Combinatorics of ordered ideal triangulations and Andersen–Kashaev volume conjecture

Ka Ho WONG

Yale University

Intelligence of Low-dimensional Topology

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(Joint work with Fathi Ben Aribi and Antonin Guilloux)

Background: two approaches of studying links and 3-manifolds

Hyperbolic geometry	**Quantum topology**
Geometric invariants	Quantum invariants
(hyperbolic volume)	(Jones polynomial)
Hyperbolic metric	Quantum groups

Kashaev–Murakami–Murakami volume conjecture: for any hyperbolic knot $K \subset \mathbb{S}^3$,

$$\lim_{N\to\infty} \frac{2\pi}{N} \log |\langle K \rangle_N| = \lim_{N\to\infty} \frac{2\pi}{N} \log \left| \mathrm{J_N}\left(K, e^{\frac{2\pi i}{N}}\right) \right| = \mathrm{Vol}(\mathbb{S}^3 \backslash K),$$

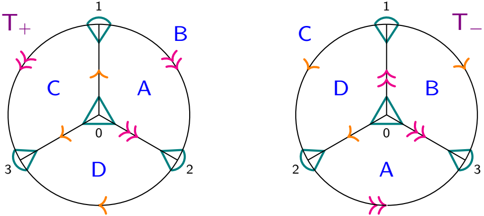
where $\langle K \rangle_N$ and $\mathrm{J_N}(K,t)$ are the N-th *Kashaev invariant* and the colored Jones polynomials of K respectively.

Today: the Andersen–Kashaev volume conjecture of Teichmüller TQFT

- Construction of a partition function and its volume conjecture
- The conjecture and its generalizations hold if the triangulation is *geometric* and *FAMED* ("Face Adjacency Matrices with Edge Duality")

Ordered triangulations

- An ordered triangulation is a triangulation $X = \{T_1, \ldots, T_N\}$ such that
  - the vertices of each tetrahedron are ordered by 0, 1, 2, 3 and
  - the face pairings preserve the orientations of the edges.
- Example: Thurston's ideal triangulation of the figure eight knot complement.



- Ordering implies that there is only one way to glue two faces together
- Sign of a tetrahedron $\epsilon(T)$: positive (left) or negative (right)
- $x_k(T)$: face opposite to k-th vertex of T

  e.g. $x_0(T_+) = \mathrm{B}$, $x_1(T_+) = \mathrm{D}$, $x_2(T_+) = \mathrm{C}$, $x_3(T_+) = \mathrm{A}$

Construction of quantum invariant

Construction: pick $N \in \mathbb{N}$, $q = e^{\frac{2\pi i}{N}}$

- Each tetrahedron has two incoming faces and two outgoing faces.
- For each $N \in \mathbb{N}$, regard each $+$ve/$-$ve tetrahedron as some "nice" linear map $S^{\pm} : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ with

$$S^{\pm}(e_i \otimes e_j) = \sum_{k,l=1}^{N} (S^{\pm})_{i,j}^{k,l} e_k \otimes e_l$$

- Gluing faces corresponds to identifying the variable and taking the sum
- For Thurston's ideal triangulation of $\mathbb{S}^3 \setminus 4_1$, we get a number that "looks like"

$$\sum_{A,B,C,D=1}^{N} (S^{+})_{A,D}^{B,C} (S^{-})_{B,C}^{A,D}$$

Construction of quantum invariant

Special cases: ($S^{\pm}$ comes from cyclic representations of Borel subalgebra of $U_q(sl_2(\mathbb{C}))$)

- Hamiltonian triangulation for $(\mathbb{S}^3, K)$

  (Triangulation of $\mathbb{S}^3$ with a subset of edges that contains all vertices and forms K)

  $\rightsquigarrow$ Kashaev invariants $|\langle K \rangle_N|$ (equal to $|J_N(K, e^{\frac{2\pi i}{N}})|$ by Murakami-Murakami)

- Ideal triangulation for $\mathbb{S}^3 \setminus K$ (with additional data)

  $\rightsquigarrow$ Quantum Hyperbolic Invariant (QHI) by Baseilhac–Benedetti

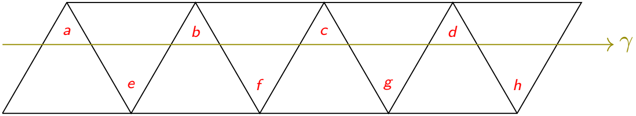
Andersen–Kashaev: infinite dimensional representations

