Transition of Ground State of Boson-Fermion Model and Renormalizable Field Theory

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1 Introduction

Recently, mathematical techniques for non-perturbative way to analyze models in quantum electrodynamics (QED) are developing gradually. In the development, we face some cases so that we cannot analyze the ground state energy of models in QED by the regular perturbation theory [LL, Hi, HS]. Especially, Lieb and Loss showed in [LL, Theorem 1.1] the curious results on the upper and lower estimates of the ground state energy of so-called Pauli-Fierz model describing electrons interacting with the radiation field. Their result means that the renormalized mass of the model cannot be calculated by the perturbation theory not only in the case of large coupling length but also small one. They showed that the order in the coupling length is less than the order of the square derived by the regular perturbation theory. Its physical reason has not clarified yet to author's best knowledge. Moreover, Griesemer, Lieb, and Loss showed in [GLL] that the Pauli-Fierz model has a ground state for all values of the coupling length. Thus, their results mean that the Pauli-Fierz model has a non-perturbative ground state for the all coupling length. Considering the history of physics, we should have succeeded in the mass renormalization for non-relativistic treatment by Pauli and Fierz. What on physics may have happened to the Pauli-Fierz model? We are much interested in the physical reason for the existence of such a ground state refusing the perturbation theory, and also we are interested in the influence on the renormalizable field theory.

On the other hand, for the Weisskopf-Wigner (WW) model (i.e., the Dicke model in the rotating wave approximation), we know that a non-perturbative ground state appears in the case with the large coupling length [Hi], and the ground state energy is so low that the regular perturbation theory cannot give it. Here WW model describes a two level system coupled with a Bose field, and it was actively argued as a simple version of the Lee model [Le], the Dicke model for superradiance [Di], a simple model of spin-photon model of quantum optics and NMR, and the model describing the elementary process of the decay from neutron to proton and π^--meson. And also it is to argue the spontaneous emission in the Weisskopf-Wigner theory [WW]. For the emission and absorption of photons between the two-level system, we face the difficulty of the resonance scattering in the regular perturbation theory. So, the Weisskopf-Wigner theory is for the higher order revision for the regular perturbation theory, and WW model has the effect of this revision. Then, the ground state energy with the order in the coupling length is less than the order of square coming from the regular perturbation theory. For WW model this order less than the order of square is available even for the sufficiently small coupling length i.e., in the region of the perturbation theory. But in [Hi, Lemma 2.2] we knew that growing the coupling length restore the same order as the regular perturbation theory.

As mentioned above the WW model is a simplification of the Lee model, and moreover the Lee model can be decomposed into a direct sum of the Hamiltonian H_1 equivalent to WW model and a free Hamiltonian H_2 (see (2.4) below), so we can expect that Lee model has the similar non-perturbative ground state in the case with the large coupling length. Moreover, it is well known that, for the Lee model, renormalizations with perturbative way and non-perturbative way imply the same result. Thus, in this paper, we reconsider the Lee model in the light of the renormalizable field theory in the early stage and quantum optics for the case including the large coupling length. More precisely, we return to the early stage of the renormalizable field theory developed by Lee, Källén, and Pauli, and we show the limit of the successful result of renormalizations with perturbative and non-perturbative ways. We show also the existence of a non-perturbative ground state when we are beyond perturbation theory. As for the ground state energy of the Lee model, as well as WW model, the ground state energy for the small coupling length has the order less than that of square because of the higher order revision for the regular perturbation theory following the Weisskopf-Wigner theory. But the non-perturbative ground state energy recovers the order of the square, which is the same order as that by regular perturbation theory, in the coupling length when the Lee model is outside the region of the regular perturbation theory. We investigate the behavior of the ground state energy with the Jaynes-Cummings model [Mi, §6.4] in quantum optics.

As to such a low energy of the non-perturbative ground state beyond the regular perturbation theory, a similar non-perturbative ground state is shown in physics by Preparata [Pr90, Pr95] and Enz [En]. It is called *superradiant* ground state from the point of view of superradiance of soft photons. The superradiance was, of course, found by Dicke in [Di]. Its existence is proved with the path-integral method by Preparata, and with another manner by Enz. But it has not yet been clarified whether the ground state showed in [Hi] is superradiant or not. By the way, in [Hi] we had adhered to the coupling length. But, following the recent result [Bi01] by Billionnet, we should consider the condition of physical parameter $\mathsf{B}_{g,\mu}$ which represents a relation of the coupling length and an infrared or ultraviolet singularity condition. We apply the same method as [Hi] to a special Lee Model and prove there also exists the similar non-perturbative ground state being still in the standard state space. Thus, we show that the non-perturbative ground state is stable, and moreover the ground state energy is also lower than the normal renormalized mass showed in [Le] by Lee.

In Lee's renormalization argument, there is the possibility that such a ground state becomes a ghost. Actually, Lee noted briefly in [Le, footnote 4] the existence of another state from the state with the normal renormalized mass. And moreover, in the process of developing the renormalizable field theory, Källén and Pauli investigated precisely in [KP] the existence of another state than the normal state, and they showed concrete form of the state and it has lower energy than the normal renormalized mass. But we cannot understand their extra ground state in the standard Hilbert space theory because it has negative 'norm' coming from complex renormalization constant. We are interested in the relation among the states which we show in this paper, Preparata found, and Lee, Källén and Pauli found.

For a while, let us review Lee's renormalization argument [Le], Källén and Pauli's [KP] renormalization argument, and the Weisskopf-Wigner model [Hi].

