Absence of Phase Transitions in Two-Dimensional $O(N)$ Spin Models with Large N

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Quark confinement in four-dimensional (4D) non-abelian lattice gauge thoeries and spontaneous mass generations in two-dimensional (2D) non-abelian sigma models remain as unsloved problems in physics of the last century [1]. We extend the methods of [2,3] to construct a new block spin transformation (BST) of the model which yields small non-local terms only. Though it cuurently remains [4] to control some nonlocal terms, I here announce

Provisional Results *There exists no phase transition in 2D $O(N)$ invariant Heisenberg model for all β if N is large enough.*

We sketch our proof in this note. The ν dimensional $O(N)$ spin (Heisenberg) model is determined by the Gibbs expectation value

$$< F >\equiv \frac{1}{Z_\Lambda(\beta)}\int F(\phi)e^{-H_\Lambda(\phi)}\prod_i \delta(\phi_i^2-N\beta)d\phi_i \tag{1}$$

where $\Lambda=[-(L/2)^M,(L/2)^M)^2\subset \mathbf{Z}^2$ is the large square with center at the origin, where L is a positive interger (e.g. $L=3$) and M is an arbitrarily large integer. Moreover $\phi(x)=(\phi(x)^{(1)},\cdots,\phi(x)^{(N)})$ is the vector valued spin at $x\in\Lambda$, Z_Λ is the partition function defined so that $<1>=1$, and H_Λ is the Hamiltonian given by

$$H_\Lambda \equiv -\frac{1}{2}\sum_{|x-y|=1}\phi(x)\phi(y), \tag{2}$$

$\beta(N)\equiv N\beta$ is the inverse temperature (β is scaled by N).

1. Auxiliary Field. We first release ϕ from the non-analytic constraint $|\phi|=1$, which makes the traditional BST of Wilson-Kadanov type impossible. To do this, we substitute

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$2\pi\delta(\phi^2 - N\beta) = \int \exp[-ia(x)(\phi(x)^2 - N\beta)]da(x)$ and put $\mathrm{Re}a(x) = N^{-1/2}\psi(x)$, $\mathrm{Im}a(x) = -(2 + m^2/2)$ so that

$$Z_\Lambda = \int \exp[-W_0(\phi,\psi)] \prod d\phi(x)d\psi(x), \tag{3}$$

$$W_0 = \frac{1}{2} < \phi, (-\Delta + m^2)\phi > -i < J_0, \psi >, \tag{4}$$

$$J_0(x) = -\frac{1}{\sqrt{N}} : \phi^2(x) :_{G_0} = \sqrt{N}\beta - \frac{1}{\sqrt{N}}\phi^2(x), \tag{5}$$

where $\Delta_{xy} = -4\delta_{xy} + \delta_{|x-y|,1}$ is the lattice laplacian on Z^2, $G_0(x,y) = (-\Delta + m^2)^{-1}_{xy}$ and we have chosen $m > 0$ so that $G_0(0) = \beta$, (thus $m^2 \sim 32e^{-4\pi\beta}$). Here $: A :_{G_0}$ is the Wick product of A with respect to the Gaussian probability measure $d\mu_0(\phi)$ of mean zero and covariance G_0^{-1}, and $< f, g >= \sum_x f(x)g(x)$. We note that

$$Z_\Lambda = c^{|\Lambda|} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi}, \tag{6}$$

$$\begin{aligned} F(\psi) &= \det{}^{-N/2}(1 + \frac{2iG_0}{\sqrt{N}}\psi) \exp[i\sqrt{N}\beta \sum_j \psi_j] \\ &= \det{}_3^{-N/2}(1 + \frac{2iG_0}{\sqrt{N}}\psi) \exp[-\mathrm{Tr}(G_0\psi)^2] \end{aligned} \tag{7}$$

where $\det_3(1 + A) = \det[(1 + A)e^{-A+A^2/2}]$. Thus, the system is regarded as a massive gaussiam system perturbed by $\det_3^{-N/2}(1 + \frac{2iG_0}{\sqrt{N}}\psi)$. It is easy to see [2] that if the subtracted determinant $F(\psi)$ is positive and integrable (i.e. $N \geq 3$), the correlation functions decay exponentially fast. In the present case, $F(\psi)$ is almost positive if $G_0\psi/\sqrt{N}$ is small. But the approximate positivity of $F(\psi)$ cannot be be justified for large β.

We then apply the BST to the integral and decompose the determinant into the product of the determinants each of which comes from the integration over the fluctuation field of ϕ. The fluctuation fields have short correlation lengths, then the determinants can be expandable and are approximately positive. Thus $F(\psi)$ is still approximately positive. (The main contribution of the ψ integral comes from about $< \beta^{-1/2}$.)

2. BST. To do this, we decompose $\Lambda \subset Z^2$ into blocks $\square_x$ of size $L \times L$, centered at $x \in LZ^2$, and repeat the following steps ($\phi_0 \equiv \phi$, $\psi_0 \equiv \psi$):

(1) integrate by ϕ_{n-1} keeping their block averages at ϕ_n,

(2) integrate by ψ_{n-1} keeping their block sums at ψ_n.

