Universal natures and rich structures in infinite-order phase transitions
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1 Introduction

The second order derivative of the free energy with respect to a environmental parameter g diverges at the critical point, when an ordinary second-order phase transition occurs. The correlation length of the system has a singularity at the critical point g_c

$$\xi \sim |g - g_c|^{-\nu}. \tag{1}$$

In the renormalization group (RG) method, the critical point is given as a fixed point of the RG. The maximal eigenvalue b_1 in the linearized RG flow near the fixed point gives the inverse of the critical exponent

$$b_1 = 1/\nu.$$

Here, we do not have to solve any recursion relation or differential equation explicitly to obtain critical exponents. one has to only diagonarize the scaling matrix at the fixed point of RG. On the other hand, in an infinite-order phase transition, the free energy has an essential singularity, and any order derivative of the free energy does not diverge. The correlation length shows strong divergence at the critical point with

$$\xi \sim \exp A|g - g_c|^{-\tilde{\nu}}.$$

In this case, a thermodynamic quantity scaled with a positive power of the correlation length does not diverge at any order derivative, such as a free energy, while that with a negative power diverges. The Kosterlitz-Thouless (KT) transition is the well-known example as an infinite-order phase transitions. This transition appears in $c = 1$ conformal field theories with a marginal perturbation. In this case, the critical exponent is $\tilde{\nu} = 1$, or $1/2$ universally. In this case, the scaling matrix at the critical point vanishes, and then the renormalization group equation becomes nonlinear differential equation. Commonly the critical exponent $\tilde{\nu}$ is obtained by integrating the differential equation of the renormalization group explicitly. In general situation, however, the renormalization group equation cannot be integrated explicitly. In this talk, I present a method of RG for RG, which enables us to extract the universal critical exponent $\tilde{\nu}$ from the nonlinear differential equation in an algebraic way [1]. It will be shown that the inverse of the critical exponent $1/\tilde{\nu}$ is given by the maximal eigenvalue of the scaling matrix in the linearized RG for RG. In section 2, I describe the method of RG for RG briefly. In section 3, I give several non-trivial examples of quantum spin systems which differs from the universality class of the KT transition.

2 Renormalization group for renormalization group

Here, I study a system with coupling constants $\mathbf{g} = (g_1, g_2, \cdots, g_n)$. The running coupling parameter $\mathbf{x}(t, \mathbf{g})$ obeys the following RG differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}) \tag{2}$$

with an initial condition $\mathbf{x}(0, \mathbf{g}) = \mathbf{g}$. The real parameter t is logarithm of a scale parameter in the RG transformation. Here I call t time. The vector field $\mathbf{V}(\mathbf{x})$ is sometimes called beta function. Let the origin be a fixed point of this RG $\mathbf{V}(\mathbf{0}) = \mathbf{0}$. The correlation length ξ in the system is considered as the scale determined by the time when the solution $\mathbf{x}$ spends near the fixed point. If the beta function is expanded in x_i at the fixed point,

$$V_i(\mathbf{x}) = \sum_j A_i^j x_j + \sum_{jk} C_i^{jk} x_j x_k + \cdots$$

the maximal eigenvalue of the scaling matrix A_i^j gives the inverse of the critical exponent $1/\nu$. This well-known fact implies that one does not have to integrate the differential equation explicitly in order to obtain the leading behavior in critical phenomena. Here, I consider the case that the first derivative of the beta function vanishes at the fixed point. This situation yields infinite-order phase transition. For example in the KT transition which is famous as an infinite-order transition, the RG equation of the KT transition is

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2^2 \\ \frac{dx_2}{dt} &= -x_1 x_2, \end{aligned} \tag{3}$$

which can be integrated explicitly. The spending time of the running coupling near the fixed point is evaluated and the characteristic length of the system ξ is obtained as a function of the initial data

$$\xi \sim \exp A|\mathbf{g} - \mathbf{g}_c|^{-1/2}.$$

Since the RG equation cannot be solved explicitly in general, I apply a RG method to the RG nonlinear differential equation. Here, we consider the RG differential equation

$$\frac{dx_i}{dt} = \sum_{jk} C_i^{jk} x_j x_k.$$

If the function $\mathbf{x}(t, \mathbf{g})$ is a solution of this equation, $e^\tau \mathbf{x}(e^\tau t, \mathbf{g})$ becomes a solution of this equation. On the basis of this scaling relation, I define a renormalization group transformation for the initial parameter. First, fix a surface S in the coupling constant space and consider the problem with an initial parameter on this surface. Let us define a transformation $R_\tau : \mathrm{S} \to \mathrm{S}$ for an arbitrary real parameter τ

$$R_\tau(\mathbf{g}) = e^\tau \mathbf{x}(s(\tau), \mathbf{g}),$$

where $s(\tau)$ is determined for a given τ in such away that the point $e^{\tau}\mathbf{x}(s(\tau),\mathbf{g})$ is on the surface S. Here, I call R_τ RG transformation for RG. I show the following properties of RG for RG.