1.1 Lee's Renormalization Argument

The Lee model describes the interacting system between two neutral fermion fields V and N and a neutral scalar boson field θ. In this paper, we use the natural units, $\hbar = c = 1$. Let $\psi_{V}, \psi_{V}^{\dagger}$ and $\psi_{N}, \psi_{N}^{\dagger}$ be annihilation and creation operators of V-particle and N-particle, respectively, and let $\alpha_{\theta}, \alpha_{\theta}^{\dagger}$ be annihilation and creation operators of θ-particle, respectively. Then, the Hamiltonian of the Lee model is given by

$$H := H_0 + gH_I, \tag{1.1}$$

$$H_0 := m_V \int_{\mathsf{R}^d} d^dp\ \psi_V^{\dagger}(p)\psi_V(p) + m_N \int_{\mathsf{R}^d} d^dp\ \psi_N^{\dagger}(p)\psi_N(p) + \int_{\mathsf{R}^d} d^dk\ \omega(k)\alpha_\theta^{\dagger}(k)\alpha_\theta(k) \tag{1.2}$$

$$H_I := \int_{\mathsf{R}^d\times\mathsf{R}^d} d^dp d^dk\ \frac{\rho(\omega(k))}{\sqrt{2\omega(k)}}\left(\psi_V^{\dagger}(p)\psi_N(p-k)\alpha_\theta(k) + \psi_V(p)\psi_N^{\dagger}(p-k)\alpha_\theta^{\dagger}(k)\right), \tag{1.3}$$

where $\omega(k)$ gives the dispersion relation defined by $\omega(k) := \sqrt{k^2+\mu^2}$ ($\mu \geq 0$), and $m_V > 0$, $m_N > 0$, and $\mu \geq 0$ are bare masses of V-particle, N-particle, and θ-particle, respectively. And g is the bare coupling constant, ρ is introduced as a cutoff function of energy. We note that the following: although V-particle and N-particle have momenta, they do not have kinetic energies. Thus, we here understand that the masses m_V and m_N are so heavy that we can ignore the kinetic energies.

The interaction Hamiltonian H_I represents the reaction

$$V \ \square\ N + \theta. \tag{1.4}$$

Namely, a V-particle emits a θ-particle, and changes into an N-particle. On the other side, an N-particle absorbs a θ-particle, and changes into a V-particle. Moreover, m_V has a renormalization because of the process of $V \to N + \theta \to V$, and the process of $N + \theta \to V \to N + \theta$ means the scattering of θ-particle by N-particle. Thus, the physical system described by H possesses two conservation laws:

$$N_V + N_N = \text{constant}, \tag{1.5}$$
$$N_V + N_\theta = \text{constant}, \tag{1.6}$$

where N_V, N_N, and N_θ are the total number of V-particles, N-particles, and θ-particles, respectively. Because of these conservation laws (1.5) and (1.6), the eigenstate of H contains only a finite number of particles. So, the eigenstate can be solved directly, and Lee performed that in [Le].

Let $|V(p)$ and $|N(p)$ be the state of the bare V-particle and N-particle, respectively. We denote the state of the corresponding physical particles by $|\mathbf{V}(p)$ and $|\mathbf{N}(p)$. Then, by (1.4), (1.5), and (1.6), we have

$$|\mathbf{N}(p) = |N(p) , \tag{1.7}$$
$$|\mathbf{V}(p) := Z_V^{1/2}\left\{|V(p) + g\int_{\mathsf{R}^d} d^dk\ f(k)\,\alpha_\theta^\dagger(k)|N(p-k)\right\}, \tag{1.8}$$

where $Z_V^{1/2}$ is a normalization constant, and the function, $f(k)$, is determined latter for an ultraviolet cutoff.

We now follow the theory of renormalization by the power series method [Dy, Sa, Wa] in the perturbative way. We denote the renormalized mass of V-particle, renormalized constant of wave function, and renormalized coupling constant by m_c, Z_2, and g_c. Then, as Lee proved in [Le], the self-energy is given by

$$\Sigma(p_0) = g^2\int_{\mathsf{R}^d} d^dk\ \frac{|\rho(\omega(k))|^2}{2\omega(k)}\ \frac{1}{(p_0 - m_N - \omega(k))}, \tag{1.9}$$

where p_0 is $-i$ times the fourth component p_4 of the momentum vector p, i.e., $p_0 = -ip_4$. So, the renormalized constant Z_2 is given by $Z_2^{-1} \equiv Z_2^{-1}(m_c) = d\Sigma(p_0)/dp_0$ at $p_0 = m_c$ on the mass shell $p^2 + m_c^2 = 0$. Namely,

$$Z_2^{-1} \equiv Z_2^{-1}(m_c) = 1 + g^2\int_{\mathsf{R}^d} d^dk\ \frac{|\rho(\omega(k))|^2}{2\omega(k)}\ \frac{1}{(m_c - m_N - \omega(k))^2}. \tag{1.10}$$

Following the way by [Sa], we put

$$g_c^2 := Z_2 g^2. \tag{1.11}$$

Then, by (1.10) we have

$$g^2 = g_c^2\left\{1 - g_c^2\int_{\mathsf{R}^d} d^dk\ \frac{(|\rho(\omega(k))|^2}{2\omega(k)}\ \frac{1}{(m_c - m_N - \omega(k))^2}\right\}^{-1}, \tag{1.12}$$

$$Z_2 = 1 - g_c^2\int_{\mathsf{R}^d} d^dk\ \frac{|\rho(\omega(k))|^2}{2\omega(k)}\ \frac{1}{(m_c - m_N - \omega(k))^2} \tag{1.13}$$

as shown in [Le, (26), (27)]. Thus, g and Z_2 are dependent of m_c, g_c, and ρ, i.e., $g = g(m_c, g_c, \rho)$, $Z_2 = Z_2(m_c, g_c, \rho)$. Then, following the primal policy of renormalization, we insert observed values into m_c and g_c respectively, and Z_2 has to be finite as $\rho \to 1$ for the fixed m_c and g_c. Then, when we regard m_c and g_c as independent variables, we can define a function, $Z_2^{ren}(m_c, g_c, \rho)$, from Z_2, i.e.,

