Nielsen equivalence classes on mapping class groups

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§1. Main result.

§2. Related results on Nielsen equivalence

§3. Outline of the proof of the main result.

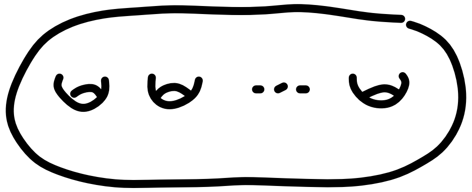
§4. Topology & Nielsen equivalent

§5. Stabilization.

"Today's goal is to get you comfortable with Nielsen equivalence!"

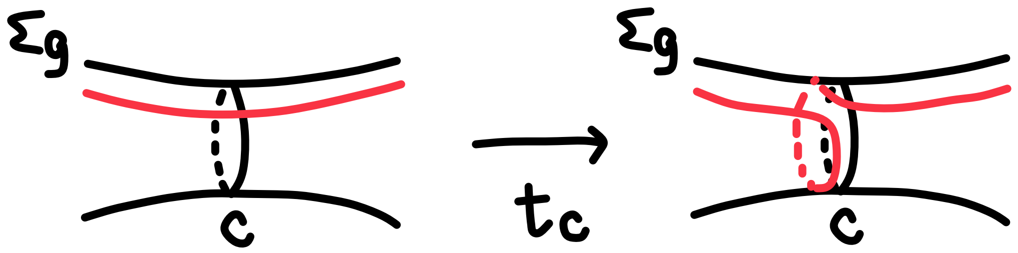
§1. Main result.

$$\mathcal{M}_g := \left\{ \begin{array}{l} \text{orientation preserving} \\ \quad \text{diffeomorphisms of } \Sigma_g \end{array} \right\} \Big/ \text{isotopy}$$

: the mapping class group of Σ_g = 

Dehn : $\mathcal{M}_g$ is finitely generated (by Dehn twists)

(t_c : the Dehn twist about c $\overset{\text{def}}{\Longleftrightarrow}$



)

Lickorish : $\mathcal{M}_g$ is generated by $3g-1$ Dehn twist.

↓

Humphries : $\begin{cases} \text{〃} \quad 2g+1 \quad \text{〃} \quad \text{for } g \geq 2. \\ 2g \text{ (or less) Dehn twists can not generate } \mathcal{M}_g \text{ for } g \geq 2. \end{cases}$

Wajnryb: $\mathcal{M}_g$ is generated by two elements.

⋮

Hirose - M. (2026) ∞-ly many generating pairs of $\mathcal{M}_g$ were constructed for $g \geq 9$.

$(x_1, x_2) \in G \times G$ s.t. $G = \langle x_1, x_2 \rangle$.

Question.1 (M. Linton & M. Sakuma)

Are the generating pairs given in Hirose-M. (2026) Nielsen equivalent?

G : a finitely generated group

$\mathcal{X} = (x_1, \cdots, x_n)$, $\mathcal{X}' = (x_1', \cdots, x_n') \in \underbrace{G \times \cdots \times G}_{n}$

Def $\mathcal{X}$ is Nielsen equivalent to $\mathcal{X}'$ $(\mathcal{X} \sim_N \mathcal{X}')$

$\Leftrightarrow$ $\mathcal{X}'$ is transformed into $\mathcal{X}$ by a finite sequence of (1) ~ (3) :

$$\begin{cases} (1) \quad (\cdots, x_i, \cdots) \longleftrightarrow (\cdots, x_i^{-1}, \cdots) & \\ (2) \quad (\cdots, x_i, \cdots, x_j, \cdots) \longleftrightarrow (\cdots, x_j, \cdots, x_i, \cdots) & (i \neq j) \\ (3) \quad (\cdots, x_i, \cdots) \longleftrightarrow (\cdots, x_i x_j^{\pm 1}, \cdots) & (i \neq j) \end{cases}$$

Question.1 (M. Linton & M. Sakuma)

Are the generating pairs given in Hirose-M. (2026) Nielsen equivalent ?

We could not answer Question 1 ..., but

Thm.2

"Nielsen class" for short

$\mathcal{M}_g$ has infinitely many Nielsen equivalence classes of generating pairs for $g \geqslant 8$.

$$\#\left(\{(x_1, x_2) \in \mathcal{M}_g \times \mathcal{M}_g \mid \mathcal{M}_g = \langle x_1, x_2 \rangle\} / \sim_N\right) = \infty$$

for $g \geqslant 8$.

§2. Related results on Nielsen equivalence

Thm (Nielsen)

The free group F_n of rank n has exactly one Nielsen class of generating n-tuples. //

$$\left(\begin{array}{l} \text{i.e. } F_n = \langle x_1, \cdots, x_n \rangle = \langle y_1, \cdots, y_n \rangle \\ \qquad (x_1, \cdots, x_n) \sim_N (y_1, \cdots, y_n). \end{array} \right)$$

$\rightsquigarrow$ The automorphism group $\mathrm{Aut}(F_n)$ is generated by (1) ~ (3).