- Finite sum becomes improper integral
- Issue about convergence

Angle structures for an ideal triangulation

- Given an ideal triangulation $X = \{T_1, \ldots, T_N\}$, an angle structure is an assignment of dihedral angles in $(0, \pi)$ to edges of the tetrahedra such that
  - opposite edges share the same dihedral angle,
  - the angle sum of each truncated triangle is equal to π,
  - the angle sum around each edge is equal to 2π.
- The set of all angle structure is denoted by $\mathscr{A}_X$.
- For each T, the angles at the edges $01, 02, 03$ are $\alpha_1(T), \alpha_2(T), \alpha_3(T)$ respectively.
- The truncated triangles are glued together to form a triangulation of $\partial(N(K))$.
- For any peripheral curve $\gamma \in \pi_1(\partial(N(K))$, the *angular holonomy* of γ is defined to be

$$\text{sum of angles on the left} - \text{sum of angles on the right.}$$



Faddeev's quantum dilogarithm function

Let $\mathrm{b} > 0$ and $\hbar = (\mathrm{b} + \mathrm{b}^{-1})^{-2} > 0$.

Definition: (Faddeev's quantum dilogarithm)

The *Faddeev's quantum dilogarithm* is the holomorphic function defined by

$$\Phi_{\mathrm{b}}(z) = \exp\left(\frac{1}{4}\int_{w \in \mathbb{R}+i0^+} \frac{e^{-2izw}\,dw}{\sinh(\mathrm{b}w)\sinh(\mathrm{b}^{-1}w)w}\right) \quad \text{for } z \in \mathbb{R} + i\left(\frac{-1}{2\sqrt{\hbar}}, \frac{1}{2\sqrt{\hbar}}\right).$$

Properties:

- In the semi-classical limit $\mathrm{b} \to 0$,

$$\Phi_{\mathrm{b}}\left(\frac{z}{2\pi \mathrm{b}}\right) = \exp\left(\frac{-i}{2\pi \mathrm{b}^2}\,\mathrm{Li}_2(-e^z)\right)\left(1 + O_{\mathrm{b}\to 0^+}(\mathrm{b}^2)\right).$$

- For any $\mathrm{b} \in \mathbb{R}_{>0}$ and any $d \in \left(\frac{-1}{2\sqrt{\hbar}}, \frac{1}{2\sqrt{\hbar}}\right)$,

$$|\Phi_{\mathrm{b}}(x + id)| \underset{\mathbb{R}\ni x \to -\infty}{\sim} 1, \quad |\Phi_{\mathrm{b}}(x + id)| \underset{\mathbb{R}\ni x \to +\infty}{\sim} e^{-2\pi x d}.$$

Partition functions

Definition/Theorem: (Partition functions)

Let X be an ordered ideal triangulation of $\mathbb{S}^3 \setminus K$ with an angle structure $\alpha \in \mathscr{A}_X$. The partition function of (X, α) is defined by

$$\mathscr{Z}_\hbar(X,\alpha) = \int_{\mathbf{x}\in\mathbb{R}^{2N}} \prod_{\text{tetra.}\,T} \frac{\delta(x_0(T)-x_1(T)+x_2(T)) e^{(2\pi i x_0(T)+\epsilon(T)\alpha_3(T)/\sqrt{\hbar})(x_3(T)-x_2(T)))}}{\Phi_{\mathsf{b}}\Big(\epsilon(T)((x_3(T)-x_2(T)) - \frac{i}{2\pi\sqrt{\hbar}}(\alpha_2(T)+\alpha_3(T)))\Big)^{\epsilon(T)}} d\mathbf{x},$$

where

- Φ_{b} is the Faddeev's quantum dilogarithm function;
- $x_k(T)$ is the face opposite to the k-th vertex of T;
- $\alpha_2(T), \alpha_3(T)$ are angles on $02, 03$ edges of T;
- $\epsilon(T) \in \{\pm 1\}$ is the sign of T; and
- δ is the Dirac delta distribution.