$Z_2^{ren}(m_c, g_c, \rho)$ is defined by (1.13) for independent variables, m_c, g_c $\mathbb{R}$ (1.14)

$\nmid\ \leq,\ \ (k))$

Since the physical meaning of Z_2^{ren} is the probability of existence of a state, we have to avoid a ghost ($Z_2^{ren} < 0$). Thus, we cannot take such a limit freely, and we have to keep (m_c, g_c, ρ) really so that Z_2^{ren} can be between 0 and 1 (see the conclusion of [KP]). This is one of Lee's statements in [Le, KP] as to the non-unitary-equivalence between the bare particle states and physical particle states. Set

$$g_{crit} := \left\{\int_{\mathsf{R}^d} d^dk\ \frac{|\rho(\omega(k))|^2}{2\omega(k}\right.$$

Then,

$$Z_2^{ren} = 1 - \frac{g_c^2}{g_{crit}^2}. \tag{1.16}$$

And the renormalized coupling constant, g_c, has to satisfy $|g_c| \leq g_{crit}$ to Z_2^{ren} lies between zero and one.

On the other hand, we have

$$g_c = g \left\{ 1 + g^2 \int_{\mathsf{R}^d} d^d k \ \frac{|\rho\left(\omega\left(k\right)\right)|^2}{2\omega\left(k\right)} \ \frac{1}{\left(m_c - m_{_N} - \omega\left(k\right)\right)^2} \right\}^{-1}. \tag{1.17}$$

So, when we regard g and m_c as independent variables, we can define a function, g_c^{ren}, from g_c, i.e.,

$$g_c^{ren}(m_c, g, \rho) \text{ is defined by (1.17) for independent variables } m_c, g \in \mathbb{R}. \tag{1.18}$$

So, it is important to check the normal zone, $\mathsf{G}_{m_c,\rho}$, which is given by the range of the function, $g_c(g) = g_c(m_c, g, \rho)$, of $g \in \mathbb{R}$ for fixed m_c and ρ arbitrarily, i.e., $\mathsf{G}_{m_c,\rho} := \{g_c(m_c, g, \rho) \,|\, -\infty < g < \infty\}$. Because, if the observed coupling constant, g_{obs}, is more than $g^{\uparrow}(g, \rho) := \sup_g g_c(m_c, g, \rho)$ (i.e., $g_{obs} > g^{\uparrow}(g, \rho)$), we cannot take g_c as $g_c = g_{obs}$. For the fixed m_c, by (1.17)

$$|g_c| \leq \left\{ 4 \int_{\mathsf{R}^d} d^d k \ \frac{|\rho\left(\omega\left(k\right)\right)|^2}{2\omega\left(k\right)} \ \frac{1}{\left(m_c - m_{_N} - \omega\left(k\right)\right)^2} \right\}^{-1/2}, \tag{1.19}$$

$$g_c \to 0 \ \text{ as } |g| \to \infty. \tag{1.20}$$

Thus, $|g_c| < g_{crit}$ now.

On the other hand, for the Lee model we can determine m_c independently of the perturbative way. As we did in [AH00, §6.2] and [Hi, (2.11)] we introduce a function, $D(z;\alpha)$, of z by

$$D(z;\alpha) := -z + m_{_V} - \alpha^2 \int_{\mathsf{R}^d} d^d k \ \frac{|\rho\left(\omega\left(k\right)\right)|^2}{2\omega\left(k\right)} \ \frac{1}{\omega(k) + m_{_N} - z} \tag{1.21}$$

defined for all $z \in \mathbb{C}$ and every $\alpha \in \mathbb{R}$ such that $|\rho\left(\omega\left(k\right)\right)|^2/\omega\left(k\right)|z - m_{_N} - \omega(k)|$ is Lebesgue integrable on $\mathbb{R}^d$. In the same way as [AH00, §6.2], $D(z;\alpha)$ is defined in the cut plane $\mathbb{C}_{m_{_N},\mu} := \mathbb{C} \setminus [m_{_N} + \mu, \infty)$, $\mu \geq 0$, and analytic there. It is easy to see that $D(x;\alpha)$ is monotone decreasing in $x < m_{_N} + \mu$. Hence, the limit $d_\mu(\alpha) := \lim_{x \uparrow m_{_N} + \mu} D(x;\alpha)$ exists, and

$$d_\mu(\alpha) = -\mu - m_{_N} + m_{_V} - \alpha^2 \lim_{t \downarrow 0} \int_{\mathsf{R}^d} d^d k \ \frac{|\rho\left(\omega\left(k\right)\right)|^2}{2\omega\left(k\right)} \ \frac{1}{\omega(k) - m_{_N} - \mu + t}. \tag{1.22}$$

In the case of $d_\mu(g) < 0$, $D(z;g) = 0$ has a solution, $z = m_{V_c}$. Thus, by this solution, m_{V_c}, and (1.8), we know that

$$|\mathbf{V}(p)\rangle = Z_{V_c}^{1/2} \left\{ |V(p)\rangle + g \int_{\mathsf{R}^d} d^d k \ \frac{\rho\left(\omega\left(k\right)\right)}{\sqrt{2\omega\left(k\right)}} \ \frac{1}{m_{V_c} - m_{_N} - \omega\left(k\right)} \theta^{\dagger}(k) |N(p-k)\rangle \right\} \tag{1.23}$$

is an eigenstate of H with $H|\mathbf{V}(p)\rangle = m_{V_c}|\mathbf{V}(p)\rangle$, where we took $Z_{_V}$ as $Z_{_V} = Z_{V_c} \equiv Z_2(m_{V_c})$. Therefore, a candidate for m_c is m_{V_c}, i.e., $m_c = m_{V_c}$. Moreover,

$$m_{V_c} < m_{_N} + \mu \tag{1.24}$$

for every $|g|$ satisfying $D(0;g) < 0$.