1. A one parameter semi group property of RG for RG
$R_{\tau_2}R_{\tau_1} = R_{\tau_1+\tau_2}$.

2. A straight flow line in the original RG corresponds to a fixed point of this RG for RG.

3. The maximal eigenvalue of the scaling matrix in the RG for RG gives the inverse of the critical exponent $1/\tilde{\nu}$.

Therefore, one can obtains the critical exponent $\tilde{\nu}$ without solving the differential equation explicitly.

3 Examples

In two parameter systems, by solving the RG equation explicitly, one can check the method of RG for RG, such as a critical exponent $\tilde{\nu} = 1/2$ in the KT transition. Here, I present three other nontrivial examples of one dimensional quantum spin systems, a spin 1 bilinear-biquadratic model [3], a spin-orbital model [4] and a zigzag chain model [5], which shows infinite-order transitions different from the KT universality class. The phase diagram of each model has a rich structure. A spin 1 bilinear-biquadratic model is well-known as a system with the Haldane gap. A Bethe ansatz solvable point is a critical point, where the system is described in SU(3) Wess-Zumino-Witten (WZW) model with $c = 2$. This system shows an infinite-order transition from the Haldane gap phase to a gapless phase at this critical point. The critical exponent $\tilde{\nu} = 3/5$ is obtained both in integrating the RG equation and the RG for RG method. In a one dimensional spin-orbital model, one non-trivial critical point is a Bethe ansatz solvable point where the system is described in the SU(4) WZW model with $c = 3$. There are an extended gapless phase and dimer gap phase, where the transition between two phases is infinite-order. The critical exponent $\tilde{\nu} = 2/3$ or 1 are obtained in both ways. In the zigzag chain model, an interesting new phenomenon is discovered recently. In the Hamiltonian of the spin 1/2 zigzag chain model

$$H = \sum_i (J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2}), \tag{4}$$

there are three critical points $J_1 = -4J_2$, $J_1 = 4.149\cdots J_1$, and $J_1 = 0$. The transition at $J_1 = -4J_2$ corresponds to the ferromagnetic transition that is first order. The next one at $J_1 = 4.149\cdots J_2$ is the transition from antiferromagnetic gapless phase to the dimer gapped phase which is the KT type transition with $\tilde{\nu} = 1$. The transition at $J_1 = 0$ is a non-KT type infinite-order transition with $\tilde{\nu} = 2/3$ obtained by RG for RG method. Around this point, the system is described in $(c = 1\mathrm{CFT})^2$ with five marginal perturbations. The RG equation of this system in the one-loop approximation is

$$
\begin{aligned}
l\frac{dx_1}{dl} &= x_1^2 - x_3x_4 - x_4^2, \\
l\frac{dx_2}{dl} &= x_2^2 + x_3x_4 + x_3^2, \\
l\frac{dx_3}{dl} &= -\frac{1}{2}x_1x_3 + \frac{3}{2}x_2x_3 + x_2x_4, \\
l\frac{dx_4}{dl} &= x_1x_3 + \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_4, \\
l\frac{dx_5}{dl} &= \frac{1}{2}x_3x_4,
\end{aligned} \tag{5}
$$

where the initial values of this equation are given in certain functions of J_1 and J_2. The RG equation indicates the instability of the critical point $J_1 = 0$ for the perturbation $J_1 \neq 0$. Indeed in the antiferromagnetic region $J_1 > 0$, the system is dimerized where the translational symmetry of this model is broken. In a field theory description, the corresponding chiral symmetry breaking occurs. The numerical calculation shows the finite correlation length, dimerization order parameter and the energy gap [2]. The gap scaling formula eq.(1) with $\tilde{\nu} = 2/3$ fits the data surprisingly well even for relatively large J_1. In the ferromagnetic region $J_1 < 0$, however, the gap has never been observed in numerical calculation. This fact is puzzling because the ferromagnetic perturbation seems to yield the same instability as in the antiferromagnetic one. Now, I understand this puzzle as follows [5]. This RG has a fixed line

$$
x_1 = x_2 = 0, \; x_3 + x_4 = 0. \tag{6}
$$

and the eigenvalues of the scaling matrix on this fixed line all vanish. Studying the flow near this fixed line, all perturbations is found to be marginally relevant. The flow becomes quite slow near this fixed line, however, finally the flow runs away from the fixed line. Since the running coupling $\mathbf{x}(t)$ spends long time near the fixed line, the characteristic length scale of the system becomes always an astronomical length scale. Therefore, the correlation length is finite but quite long in an extended region. At the same time the energy gap is finite, but very tiny without fine-tuning of the coupling J_1. The scaling formula eq.(1) of the correlation length holds only for small $|J_1|$. This spin model is a rare example of a strong scale reduction without fine-tuning of the coupling constant.

References

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