Partition functions

Remarks:

- [Andersen–Kashaev 11'] The integral converges absolutely.
- Every two ideal triangulations are related by a sequence of 2-3 Pachner moves
- [Andersen–Kashaev 11'] $|\mathscr{Z}_\hbar(X, \alpha)| \in \mathbb{R}$ is invariant under angled 2-3 Pachner move. (*Charged pentagon identity* satisfied by the tetrahedral operators).
- Not all triangulation supports angle structures.
- It is not known how the partition function depends on the angle structure
- The construction is very similar to that of Kashaev's invariants and of Baseilhac–Benedetti's Quantum Hyperbolic Invariant

Andersen–Kashaev volume conjecture for $\mathbb{S}^3 \setminus K$

Conjecture: [Andersen–Kashaev 11', Ben Aribi–Guéritaud-Piguet-Nakazawa 20']

There exists an ordered ideal triangulation X and a Jones function J_X such that :

1. for any angle structures $\alpha \in \mathscr{A}_X$ and all $\hbar > 0$, we have:

$$|\mathscr{Z}_\hbar(X, \alpha)| = \left| \int_{\mathbb{R}} J_X(\hbar, x) e^{\frac{1}{2\sqrt{\hbar}} x \lambda_X(\alpha)} dx \right|,$$

   where $\lambda_X(\alpha)$ is the angular holonomies of the preferred longitude of K.

2. In the semi-classical limit $\hbar \to 0^+$, we retrieve the hyperbolic volume of K as:

$$\lim_{\hbar \to 0^+} 2\pi\hbar \log |J_X(\hbar, 0)| = -\operatorname{Vol}(\mathbb{S}^3 \setminus K).$$

The conjecture is proved for

- 4_1 and 5_2 [Andersen–Kashaev 11'], 6_1 [Andersen–Nissen 17'], 7_3 [Uemura 23']
- all hyperbolic twist knots [Ben Aribi–Guéritaud–Piguet-Nakazawa 20']

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$$\mathscr{Z}_\hbar(X,\alpha) = \int_{\mathbf{x}\in\mathbb{R}^{2N}} \prod_{\text{tetra. } T} \frac{\delta(x_0(T) - x_1(T) + x_2(T)) e^{(2\pi i x_0(T) + \epsilon(T)\alpha_3(T)/\sqrt{\hbar})(x_3(T) - x_2(T)))}}{\Phi_{\mathsf{b}}\left(\epsilon(T)((x_3(T) - x_2(T)) - \frac{i}{2\pi\sqrt{\hbar}}(\alpha_2(T) + \alpha_3(T)))\right)^{\epsilon(T)}} d\mathbf{x},$$

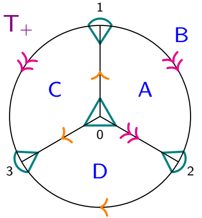
where

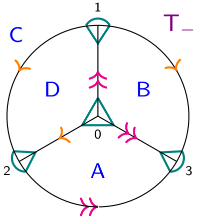
- Φ_{b} is the Faddeev's quantum dilogarithm function;
- $x_k(T)$ is the face opposite to the k-th vertex of T;
- $\alpha_2(T), \alpha_3(T)$ are angles on $02, 03$ edges of T;
- $\epsilon(T) \in \{\pm 1\}$ is the sign of T; and
- δ is the Dirac distribution.

$$x_0(T_+) = \mathrm{B},\quad x_1(T_+) = \mathrm{D},\quad x_2(T_+) = \mathrm{C},\quad x_3(T_+) = \mathrm{A},$$

$$x_0(T_-) = \mathrm{C},\quad x_1(T_-) = \mathrm{A},\quad x_2(T_-) = \mathrm{B},\quad x_3(T_-) = \mathrm{D}$$