We represent $\phi(x) \equiv \phi_0(x)$ and $\psi(x) \equiv \psi_0(x)$ in terms of block spins $\phi_1(x) = (C\phi_0)(x)$ and $\psi_1(x) = (C'\psi_0)(x)$, and fluctuations $\xi_0(\zeta)$ of ϕ_0 and $\tilde{\psi}_0(\zeta)$ of ψ_0, where $x \in \Lambda_1$, $\Lambda_n \equiv Z^2 \cap L^{-n}\Lambda$ and $\zeta \in \Lambda - L\Lambda_1$. The operator C takes the arithmatic averages of $\phi(x)$ over the blocks and the operator C' takes sums of $\psi(x)$ over the blocks, and the both subsequently scale the coordinates by L^{-1}:

$$(C\phi)(x) = L^{-2} \sum_{\zeta \in \square} \phi(Lx + \zeta), \tag{8}$$

$$(C'\psi)(x) = L^2 (C\psi)(x) = \sum_{\zeta \in \square} \psi(Lx + \zeta) \tag{9}$$

where $x \in \Lambda_1$ and $\square$ is the box of size $L \times L$ center at the origin. These transformation rules mean that we assume that the boson fields ϕ_n (as well as ϕ_n^2) are relevant, but the auxiliary fields ψ_n (as well as $\phi_n^2(x)\psi_n(x)$) are marginal. The latter reflects the fact that the ψ field interacts almost antiferromagnetically. Conversely we have

$$\phi_n(x) = (A_{n+1}\phi_{n+1})(x) + (Q\xi_n)(x), \quad \psi_n(x) = (\tilde{A}_{n+1}\psi_{n+1})(x) + (Q\tilde{\psi}_n)(x).$$

where $Q\xi$ and $Q\tilde{\psi}$ are zero-average fluctuations ($CQ = 0$) and A_n and $\tilde{A}_n$ are transformation matrices ($CA_n = C'\tilde{A}_n = 1$) chosen to decouple the main parts of the Hamiltonian.

3. Recursion Relation. We inductly assume the form of W_n

$$\exp\left[-\frac{1}{2} < \phi_n, G_n^{-1}\phi_n > - < \psi_n H_n^{-1} \psi_n > + i < J_n, \Psi_n > - \mathcal{F}_n \right] \tag{10}$$

where

$$J_n = -\frac{1}{\sqrt{N}} : \varphi_n^2 :_{G_n^{(r)}}, \quad \mathcal{F}_n = \frac{1}{4} < J_n, f_n J_n >, \quad f_{n+1} = O((N\beta_n)^{-1}) \tag{11}$$

$$\Psi_n = \tilde{\mathcal{A}}_n \psi_n \sim \frac{1}{L^{2n}} \psi_n, \quad H_n^{-1}(x, y) \sim \delta_{x,y}, \quad x, y \in \Lambda_n, \tag{12}$$

and $\tilde{\mathcal{A}}_n \equiv \tilde{A}_0 \cdots \tilde{A}_n$ (resp. $\mathcal{A}_n \equiv A_0 \cdots A_n$)is the transformation matrices of the ψ_n fields (resp. ϕ_n fields) chosen so that the main biblinear parts are decoupled:

$$\Psi_n(x) = \Psi_{n+1} + \tilde{\mathcal{A}}_n Q \tilde{\psi}_n, \quad \Psi_0 = \psi, \quad \Psi_n = \tilde{\mathcal{A}}_n \psi_n, \tag{13}$$

$$\varphi_n(x) = \varphi_{n+1} + \tilde{\mathcal{A}}_n Q \xi_n, \quad \varphi_0 = \phi, \quad \varphi_n = \mathcal{A}_n \phi_n. \tag{14}$$

Note that $< J_n, f_n J_n >$ is the reminiscence of $\lambda(\phi^2 - \beta)^2$, $\lambda \to \infty$ (but $f_n = O((N\beta)^{-1})$.

We have to show that this form is kept (scaling) throught the BST's each of which consists of two successive integrations by the fluctuations:

$$e^{-W_{n+1}} = \int \prod d\tilde{\psi}(x) \left\{ \int e^{-W_n(\varphi_{n+1}+z, \Psi_{n+1}+\zeta)} \prod d\xi_n(x) \right\} \tag{15}$$

where $z = \mathcal{A}_n Q \xi_n$ and $\zeta = \tilde{\mathcal{A}}_n Q \tilde{\psi}_n$. This is the main part of this work. We remark that

(1) Non-analytic parts are absobed by the large field configuartions which have small probabilities to exist,

(2) the system is very close the system described by the hierachical approximation of Dyson-Wilson type. Thus fluctuations paralelle with the block spins are very small. This is reflected by the fact that f_n is always small uniformly in n.

(3) we need to develope a polymer expansion with back groud fields which have $O(N)$ symmetry. This is possible if there are no strong or large domain walls.

Since $: \varphi_n(x) :_{G_n} = \varphi_n^2(x) - NG_n(0)$, the approximate flow is represented by by $\beta_{n+1} = \beta_n - (1 - \frac{1}{N})\mathcal{T}_n(x,x)$, or equivalently by $\beta_{n+1} = \beta_n - (1 - \frac{1}{N})\tau$, $\tau = const. \log L$. The effect of H_n^{-1} is small in the recursion relations since ψ_n^2 is irrelevant. Finally (11) and (12) enable us to integrate by ψ_n. Then we have the double-well potential of the form

$$\frac{1}{N}(\varphi_n^2 - N\beta_n^{(r)})^2$$

which was the flow obtained two decades ago in the hierarchical approximation of Dyson-Wilson type. (The flow of the hierarchical model of Gallavotti type differs from this.)

References

1. A.Polyakov, Phys. Lett. **59B**, 79 (1975); K. Wilson, Phys. Rev. D **10**, 2455 (1974).

2. K. R. Ito and H.Tamura, Commun. Math. Phys., **202**, 127 (1999).

3. K. R. Ito, "Renormalization Group Recursion Formulas and Flow in 2D $O(N)$ Spin Models", Preprint (2001,March).

4. K. R. Ito, Paper in preparation.