Ex.3 $G = \mathbb{Z}_m = \langle x \mid x^m = 1 \rangle$

- $m = 2, 3 \Rightarrow \mathbb{Z}_m$ has exactly one Nielsen class of generating 1-tuples.

$$\left(\begin{array}{l} (\because) \quad (x) \in \mathbb{Z}_2 \underset{(1)}{\longleftrightarrow} (x^{-1}) = (x) \\ \\ \qquad (x) \in \mathbb{Z}_3 \underset{(1)}{\longleftrightarrow} (x^{-1}) = (x^2) \qquad // \end{array} \right.$$

- $m \geqq 3 \Rightarrow \quad L := \#\{ \ell \mid \gcd(\ell, m) = 1, \quad 1 \leqq \ell < m \}$

$\mathbb{Z}_m$ has $L/2$ Nielsen classes of generating 1-tuples.

$$\left(\begin{array}{l} (\because) \quad (x^\ell) : \text{a generating 1-tuple of } \mathbb{Z}_m \Leftrightarrow \gcd(\ell, m) = 1, 1 \leqq \ell < m \\ \qquad (x^\ell) \underset{(1)}{\longleftrightarrow} (x^{-\ell}) = (x^{m-\ell}) \qquad // \end{array} \right.$$

Ex.4 (B.H. & Hanna Neumann)

The alternating group A_5 of degree 5 has 3 Nielsen classes of generating pairs :

$$[(12345),(124)],\quad [(12354),(125)],\quad [(12345),(12354)]$$

To distinct them, an invariant is needed.

(∵) ex $(x,y) \xleftrightarrow[(1)]{} (x,y^{-1}) \xleftrightarrow[(3)]{} (x,y^{-1}x) \xleftrightarrow[(1)]{} (x,x^{-1}y)$

$$((12345),(12354)) \sim_N ((12345),\underbrace{(12345)^{-1}(12354)}_{\overset{\shortparallel}{(345)}})$$

$$\xleftrightarrow[(3)]{} ((\underbrace{(12345)(345)^{-1}}_{\overset{\shortparallel}{(123)}},(345))$$

Lem.5 $(x_1, x_2) \sim_N (y_1, y_2)$ $\quad \big((x_1, x_2), (y_1, y_2) \in G^2 \big)$

$\Rightarrow$ $[y_1, y_2] = y_1 y_2 y_1^{-1} y_2^{-1}$ is conjugate to either $[x_1, x_2]$ or $[x_1, x_2]^{-1}$. //

the Higman invariant := the conjugacy classes of $[x_1, x_2]^{\pm 1}$.

$\rightsquigarrow$ The proof of Ex.4 is completed by Lem.5.

Thm (Gurshko's theorem)

$G = G_1 * G_2$: the free product of G_1 and G_2.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{a generating} \\ n\text{-tuple of } G}}{(x_1, \cdots, x_n)} \sim_N (\underset{\substack{\uparrow \\ \text{a generating} \\ k\text{-tuple of } G_1}}{y_1, \cdots, y_k}, \underset{\substack{\uparrow \\ \text{a generating} \\ (n-k)\text{-tuple of } G_2}}{z_1, \cdots, z_{n-k}})$$

Ex $PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ has exactly one Nielsen class of generating pairs. //

(∵) By Gurshko's theorem,

$$\underset{\substack{\uparrow \\ \text{a generating} \\ \text{pair of } PSL(2,\mathbb{Z})}}{(x_1, x_2)} \sim_N (s, t) \in \underset{\substack{\parallel \\ \langle s \mid s^2 = 1 \rangle}}{\mathbb{Z}_2} \times \underset{\substack{\parallel \\ \langle t \mid t^3 = 1 \rangle}}{\mathbb{Z}_3} \overset{\text{Ex. 1}}{\sim_N} (s, t^2) \quad //$$

Ex. $SL(2,\mathbb{Z})$ $(\cong \mathcal{M}_1)$ has

exactly one Nielsen class of generating pairs. //

(∵) $S, T \in SL(2,\mathbb{Z})$: matrices s.t. $\langle s,t \mid s^2 = t^3 = I \rangle$

$$\left\{ \begin{tikzcd} SL(2,\mathbb{Z}) \arrow[r, "p"] & SL(2,\mathbb{Z})/\{\pm I\} = PSL(2,\mathbb{Z}) \\ S, T \arrow[u, phantom, "\cup" sloped] \arrow[r, mapsto] & s, t \arrow[u, phantom, "\cup" sloped] \end{tikzcd} \right.$$