$$w(T) = x_0(T) - x_1(T) + x_2(T), w'(T) = x_2(T) - x_3(T)$$

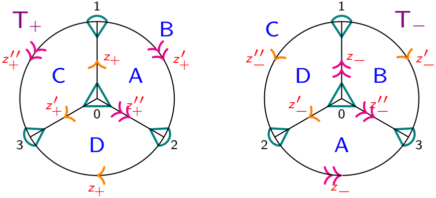




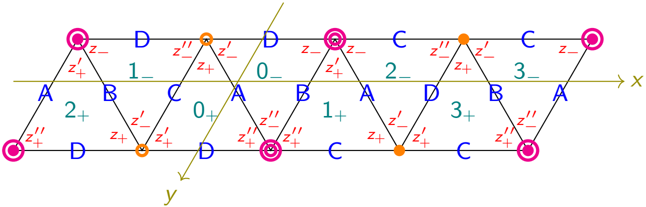
$$\mathscr{X}_0 = \begin{array}{c} \\ T_+ \\ T_- \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{A} = \begin{array}{c} \\ w(T_+) \\ w(T_-) \\ w'(T_+) \\ w'(T_-) \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}, \quad \mathscr{B} = \begin{array}{c} \\ w(T_+) \\ w(T_-) \\ w'(T_+) \\ w'(T_-) \end{array} \begin{array}{c} \begin{array}{cc} T_+ & T_- \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

Neumann-Zagier matrices for Thurston's triangulation of $\mathbb{S}^3 \setminus 4_1$

- Put shape parameters z on 01 edge. Put $z' = 1/(1-z)$ and $z'' = 1 - 1/z$ counter-clockwisely. They satisfy $\operatorname{Log} z + \operatorname{Log} z' + \operatorname{Log} z'' = \pi i$.



- The truncated green triangles are glued together to form a triangulation of the boundary torus.



Neumann-Zagier matrices for Thurston's triangulation of $\mathbb{S}^3 \setminus 4_1$

- The *logarithmic holonomies* of the orange edge e and the preferred longitude $l = x + 2y$ are given by

$$\mathrm{H}(e) = 2\,\mathrm{Log}\, z_+ + \mathrm{Log}\, z'_+ + 2\,\mathrm{Log}\, z'_- + \mathrm{Log}\, z''_-$$

$$\mathrm{H}(l) = 2i\pi - 4\mathrm{Log}(z'_-) - 2\mathrm{Log}(z''_-)$$

- The Neumann-Zagier matrices are given by

$$\mathbf{G} = \begin{array}{c} \\ (\rightarrow) \\ l \end{array} \begin{array}{c} \begin{matrix} z_+ & z_- \end{matrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \mathbf{G}' = \begin{array}{c} \\ (\rightarrow) \\ l \end{array} \begin{array}{c} \begin{matrix} z'_+ & z'_- \end{matrix} \\ \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} \end{array}, \quad \mathbf{G}'' = \begin{array}{c} \\ (\rightarrow) \\ l \end{array} \begin{array}{c} \begin{matrix} z''_+ & z''_- \end{matrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \end{array}$$

- Define

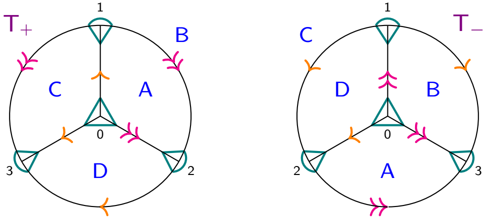
$$\mathbf{A} := \mathbf{G} - \mathbf{G}' \quad \text{and} \quad \mathbf{B} := \mathbf{G}'' - \mathbf{G}'.$$

Face adjacency matrices for Thurston's triangulation of $\mathbb{S}^3 \setminus 4_1$

$$x_0(T_+) = \mathrm{B}, \quad x_1(T_+) = \mathrm{D}, \quad x_2(T_+) = \mathrm{C}, \quad x_3(T_+) = \mathrm{A},$$

$$x_0(T_-) = \mathrm{C}, \quad x_1(T_-) = \mathrm{A}, \quad x_2(T_-) = \mathrm{B}, \quad x_3(T_-) = \mathrm{D}$$

$$w(T) = x_0(T) - x_1(T) + x_2(T), w'(T) = x_2(T) - x_3(T)$$



$$\mathscr{X}_0 = \begin{array}{c} \\ T_+ \\ T_- \end{array} \begin{array}{c} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}, \quad \mathscr{A} = \begin{array}{c} \\ w(T_+) \\ w(T_-) \\ w'(T_+) \\ w'(T_-) \end{array} \begin{array}{c} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}, \quad \mathscr{B} = \begin{array}{c} \\ w(T_+) \\ w(T_-) \\ w'(T_+) \\ w'(T_-) \end{array} \begin{array}{c} \begin{matrix} T_+ & T_- \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