By the way, using the fact that scattering state satisfies the Lippmann-Schwinger equation, it is known that the scattering amplitude is given by

$$\begin{aligned} & g_{V_c}^2 \frac{\rho\left(\omega\left(k\right)\right)}{\sqrt{2\omega\left(k\right)}} \frac{\rho\left(\omega\left(k'\right)\right)}{\sqrt{2\omega\left(k'\right)}} \ \delta(p + k - p' - k') \frac{1}{m_{_N} + \omega\left(k\right) - m_{V_c}} \\ & \quad \times \left\{ 1 - g_{V_c}^2 \int_{\mathsf{R}^d} d^d k' \ \frac{|\rho\left(\omega\left(k'\right)\right)|^2}{2\omega\left(k'\right)} \ \frac{m_{_N} + \omega\left(k\right) - m_{V_c}}{\left(\omega\left(k\right) - \omega\left(k'\right) + i\epsilon\right)\left(m_{V_c} - m_{_N} - \omega\left(k'\right)\right)^2} \right\}^{-1} \end{aligned} \tag{1.25}$$

(e.g., see [Ta, (53)]), where p and k denote the momenta of scattering state of N-particle and θ-particle respectively, p' and k' are those of N-particle and θ-particle coming into a detector respectively, and moreover $g_{V_c}^2 := Z_{V_c} g^2$, $i\epsilon$ $(\epsilon > 0)$ comes from the adiabatic factor in the Lippmann-Schwinger equation, and $i\epsilon$ means the outgoing plane wave. Thus, since differential cross-section is given by the square of the absolute value of the scattering amplitude, (1.24) means that

$$V\text{-particle is stable for every } g \text{ with } D(0; g) < 0, \text{ i.e.,} \tag{1.26}$$

V-particles do not decay into N-particles and θ-particles spontaneously beyond (1.24) because (1.5) holds and the resonance scattering hardly occurs since $\omega(k') \geq \mu > m_{V_c} - m_N$, which comes from all higher order revisions

$$N + \theta \to V \to N + \theta \to V \to N + \theta \to \cdots$$

for the regular perturbation theory following the Weisskopf-Wigner theory. On the other hand, even if $m_N < m_V$ first, we have (1.24) as long as the coupling constant g satisfies $D(0; g) < 0$. Thus, in the process from $m_N < m_V$ to (1.24),

$$N\text{-particle is unstable for every } g \text{ satisfying } D(0; g) < 0, \text{ i.e.,} \tag{1.27}$$

N-particles decay into V-particles by absorbing θ-particles.

1.2 Källén and Pauli's Renormalization Argument

In this subsection, we review Källén and Pauli's renormalization argument in [KP] in terms of our situation.

We set

$$m_V = m_N = m > 0, \tag{1.28}$$
$$\delta m := m_{V_c} - m. \tag{1.29}$$

Then, by (1.11), (1.13), and (1.22) we know that $z = m_{V_c}$ is a solution of

$$D(z; g) = -z + m - \delta m - \frac{g_c^2}{2Z_2} \int_{\mathbb{R}^d} d^d k \; \frac{|\rho(\omega(k))|^2}{\omega(k)} \; \frac{1}{\omega(k) - \left(z - m_{V_c} + \delta m\right)} = 0. \tag{1.30}$$

Källén and Pauli derived

$$\begin{aligned} & h(z - m_{V_c}) \\ := \; & (z - m_{V_c}) \left[1 + \frac{g_c^2}{2} \int_{\mathbb{R}^d} d^d k \; \frac{|\rho(\omega(k))|^2}{\omega(k)} \frac{z - m_{V_c}}{(\omega(k) - \delta m)^2 \left(\omega(k) - \delta m - \left(z - m_{V_c}\right)\right)} \right] \\ = \; & 0. \end{aligned} \tag{1.31}$$

from (1.30) by using (1.29) and (1.16) in [KP]. Of course, $z = m_{V_c}$ is a solution of (1.31), but Källén and Pauli found that there exists another solution $z = \lambda_{KP}$, living on the real axis with $\lambda_{KP} < m_{V_c}$ [KP, §II and Appendix I]. And, they gave the concrete form of the state with energy λ_{KP} as

$$\begin{aligned} |\mathbf{V}_{KP}(p)\rangle = \frac{1}{\sqrt{|h'(\lambda_{KP})|}} \Bigg\{ & Z_{V_c}^{ren\;1/2} |V(p)\rangle + g_c \int_{\mathbb{R}^d} d^d k \; \frac{\rho(\omega(k))}{\sqrt{2\omega(k)}} \\ & \times \frac{1}{\omega(k) - \lambda_{KP} - \delta m} \theta^\dagger(k) |N(p-k)\rangle \Bigg\}, \end{aligned}$$

where $|\mathbf{V}_{KP}(p)\rangle$ has not yet been normalized. Then, the normalization becomes negative because $Z_{V_c}^{ren}$ makes a ghost (i.e., $Z_{V_c}^{ren\;2} < 0$) for so large coupling as to exist the solution. Here we regarded Z_{V_c} as the function $Z_{V_c}^{ren}$ of independent variables m_{V_c}, g_c running over $\mathbb{R}$ respectively, and ρ in the sense of (1.14). So, such a mathematically strange situation occurs. In order to cope with this trouble, they introduced an indefinite metric in the Hilbert space [KP, §III].