$PSL(2,\mathbb{Z}) = \langle s,t \mid s^2 = t^3 = I \rangle$

$S, T \rightsquigarrow S^2 = T^3 = -I$

a generating pair of $SL(2,\mathbb{Z})$

$$\begin{tikzcd} (A,B) \arrow[r, leftrightarrow, "(i)"'] \arrow[d, "p"] & \cdots \arrow[r, leftrightarrow, "(j)"'] & \text{either } \underbrace{(S,T), (S,-T), (-S,T), \text{ or } (-S,-T)}_{\text{They are Nielsen equiv. to each others.}} . \\ (p(A), p(B)) \arrow[r, leftrightarrow, "(i)"'] & \cdots \arrow[r, leftrightarrow, "(j)"'] & (s,t) \arrow[u, "p^{-1}"'] \end{tikzcd}$$

$i, j \in \{1,2,3\}$

$\left(\underline{\text{ex}} \quad (S,T) \underset{(3)\times 2}{\leftrightarrow} (S, TS^2) = (S, -T) \right)$

<u>Thm</u> (Dunwoody - Pietrowski)

The trefoil knot group $G = \langle x, y \mid x^2 = y^3 \rangle$ has ∞-ly many Nielsen classes of generating pairs. //

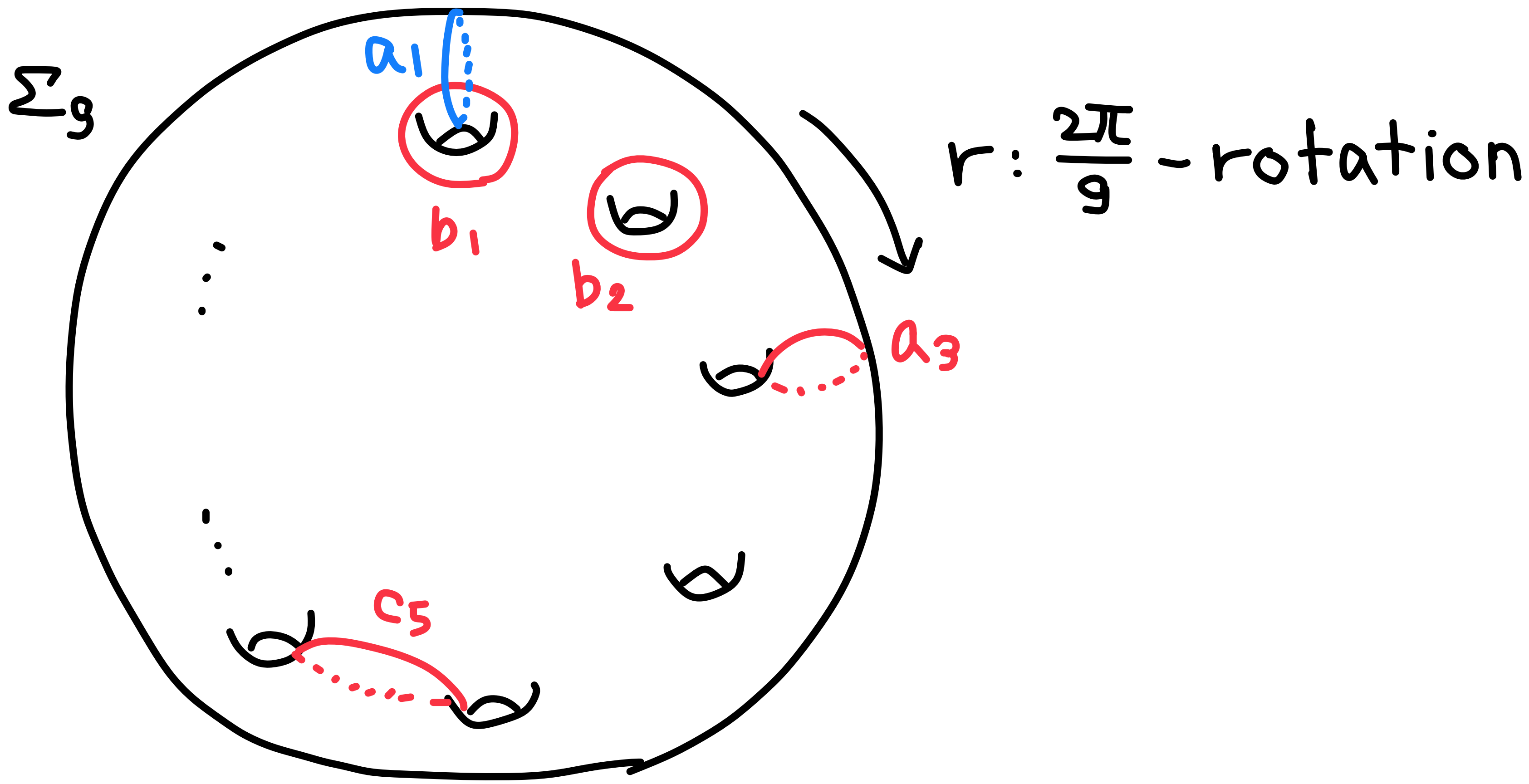
§3. Outline of the proof of Thm. 2

Thm. 2 (Hirose - M.)

