_{13}\, \sigma_{23}\, T_{13}, \qquad R^4 := S_{24}\, \sigma_{12}\, \sigma_{34}\, T_{24}$$

Dimakis - Muller '15

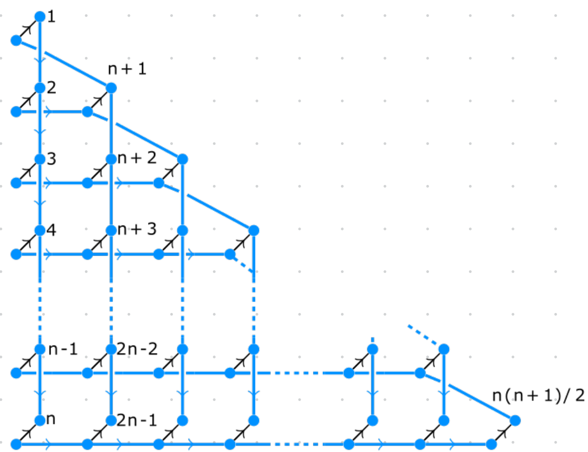
for (2k+1)-gon case, proved in Dimaks - Korepanov '20

pair of sol. of (dual) n-gon eq. + mixed relation	→	(n-1)-simplex eq.
	→ ?	(n-2)-simplex eq.

Definition of simplex eq.

Definition) (Simplex equation)

For $n \geq 2$, the n-simplex equation is



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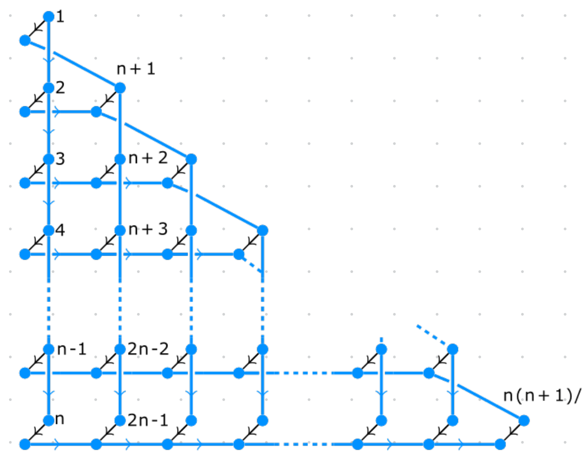
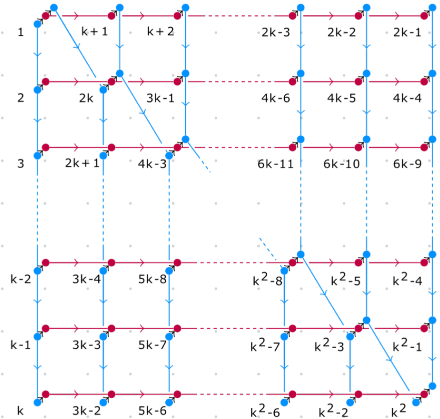


for $R : V^{\otimes n} \longrightarrow V^{\otimes n}$

Definition of mixed relation

T: (2k+1)-gon

S: dual (2k+1)-gon



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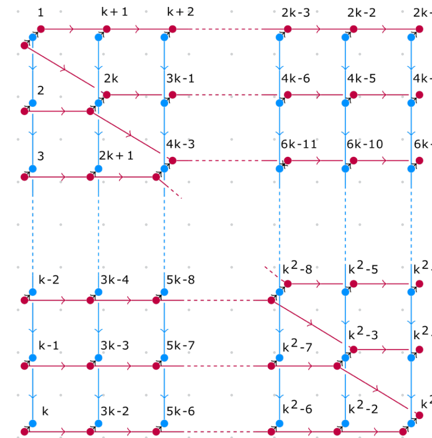
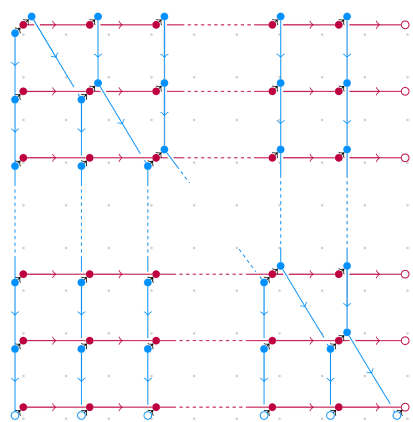