1.3 Wigner-Weisskopf Model

In this subsection, we review and modify the results on the Weisskopf-Wigner model in [Hi] to apply them to physics of π-meson.

Nuclear force is the first example with the strong interaction between elementary particles. The coupling length of the interaction between baryon and meson is in the region from 0.1 to 10, and it is very large as compared with 1/137, that of quantum electrodynamics. As is well known, nuclear force connects nucleus and nucleon. Nucleon is a generic name of proton (p) and neutron (n), and is constructed by u-quark and d-quark. The particle taking a job of nuclear force is π-meson. Physics for π-meson was investigated actively in 1940s and 1950s (see [HT]). On the other hand, as mentioned in introduction, mathematics for the non-perturbative treatment of models with large coupling length which physicists once argued such as π-meson is recently and gradually established. In this subsection, by applying mathematical techniques developed recently to the theory of π-meson, we argue rigorously existence and nonexistence of state in the elementary process of $n \ \square \ p+\pi^-$ for each total charge Q and all coupling length g. Here 'state' means that eigenvector of the Hamiltonian for our model is still alive in the Hilbert space representing the statespace.

The model with the interaction between π-meson and nucleon considering all elementary processes, $p \ \square \ p+\pi^0$, $p \ \square \ n+\pi^\dagger$, and $n \ \square \ p+\pi^-$, is described by the following Hamiltonian (see [HT]):

$$
\begin{aligned}
H &= H_0 + H', && (1.32)\\
H_0 &= \sum_{\ell,m,\alpha} \int_0^\infty d^3k\, \omega(k) a_\ell^{\dagger\,\alpha m}(k) a_\ell^{\alpha m}(k), && (1.33)\\
H' &= \frac{f}{\mu} \sum_{m\alpha} \int_0^\infty \frac{d^3k \ k^2}{(12\pi^2\omega(k))^{1/2}} \lambda(k) \tau_\alpha \sigma_m \left\{ a_1^{\alpha m}(k) + a_\ell^{\dagger\,\alpha m}(k) \right\}, && (1.34)
\end{aligned}
$$

where τ_α and σ_m are the standard τ-matrices and Pauli's σ-matrices.

We now assume $\omega(k) = \sqrt{k^2+m^2}$, where m is the mass of π-meson. We set

$$
H_{\alpha m} = \int_0^\infty d^3k\, \omega(k) a_\ell^{\dagger\,\alpha m}(k) a_\ell^{\alpha m}(k) + \frac{f}{\mu} \int_0^\infty \frac{d^3k\, k^2}{(12\pi^2\omega(k))^{1/2}} \lambda(k) \tau_\alpha \sigma_m \left\{ a_1^{\alpha m}(k) + a_\ell^{\dagger\,\alpha m}(k) \right\}.
$$

Then, regarding $\mathcal{H} \quad \mathcal{F}_\pi$, A and B_j in [AH97, (1.6)] as $\mathbb{C}^2 \quad \mathbb{C}^2 \quad \mathcal{F}_\pi$, 0 and $\tau_\alpha\sigma_m$, respectively, where $\mathcal{F}_\pi$ is the boson Fock space representing the state space for π-meson, we know that $H_{\alpha m}$ is an example of the generalized spin-boson model we called in [AH97]. By [AH97, Theorem 1.2 and Remark 1.2], $H_{\alpha m}$ has a ground state, which implied that *if $\lambda(k)$ is continuous, and $\int_{\mathbb{R}^3} d^3k \ k^4\lambda(k)^2 < \infty$, then there is a ground state for H.*

Unfortunately, the only thing we can say for H now is the above assertion with the estimates of the ground state energy in [AH97, Proposition 1.4], and we do not have physical properties for H. In order to argue physical properties for π-meson more precisely in this paper, we treat the elementary process $n \ \square \ p+\pi^-$ without $p \ \square \ p+\pi^0$ and $p \ \square \ n+\pi^+$ from now on.

We express nucleon coupling π-meson by $|p)$ and $|n)$ as $|p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Omega_\pi$ and $|n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Omega_\pi$ be the bare state of proton and neutron respectively, where Ω_π is the vacuum of π-meson. Set

$$
\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.35}
$$

Thus, the operation of $\tau_\pm$ and τ_3 acting on nucleon are

$$
\begin{aligned}
\tau_-|p) &= |n) \qquad & \tau_-|n) &= 0,\\
\tau_+|n) &= |p) \qquad & \tau_-|p) &= 0,\\
\tau_3|p) &= |p) \qquad & \tau_3|n) &= -|n).
\end{aligned} \tag{1.36}
$$

Then, following [HT], we employ the interaction which occurs $n \ \square \ p+\pi^-$ with the form,

$$
(\tau_+)\cdot(\text{creation operator of } \pi^-\text{-meson}) + (\tau_-)\cdot(\text{annihilation operator of } \pi^-\text{-meson}).
$$