$\mathcal{M}_g$ has ∞-ly many Nielsen equivalen classes of generating pairs for $g \geqslant 8$. //

Thm. 8 (Hirose - M.)

$\mathcal{M}_g = \langle h_0 := t_{b_1} t_{b_2} t_{a_3} t_{c_5} , \rho_n := r t_{a_1}^{n} \rangle$ for ${}^{\forall} n \in \mathbb{Z}_{>0}$, $g \geqslant 8$.



Rem

$$\begin{tikzcd} f : \Sigma_g \to \Sigma_g : \text{orientation preserving diffeomorphism} \arrow[d, mapsto, "\text{action on } H_1(\Sigma_g, \mathbb{Z})"] \\ f_* : H_1(\Sigma_g, \mathbb{Z}) \longrightarrow H_1(\Sigma_g, \mathbb{Z}) : \text{automorphism.} \arrow[d, mapsto, "\text{``Matrix representation''}"] \\ A_f \in GL(2g, \mathbb{Z}) \end{tikzcd}$$

Especially, $f \in \mathcal{M}_g \xmapsto{\text{(surj.) homomorphism}} A_f \in Sp(2g, \mathbb{Z}) \subset GL(2g, \mathbb{Z})$ //

$[h_0, \rho_n] = h_0 \rho_n h_0^{-1} \rho_n^{-1}$,

$\lambda_{[h_0, \rho_n]}$:= the eigen value of $A_{[h_0, \rho_n]}$. ← $\lambda_{[h_0, \rho_n]}$ can be calculated directly.

<u>claim</u> $m \neq n \Rightarrow \lambda[h_0, \rho_m] \neq \lambda[h_0, \rho_n]$ //

$\therefore$ $m \neq n \Rightarrow [h_0, \rho_m]$ is not conjugate to $[h_0, \rho_n]^{\pm 1}$

(i.e the Higman invariants are different.)