T: 2k-gon

S: dual 2k-gon



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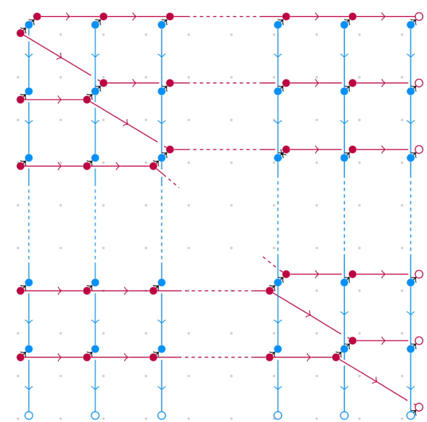

From n-gon to $(n-1)$-simplex eq.

Theorem) [Dimakis - Korepanov '20], [Mochida - M. '25]

T: solution of n-gon eq.

S: solution of dual n-gon eq. s.t. satisfies the mixed relation

$$\Rightarrow \quad R^{(n-1)} := \begin{cases} \sigma_{1,2}\,\sigma_{3,4}\cdots\sigma_{2k-1,2k}\,T_{2,4,\cdots,2k}\,S_{1,3,\cdots,2k-1} & \text{if } n=2k+1 \\ \sigma_{1,2}\,\sigma_{3,4}\cdots\sigma_{2k-3,2k-2}\,T_{2,4,\cdots,2k}\,S_{1,3,\cdots,2k-1} & \text{if } n=2k \end{cases}$$

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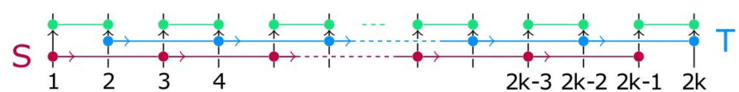
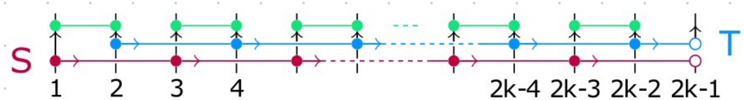


if $n=2k+1$



if $n=2k$

is a solution of $(n-1)$-simplex eq.

From n-gon to $(n-1)$-simplex eq.

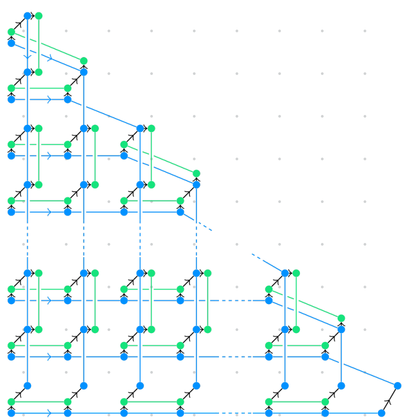
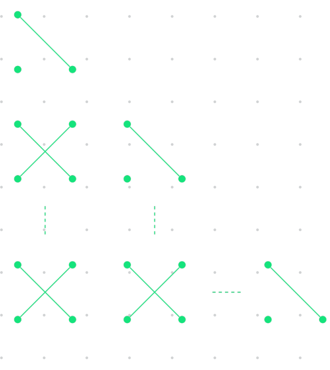
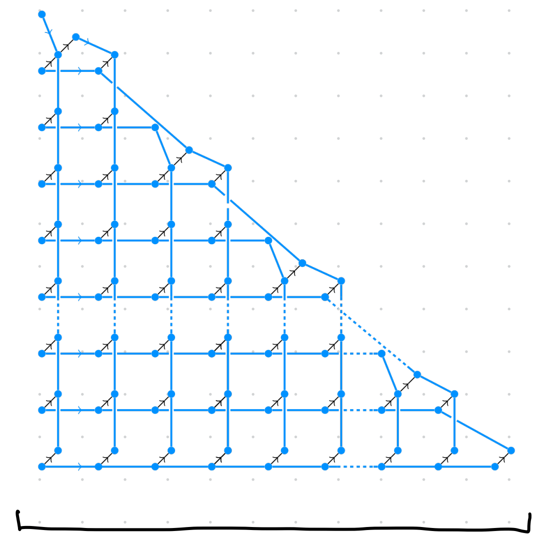
proof)