So, the Hamiltonian describing the process, $n \rightarrow p + \pi^-$, is given by

$$
\begin{aligned}
H_{\pi^-} &= H_{\pi^-,0} + H'_{\pi^-}, && (1.37)\\
H_{\pi^-,0} &= \frac{1-\tau_3}{2}E_0 + \int_{\mathbb{R}^3} d^3k\ \omega(k)a^\dagger(k)a(k) && (1.38)\\
H'_{\pi^-} &= g\int_{\mathbb{R}^3} d^3k\ \lambda(k)\left\{\tau_- a(k) + \tau_+ a^\dagger(k)\right\}, && (1.39)
\end{aligned}
$$

where $a(k)$ and $a^\dagger(k)$ are the annihilation and creation operators respectively with

$$
\begin{aligned}
\left[a(k)\,,\,a^\dagger(k')\right] &= \delta(k-k'), && (1.40)\\
\left[a(k)\,,\,a(k')\right] &= \left[a^\dagger(k)\,,\,a^\dagger(k')\right] = 0, && (1.41)
\end{aligned}
$$

and E_0 denotes the difference of masses between proton and neutron. The dispersion relation between the energy and momentum is denoted by $\omega(k)$. In the case of non-relativistic energy, $\omega(k)$ is given as $\omega(k) = m + \frac{k^2}{m}$ as in [HT]. But this dispersion relation does not make a sense for so high energy. To consider the case of higher energy, we set $\omega(k) = \sqrt{k^2+m^2}$ in the case of relativistic energy as in [Le]. For a ultraviolet cutoff function $\rho(k)$, Henley and Thirring set $\lambda(k) = \rho(k)$ in [HT] because they treated boson as non-relativistic. On the other hand, Lee set $\lambda(k) = \rho(\omega(k))/\sqrt{2\omega(k)}$ in [Le] because he treated boson as relativistic. We assume that $\lambda(k)$ is positive, continuous, spherically symmetric and $\int_{\mathbb{R}^3} d^3k\ \lambda(k)^2 < \infty$. Then, the state space for H_{π^-} as a Hilbert space $\mathcal{F}$ is given by $\mathcal{F} = \mathbb{C}^2 \otimes \mathcal{F}_\pi$. And, we get

$$
H_{\pi^-,0}\,|p) = 0 \qquad \text{and} \qquad H_{\pi^-,0}\,|n) = E_0|n). \tag{1.42}
$$

Moreover, the physical state $|\mathbf{p}\rangle$ of proton is same as its bare state, i.e., $|\mathbf{p}\rangle = |p)$, so we have

$$
H_{\pi^-}\,|\mathbf{p}\rangle = 0. \tag{1.43}
$$

The point eigenvalues $\sigma_p(H_{\pi^-,0})$ of $H_{\pi^-,0}$ are 0 and E_0, i.e., $\sigma_p(H_{\pi^-,0}) = \{0, E_0\}$.

We note here that H_{π^-} is the same Hamiltonian as in [Hi, (2.10)]. Thus, for the Pauli matrix $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have that $\sigma_1 H_{\pi^-}\sigma_1$ is the *Wigner-Weisskopf Hamiltonian* called in [Hi, (2.4)] or the *spin-boson Hamiltonian with the rotating wave approximation* called in [HS, §6].

The total charge Q is given by

$$
Q = \frac{1}{2}\tau_3 - \int_{\mathbb{R}^3} d^3k\ a^\dagger(k)a(k). \tag{1.44}
$$

Then, H_{π^-} conserves the total charge, $[H_{\pi^-}\,,\,Q] = 0$. Therefore, H_{π^-} can be written as the direct sum of $H_{Q=-(2\nu-1)/2}$'s ($\nu = 0, 1, 2, \cdots$), where $H_{Q=-(2\nu-1)/2}$ is the restricted H_{π^-} on the space of all states with $Q = -(2\nu-1)/2$. We note here that $\frac{1}{2} - U_1^* Q U_1$ is written by N_P in [HS, (6.2)] and [Hi, (2.17)] as the total number operator.

It is easy to check the states with $Q = \pm\frac{1}{2}$, but it is well-known that it is difficult to show that existence of states with $Q = -\frac{N}{2}$ for large odd number N as Henley and Thirring wrote in their text book [HT]. In this subsection, we prove that the existence of a state with $Q = -\frac{2(\nu-1)}{2}$ for $\mathbb{N} \ni \nu \geq 2$, and argue when the state appears. We know that the appearance is not standard one.

Here we introduce a physical parameter $\mathcal{B}_{g,m}$ consisting of the coupling length and the self-energy of boson part as follows:

$$
\mathcal{B}_{g,m} := \int_{\mathbb{R}^3} d^3k\ \mathcal{B}_{g,m}(k), \qquad \mathcal{B}_{g,m}(k) := g^2\,\frac{\lambda^2(k)}{\omega(k)}, \tag{1.45}
$$

which is a modification of the parameter introduced in [Bi01] by Billionnet. The importance of $\lambda^2(k)/\omega(k)$ was also pointed out in [AE, IV.A]. And, in the same way as (1.21), we introduce a function $D(z;g)$ of z by

$$
D(z;g) := -z + \mu_0 - g^2\int_{\mathbb{R}^3} d^3k\ \frac{|\lambda(k)|^2}{\omega(k)-z}. \tag{1.46}
$$

And also, we obtain a parameter as the limit $d_m(g) := \lim_{x\uparrow m} D(x;g)$ because it exists. For fixed mass m, the parameter $d_m(g)$ becomes negative, $d_m(g) < 0$, when $|g|$ grows.

We treat now the case of $m > 0$, then the existence of the ground state comes from [AH97, Theorem 1.2]. In the case of $m = 0$, the existence is due to Gérard's work [Gé], which is explained in Section 2. We denote the ground state and ground state energy by $|\Psi_{grd}$ and E_{grd}, respectively:

$$H_{\pi^-}|\Psi_{grd} \ = E_{grd}|\Psi_{grd} \ . \tag{1.47}$$

To restrain a state from appearing for $N_P \geq 2$, we define a differential operator D_{HS} by

$$D_{HS} := \frac{1}{2}\left(\frac{1}{| \ \ {}_k\omega|^2} \ \ {}_k\omega \cdot \ \ {}_k + \ \ {}_k \cdot \ \ {}_k\omega\frac{1}{| \ \ {}_k\omega|^2}\right). \tag{1.48}$$

The operator D_{HS} was introduced by Hübner and Spohn in [HS, (2.9)] to apply the Mourre estimate, and it is called conjugate operator by mathematicians.