By Lem. 5,

$m \neq n \Rightarrow (h_0, \rho_m)$ is not Nielsen equivalent to (h_0, ρ_n) //

§4. Topology & Nielsen equivalence

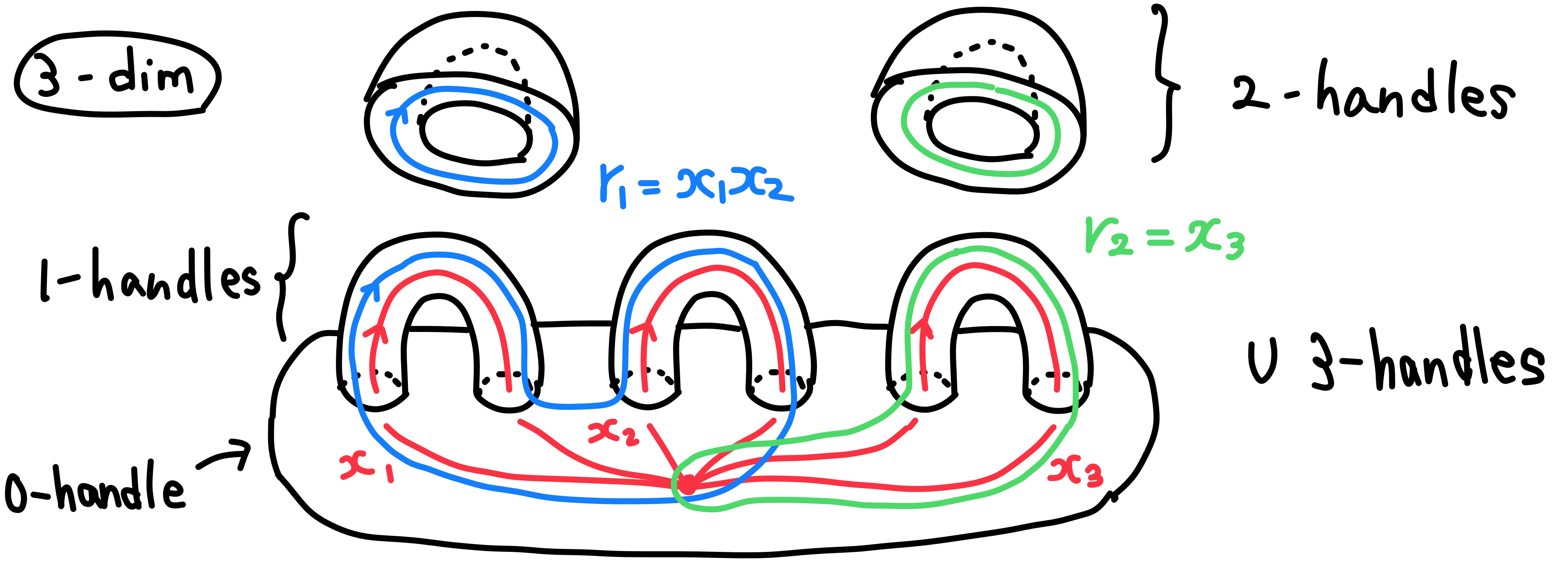
M : a connected oriented mfd

admitting a handle decomposition with

$$\underset{\text{0-handle}}{h^{(0)}} \cup \underbrace{h_1^{(1)} \cup \cdots \cup h_n^{(1)}}_{n\ \text{1-handles}} \cup \underbrace{h_1^{(2)} \cup \cdots \cup h_m^{(2)}}_{m\ \text{2-handles}} \cup \underset{\cdots}{h_1^{(3)} \cup \cdots}$$

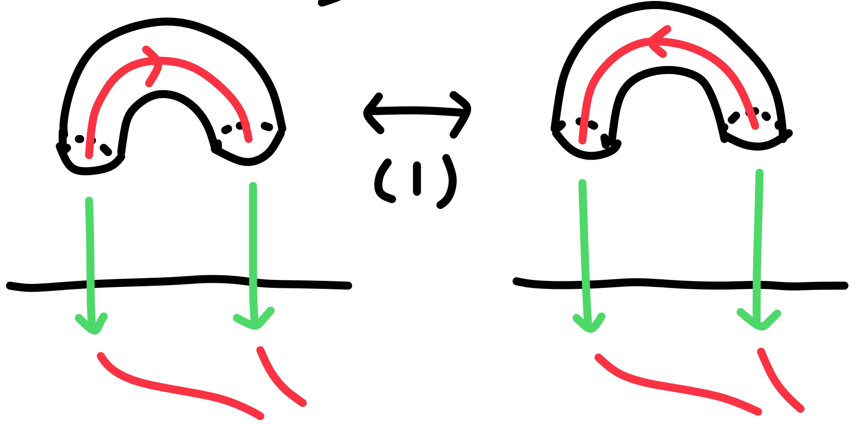
$$\rightsquigarrow$$

$$\pi_1(M) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$$



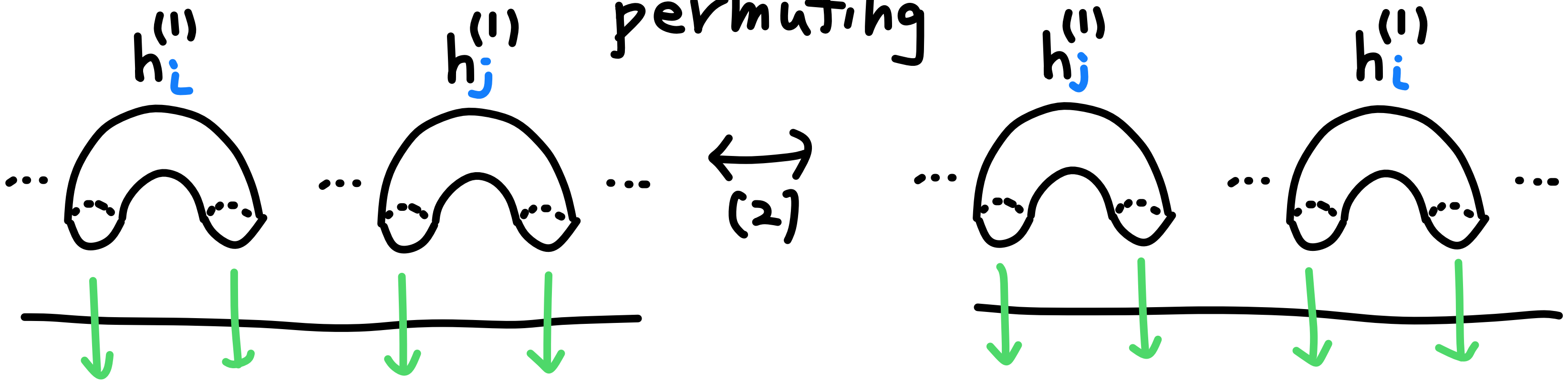
- $(\cdots, x_i, \cdots) \underset{(1)}{\longleftrightarrow} (\cdots, x_i^{-1}, \cdots)$