We show for the $n=2k$ case:

Step 1) Substitute the map

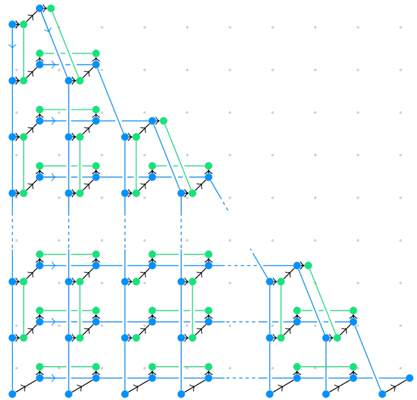
 F where $F : V^{\otimes(k-1)} \to V^{\otimes(k-1)}$

into simplex eq.

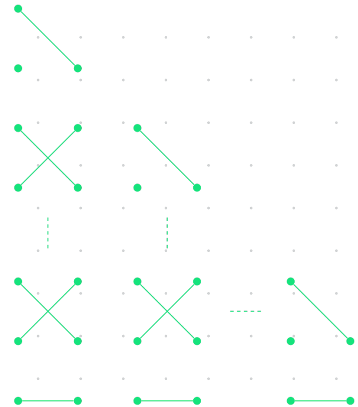
(LHS) =  =  $\circ$ 

(I)

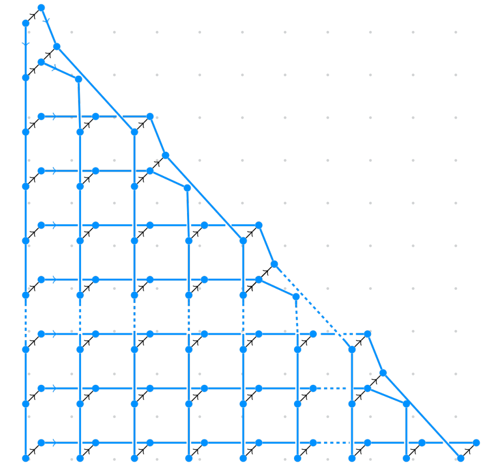
(RHS) =



=

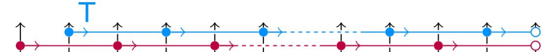


$\circ$

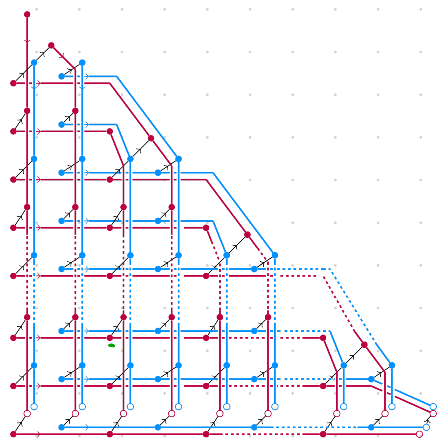


(II)

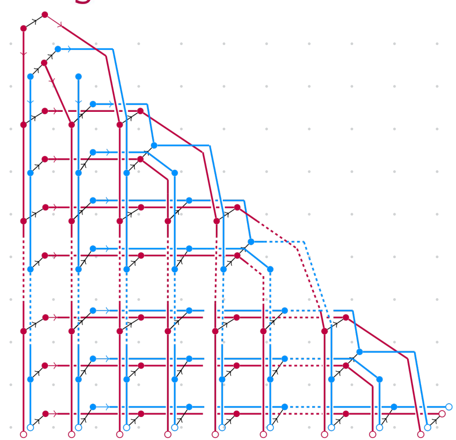
Thus, we need to verify (I) = (II) for F =



(I) =



(II) =



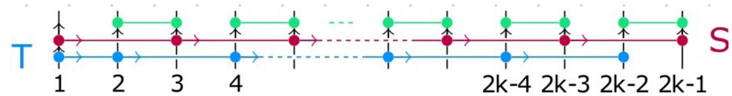
From n-gon to $(n-2)$-simplex eq.