FAMED triangulations ("Face Adjacency Matrices with Edge Duality")

Definition: (FAMED triangulations)

An ordered ideal triangulation X is FAMED with respect to the preferred longitude l if

1. the space of angle structures is non-empty,
2. $\det \mathscr{A} \neq 0$,
3. $\det \mathbf{B} \neq 0$,
4. We have the following equality for $N \times N$ matrices

$$\mathbf{B}^{-1}\mathbf{A} = \mathscr{X}_0 \mathscr{A}^{-1} \mathscr{B} + \left(\mathscr{X}_0 \mathscr{A}^{-1} \mathscr{B} \right)^\top + \frac{\mathscr{E} + \mathrm{Id}_N}{2},$$

   where $\mathscr{E}$ is the diagonal matrix that encodes the sign of the tetrahedra.

Main result I: asymptotics of Jones function

- An ideal triangulation X is *geometric* if the system of edge and completeness equations admits a solution of shape parameters with positive imaginary parts.

Theorem I: [Ben Aribi–W. 24']

The Andersen–Kashaev volume conjecture holds for FAMED geometric triangulations, i.e. if X is FAMED and geometric, then there exists a Jones function J_X such that:

1. for any angle structures $\alpha \in \mathscr{A}_X$ and all $\hbar > 0$, we have:

$$|\mathscr{Z}_\hbar(X,\alpha)| = \left| \int_{\mathbb{R}} J_X(\hbar, x) e^{\frac{1}{2\sqrt{\hbar}} x \lambda_X(\alpha)} dx \right|,$$

   where $\lambda_X(\alpha)$ is the angular holonomies of the preferred longitude of K.

2. In the semi-classical limit $\hbar \to 0^+$, we retrieve the hyperbolic volume of K as:

$$\lim_{\hbar \to 0^+} 2\pi\hbar \log |J_X(\hbar, 0)| = -\mathrm{Vol}(\mathbb{S}^3 \setminus K).$$

Main result II: asymptotics of the partition function

- Let $\theta \in \mathbb{R}$. An ideal triangulation X is *geometric at the hyperbolic cone structure with* $\mathrm{H}(l) = i\theta$ if the system of edge and holonomy equations admits a solution of shape parameters with positive imaginary parts.

Theorem II: [Ben Aribi–W. 24']

Suppose X is FAMED and geometric at the hyperbolic structure with $\mathrm{H}(l) = i\lambda_X(\alpha)$. Then we have

$$\lim_{\hbar \to 0} 2\pi\hbar \log |\mathscr{Z}_\hbar(X, \alpha)| = -\mathrm{Vol}\left(\mathbb{S}^3 \setminus K, \mathrm{H}(l) = i\lambda_X(\alpha)\right),$$

where $\mathrm{Vol}\left(\mathbb{S}^3 \setminus K, \mathrm{H}(l) = i\lambda_X(\alpha)\right)$ is the volume of the manifold $\mathbb{S}^3$ with cone angle $\lambda_X(\alpha)$ along l.

Remarks on main results

- FAMED geometric triangulations exists for all hyperbolic twist knots.
- [Ben Aribi–Guilloux–W. 25'] FAMED geometric triangulations exist for all hyperbolic examples from snappy HTLinkExteriors census up to 23 tetrahedra ($>$ 42,000 examples).
- Numerical computations suggest that if (2) holds, then (2), (3) and (4) hold. Moreover, $\det \mathscr{A} = \pm 1$ and $\det \mathbf{B} = \pm 2$.
- [Ben Aribi–W. 24'] The asymptotics of $J_X(\hbar, x)$ with $x \neq 0$ is related to the Neumann-Zagier potential function
- [W. 25'] A generalization of FAMED condition for the case with $\det \mathscr{A} = 0$
  - Numerically checked for $> 1,000,000$ ordered ideal triangulation with no counter-example found

    (+geometricity) $\implies$ asymptotics of the partition function for such triangulations
  - With additional combinatorial conditions (not always hold, but such triangulation exists for $> 1,000,000$ knots)

    (+geometricity) $\implies$ Andersen–Kashaev volume conjecture for such knots

Andersen–Kashaev volume conjecture for FAMED geometric triangulations

Thank you for your attention!