reversing the orientation



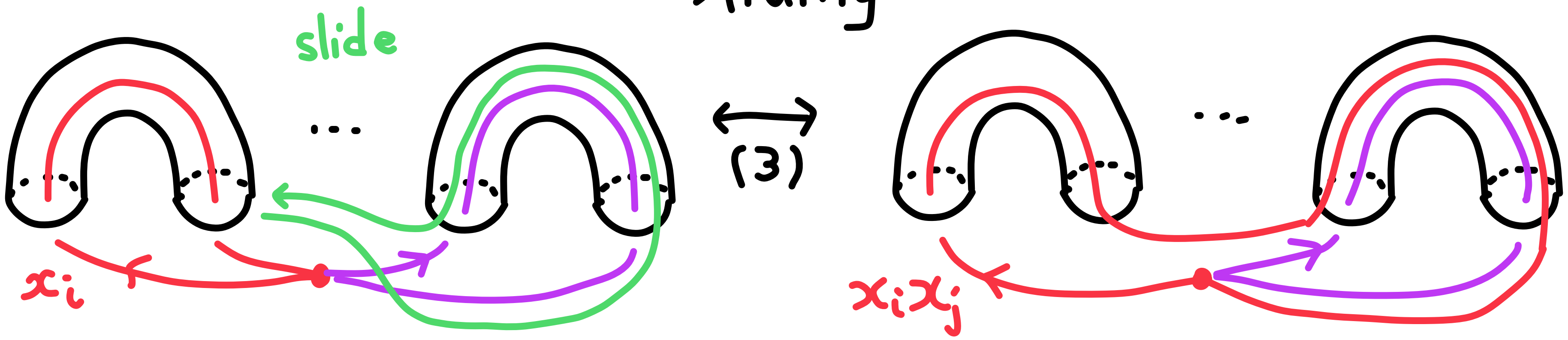
- $(\cdots, x_i, \cdots, x_j, \cdots) \underset{(2)}{\longleftrightarrow} (\cdots, x_j, \cdots, x_i, \cdots)$

permuting



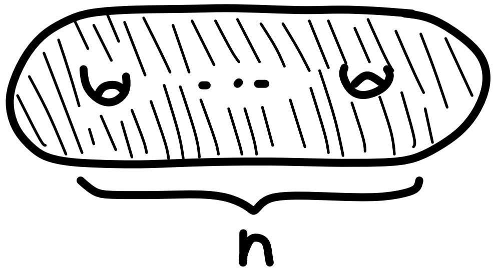
- $(\cdots, x_i, \cdots) \underset{(3)}{\longleftrightarrow} (\cdots, x_i x_j^{\pm 1}, \cdots)$

sliding



Two Heegaard splittings $M^3 = H_1^1 \cup H_2^1 = H_1^2 \cup H_2^2$ are <u>isotopic</u>

$\underset{\text{def}}{\Longleftrightarrow}$ $^{\exists}F : M \times [0,1] \to M$: isotopy s.t. $F(H_1^1, 1) = H_1^2$ (or H_2^2). //

$H_1^1 \cong H_2^1 \cong H_1^2 \cong H_2^2 \cong$ 

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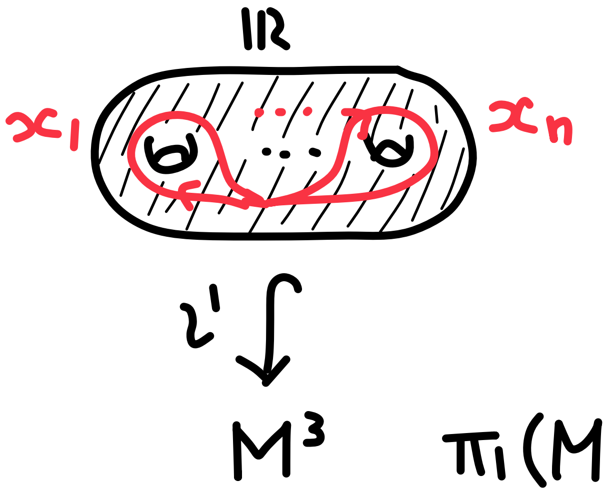


$$\begin{tikzcd}
\pi_1(H_i^1) = \langle x_1, \cdots, x_n \rangle \arrow[d, "i^1_*"'] & \pi_1(H_i^2) = \langle y_1, \cdots, y_n \rangle \ (i=1,2) \arrow[d, "i^2_*"] \\
\pi_1(M^3) = \langle x_1, \cdots, x_n \mid - \rangle \arrow[r, phantom, "="] & \langle y_1, \cdots, y_n \mid - \rangle
\end{tikzcd}$$

Lem If two splittings are isotopic,

then $(x_1, \cdots, x_n) \sim_N (y_1, \cdots, y_n)$ in $\pi_1(M^3)$ for $i=1,2$ //

The parts in green are the correction.