Theorem) [Mochida-M. '25]

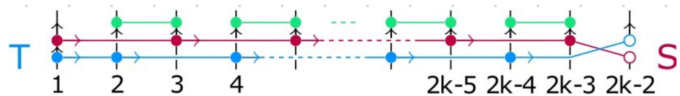
T: solution of n-gon eq.

S: solution of dual n-gon eq. s.t. satisfies the mixed relation

$\Rightarrow$ $R^{(n-2)} :=$



$n = 2k+1$



$n = 2k$

is a solution of $(n-2)$-simplex eq.

Future work (work in progress....)

(1) Cohomology of solution of set theoretic polygon equation.

Q : quandle $\rightsquigarrow$ coloring inv. of links

$\Downarrow$

$H^2(Q) \ni [\varphi]$ $\rightsquigarrow$ cocycle inv. of links

$\Downarrow$

T: sol. of n-gon $\rightsquigarrow$ coloring set of triangulation of m-manifold $(m \geq n-2)$

$\Downarrow$

$H^m(T) \ni [\omega]$ $\rightsquigarrow$ state-sum of ~~ff~~ with weight given by ω

cohomology for Yang-Baxter eq. by Carter – Elhamadadi – Saito '02

n-simplex eq. by Korepanov – Sharygin – Talalaev '14

Future work (work in progress....)

T : sol. of n-gon $\rightsquigarrow$ Col_T : semi-simp set $\rightsquigarrow$ $H^*(T)$: cohomology of T

$\uparrow$

Satisfies the "higher Segal condition"

introduced by
- Dyckerhoff & Kapranov '12
- Gálvez-Carrillo & Kock & Tonks '15
- Poguntke '17

Example of mixed pair

T : sol. of 5-gon eq.

S : sol. of dual 5-gon eq.

s.t. $T \leftrightarrow T$

$S \leftrightarrow S$

$T^{(2k+1)} := T \underset{\ell}{\circ} T \underset{\ell}{\circ} \cdots \underset{\ell}{\circ} T$, $S^{(2k+1)} := S \underset{\ell}{\circ} S \underset{\ell}{\circ} \cdots \underset{\ell}{\circ} S$

Lemma) If T and S satisfies the following, $(T^{(2k+1)}, S^{(2k+1)})$ satisfies the mixed relation.

(1) $T_{13} S_{23} = S_{23} T_{13}$

(2) $T_{23} T_{13} S_{12} = S_{12} T_{23}$

(3) $T_{12} S_{13} S_{23} = S_{23} T_{12}$

(4) $T_{12} S_{13} = S_{13} T_{12}$

(5) $T_{23} S_{34} T_{13} S_{12} = S_{34} T_{24} S_{12} T_{23}$

(6) $T_{12} S_{13} T_{34} S_{23} = S_{23} T_{12} S_{24} T_{34}$

Example of mixed pair

H: commutative and cocommutative Hopf algebra

$\rightarrow$ $T^{(2k+1)} = \Delta \circ_h M \circ_\ell \Delta \circ_h \cdots \circ_h M$: solution of $(2k+1)$-gon eq.

$S^{(2k+1)} := \Delta \circ_h M_S \circ_\ell \Delta \circ_h \cdots \circ_h M_S$: solution of dual $(2k+1)$-gon eq.

where $M_S := M \circ (id \otimes S)$, $x \otimes y \mapsto x \cdot S(y)$

antipode $S : H \to H$ of Hopf alg.

$\Rightarrow$ by the previous lemma, $(T^{(2k+1)}, S^{(2k+1)})$ satisfies mixed relation

$\rightsquigarrow$ sol. of $2k$- and $2k-1$ -simplex eq.