Our $\omega(k)$ and $\lambda(k)$ satisfy the assumptions [HS, (A.1) & (A.2)]. If $D(z;g) = 0$ has a solution, then we can make a state with $Q = -\frac{1}{2}$. But, under $d_m(g) \geq 0$, since it menas [HS, (6.3)], we cannot make any state with $Q = -\frac{1}{2}$ as Hübner and Spohn mentioned after [HS, (6.3)]. Thus, with the result in [HS, Proposition 15], we obtain the following:

$[Q = -\frac{1}{2}]$ *Suppose that*

$$g^2\int_{\mathsf{R}^3} d^3k \ |D_{HS}\lambda(k)|^2 < 1 \quad \textit{and} \quad \int_{\mathsf{R}^3} d^3k \ |D_{HS}^2\lambda(k)|^2 < \infty. \tag{1.49}$$

Then, state with $Q = \frac{1}{2}$ *exists for all* g *with* $d_m(g) \geq 0$*, and it is* $|p)$ *of which energy is* 0*. There is no state with* $Q = -\frac{2\nu-1}{2}$ *for* $\nu \quad \mathbb{N}$*. Moreover, the essential spectra of* H_{π^-} *is given as* $\sigma_{ess}(H_{\pi^-}) = [m\,,\,\infty)$*.*
Here 'essential spectra' means all continuous energy levels and point energies of infinitely degenerated eigenstates. All the results about essential spectra in this paper are due to [Ar00]. In the case of $m = 0$, we can use Skibsted's results instead of Hübner and Spohn's, which is explained in Section 2.

But, if $d_m(g) < 0$, then the condition [HS, (6.3)] breaks. Namely, $D(z;g) = 0$ has a real solution, $z = E_c$. So we can make an eigenvector with E_c as its eigenvalue. Namely, the physical state of neutron $|\mathbf{n}$ is given by

$$|\mathbf{n} \ = Z_c^{1/2}\left\{\tau_- + g\int_{\mathsf{R}^3} d^3k \ \frac{\lambda\,(k)}{E_c - \omega\,(k)}a^\dagger(k)\right\}|p) \tag{1.50}$$

with $H_{\pi^-}|\mathbf{n} \ = E_c|\mathbf{n}$, where Z_c is the normalization. Then,

$$E_c < m \qquad \text{for every } |g| \text{ satisfying } D(0;g) < 0, \tag{1.51}$$

which means that as to E_c the decay from neutron to proton and π^--meson is stable. E_c does not have the same order in the coupling length as g^2 following from the regular perturbation theory. We have that

$$E_c \quad g\sqrt{\int d^3k \ |\lambda\,(k)\,|^2} \qquad \text{as } g \to \infty, \tag{1.52}$$

where we note that the term with the order g vanishes in the regular perturbation theory.

Since $H_{Q=-1/2}$ has a state, this is the different result from the result which Hübner and Spohn mentioned after [HS, (6.3)] and before [HS, Proposition 15]. But, if we assume the hypotheses in [HS, Prosoition 15], then all $H_{Q=-(2\nu-1)/2}$ ($\mathbb{N} \quad \nu \geq 2$) have *no* state by [HS, Prosoition 15].

The following condition is to avoid the α decay in the meaning of the remark mentioned by Henley and Thirring for n □ $p + \pi^-$ in [HT]:

(Anti α**)** The function $\dfrac{|\lambda(k)|^2}{|x - \omega(k)|}$ is *not* Lebesgue integrable for all $x \quad (m\,,\,\infty)$.

And set

$$M_g := \int_{\mathsf{R}^3} d^3k \ \lambda(k)^2\,\{\omega(k) - \mu_0 + B_{g,m}\}^{-1}. \tag{1.53}$$

Then, we obtain the following (see [Hi, Theorem 2.1]):

$[Q = \pm\frac{1}{2}]$ *Assume (Anti* α*) and (1.49). Then, for all* g *with* $d_m(g) < 0$

(i) *State with $Q = -\frac{1}{2}$ exists, and it is $|\mathbf{n}$ of which energy is E_c.*

(ii) *State with $Q = \frac{1}{2}$ exists, and it is $|p)$ of which energy is 0.*

(iii) *There is no state with $Q = -\frac{2\nu - 1}{2}$ for $\mathbb{N} \quad \nu \geq 2$.*

(iv)

- *If $\mathsf{B}_{g,m} < E_0$, then $|p)$ is a unique ground state and $|\mathbf{n}$ a unique excited state.*
- If $\mathsf{B}_{g,m} = E_0$, *then $|p)$ and $|\mathbf{n}$ are 2-fold degenerate ground states.*
- *If $\mathsf{B}_{g,m} > E_0$ and*

$$2m - E_0 > \mathsf{B}_{g,m} - g^2 M_g + M_g^{-1} \int_{\mathsf{R}^3} d^3k \ \lambda(k)^2,$$

then $|\mathbf{n}$ is a unique ground state and $|p)$ a unique excited state. Moreover, the essential spectra of H_{π^-} is given as $\sigma_{ess}(H_{\pi^-}) = [\min\{0\,,\,E_c\} + m\,,\,\infty\,)$

The hypotheses in the above statement requires that $\mathsf{B}_{g,m}$ is not so large.