Stable Andrews - Curtis conjecture

For the trivial group $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle = \{1\}$,

$(r_1, \cdots, r_n)$ is transformed into $(x_1, \cdots, x_n)$ by

$(\ )$ ← the empty tuple

a finite sequence of (1) ~ (4) in $F_n = \langle x_1, \cdots, x_n \rangle$:

(1) ~ (5)

$$\begin{cases} (1) & (\cdots, r_i, \cdots) \longleftrightarrow (\cdots, r_i^{-1}, \cdots) \\ (2) & (\cdots, r_i, \cdots, r_j, \cdots) \longleftrightarrow (\cdots, r_j, \cdots, r_i, \cdots) \\ (3) & (\cdots, r_i, \cdots) \longleftrightarrow (\cdots, r_i r_j^{\pm 1}, \cdots) \\ (4) & (\cdots, r_i, \cdots) \longleftrightarrow (\cdots, s^{-1} r_i s, \cdots), \quad (s \in F_n) \end{cases}$$ //

$$(5) \quad (\cdots, r_n) \longleftrightarrow (\cdots, r_n, \underset{\underset{x_{n+1}}{\parallel}}{r_{n+1}})$$

r_{n+1} ← a trivial relator

x_{n+1} ← a generator

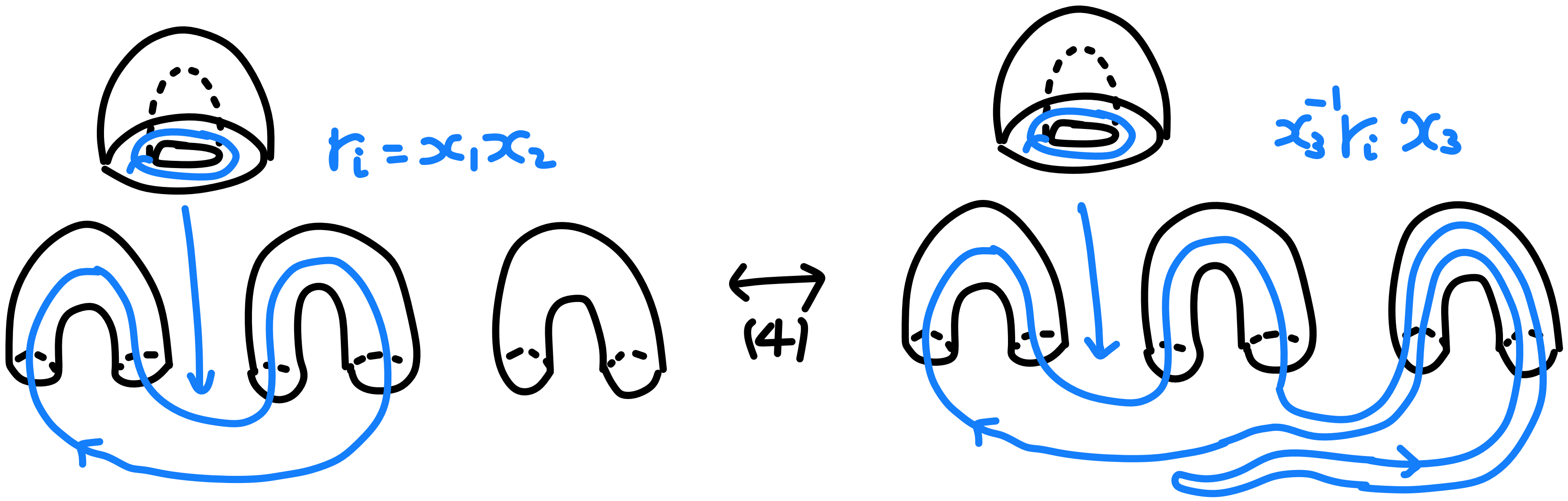
The parts in green are the correction.

(1) = reversing the orientation of $h_i^{(2)}$ ← 2-handle

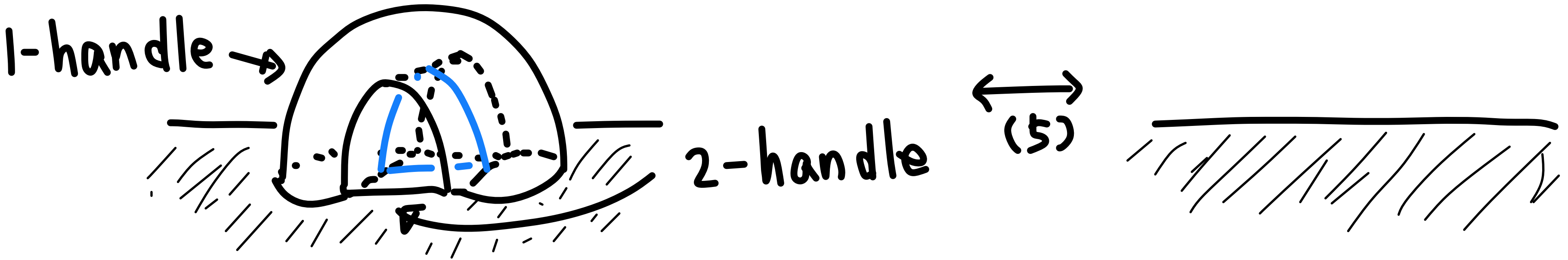
(2) = permuting the indicies of $h_i^{(2)}$ and $h_j^{(2)}$

(3) = sliding $h_i^{(2)}$ over $h_j^{(2)}$

(4) = sliding the attaching curve of $h_i^{(2)}$ over 1-handles.



(5) = adding or deleting a cancelling pair of 1/2-handles.



This slide is incorrect.

~~**Thm** (Andrews - Curtis)~~

~~M^4 : a smooth homotopy 4-sphere.~~

~~Suppose that~~

- ~~M admits a handle decomposition with~~

$$\underset{\substack{\uparrow \\ \text{0-handle}}}{h^{(0)}} \cup \underbrace{h_1^{(1)} \cup \cdots \cup h_n^{(1)}}_{n\ \text{1-handle}} \cup \underbrace{h_1^{(2)} \cup \cdots \cup h_n^{(2)}}_{n\ \text{2-handles}} \cup \underset{\substack{\uparrow \\ \text{4-handle}}}{h^{(4)}} \quad (\text{without 3-handles})$$

- ~~The Andrews - Curtis conjecture is true.~~

~~Then, $M^4 \underset{\text{diffeo.}}{\cong} \mathbb{S}^4$: the smooth 4-sphere~~ //

$\boxed{\text{This slide was added.}}$

Prop

X : a contractible smooth 4-mfd

Suppose that

- X admits a handle decomposition with

  $h^{(0)} \cup h^{(1)}_1 \cup \cdots \cup h^{(1)}_n \cup h^{(2)}_1 \cup \cdots \cup h^{(2)}_n$ (without 3, 4-handles)

- The stable Andrews-Curtis conjecture is true

Then $DX := X \cup_{\mathrm{id}_{\partial X}} \overline{X} \underset{\text{diffeo}}{\cong} S^4$: the standard 4-sphere. //

$\uparrow$ X with the opposite orientation

§5. Stabilization

Stabilization

$$G = \langle \underbrace{x_1, \cdots, x_n}_{\text{a minimal generating set.}} \rangle = \langle y_1, \cdots, y_n \rangle$$

- $(\cdots, x_i, \cdots) \underset{(1)}{\longleftrightarrow} (\cdots, x_i^{-1}, \cdots)$
- $(\cdots, x_i, \cdots, x_j, \cdots) \underset{(2)}{\longleftrightarrow} (\cdots, x_j, \cdots, x_i, \cdots)$
- $(\cdots, x_i, \cdots) \underset{(3)}{\longleftrightarrow} (\cdots, x_i x_j^{\pm 1}, \cdots)$

$$\rightsquigarrow (x_1, \cdots, x_n, \underbrace{1, \cdots, 1}_{n}) \sim_N (x_1, \cdots, x_n, y_1, \cdots, y_n)$$

n times stabilization

$$\sim_N (1, \cdots, 1, y_1, \cdots, y_n)$$

$$\sim_N (y_1, \cdots, y_n, 1, \cdots, 1)$$

Thm. (Myropolska)

$$^{\exists}G \text{ s.t. } \begin{cases} \#\left(\{(x_1,x_2)\in G^2 \mid G=\langle x_1,x_2\rangle\}/\sim_N\right)=\infty \\ \#\left(\{(x_1,x_2,1)\in G^3 \mid G=\langle x_1,x_2\rangle\}/\sim_N\right)=1 \end{cases}$$

Thm. (Kapovich – Weidmann)

$$^{\forall}n\geq 2,\ ^{\exists}G \text{ s.t. } (x_1,\cdots,x_n,\overbrace{1,\cdots,1}^{n-1}) \not\sim_N (y_1,\cdots,y_n,\overbrace{1,\cdots,1}^{n-1})$$

Question How about $\mathcal{M}_g$?

Fact (S. Kamada)

$$(\underbrace{h_0,\rho_m}_{\text{generators in Thm. 8}},1) \sim_N (h_0,\rho_n,1) \quad (^{\forall}m, ^{\forall}n)$$

(∵) $G=\langle x_1,x_2\rangle=\langle x_1,x_3\rangle$

$$(x_1,x_2,1)\sim_N(x_1,x_2,x_3)$$

$$\underset{(1)}{\longleftrightarrow}(x_1,x_3,x_2)$$

$$\sim(x_1,x_3,1)$$

Thank you very much for your attention!