Avron and Elgart argued the complex solution in the lower half plane of the analytic continuation of $D(z; g) = 0$, which is called 'resonance pole' by mathematicians, associated with the state with $Q = -\frac{1}{2}$ [AE, APPENDIX]. On the other hand, without (Anti α) there is also a possibility of the α decay in the meaning of the remark mentioned by Henley and Thirring [HT] for $n \ \square\ p + \pi^-$. Consider the following $\lambda_\alpha(k)$ instead of $\lambda(k)$ so that $\lambda_\alpha(k)$ breaks (Anti α):

$$\lambda(k) = 0 \qquad \text{for } |k| \geq \kappa \text{ with a constant } \kappa > 0. \tag{1.54}$$

Let $\mu(\kappa) := \sup_{|k| \leq \kappa} \omega(k)$. Suppose that

$$\lim_{x \downarrow \mu(\kappa)} \int_{|k| \leq \kappa} d^3k \ \frac{|\lambda(k)|^2}{|x - \omega(k)|} = +\infty. \tag{1.55}$$

Then, $D(x; g)$ restricted in $x \quad (\mu(\kappa)\,,\,\infty)$ has a unique simple zero E_c' in $(\mu(\kappa)\,,\,\infty)$, which means that the neutron becomes unstable for the decay into proton and π-meson. Thus, we have another physical state $|\mathbf{n}'$ given by

$$|\mathbf{n}' \ = Z_c'^{\,1/2} \left\{ \tau_- + g \int d^3k \ \frac{\lambda\,(k)}{E_c' - \omega\,(k)} a^\dagger(k) \right\} |p), \tag{1.56}$$

and it is the resonance state caused by the scattering between proton and π-meson. The state $|\mathbf{n}'$ is also an eigenstate of H_{π^-} with $H_{\pi^-}|\mathbf{n}' \ = E_c'|\mathbf{n}'$, where Z_c' is the normalization (see [AH00, Remark6.4] and [Bi98]). Therefore, in the same way as the previous result we have the following result, which is a rigorous proof of the statement in [HT] on the α decay for $Q = -\frac{1}{2}$:

[α Decay for $Q = -\frac{1}{2}$] *If (1.54), (1.55), and (1.49) hold, then, concerning the state with $Q = -\frac{1}{2}$, the two physical states of neutron always exist, and they are $|\mathbf{n}$ and $|\mathbf{n}'$. There is no state besides $|p)$, $|\mathbf{n}$, and $|\mathbf{n}'$. The situation about the switch between the ground state and excited state is same as previous result about $Q = -\frac{1}{2}$.*

We can prove that, for sufficiently large $\mathsf{B}_{g,m}$ (i.e., $\mathsf{B}_{g,m} \quad 1$), there is a state with $Q = -\frac{2\nu-1}{2}$ for $\mathbb{N} \quad \nu \geq 2$, and it becomes the ground state $|\Psi_{grd}$. So, it is different from any state in the above two results.

As to $\mathsf{B}_{g,m} \to \infty$, when we fix $m \geq 0$, we have, of course, $\mathsf{B}_{g,m} \to \infty$ for $|g| \to \infty$. Moreover, we consider the low energy limit (infrared catastrophe) or high energy limit (ultraviolet catastrophe) in the following sense: set

$$\mathsf{I}_m := \int_{\mathsf{R}^3} d^3k \ \frac{|\lambda\,(k)\,|^2}{\omega\,(k)} \qquad \infty \tag{1.57}$$

as $m \downarrow 0$ for fixed $\lambda(k)$ or as $\lambda(k) \to 1$ for fixed m. Then, we have $\mathsf{B}_{g,m} \to \infty$ when $m \downarrow 0$ as long as g is fixed even if $|g|$ is small or when $\lambda(k) \to 1$ for fixed g and m.

By using the manner to get [Hi, (2.73)] with a little modification, we get

$$\limsup_{\mathcal{B}_{g,m} \to \infty} \frac{E_c}{\mathsf{B}_{g,m}} = \limsup_{\mathcal{B}_{g,m} \to \infty} \ \mathsf{I}_m^{-1} \int_{\mathsf{R}^3} d^3k \ \frac{|\lambda\,(k)\,|^2}{E_c - \omega(k)}.$$

And, we can prove that the right hand side of the above equality is zero in the same way as [Hi, (2.74)] with divergent I_m by (1.57) or finite I_m because $E_c \to \infty$ as $B_{g,m} \to \infty$, but the left hand side of the above equality is not zero in the same argument as that about the inequality after[Hi, (2.74)]. This is a contradiction. Therefore,

$$E_c > E_{grd} \quad \text{for} \quad B_{g,m} \gg 1. \tag{1.58}$$

Since $E_{grd} = 0$ by (1.58), $|\Psi_{grd}\rangle$ is not a state with $Q = \frac{1}{2}$. Suppose that $|\Psi_{grd}\rangle$ is a state with $Q = -\frac{1}{2}$. Then, by [Hi, Lemma 2.1(b)] or by solving $|\Psi_{grd}\rangle$ with $Q = -\frac{1}{2}$ directly, we know that $|\Psi_{grd}\rangle$ has the same form of (1.50) and (1.56). So, E_{grd} must be a solution of $D(z; g) = 0$, since there is no solution but E_c and E_c', we have $E_c = E_{grd}$, which contradicts (1.58). Therefore, $|\Psi_{grd}\rangle$ is a state with $Q = -\frac{2\nu-1}{2}$ for some $\mathbb{N} \ni \nu \geq 2$.

It is easy check that $|p)$ and $|\mathbf{n}\rangle$ (also $|\mathbf{n}'\rangle$ if it exists) are still state of H_{π^-}.

Namely, we obtain

[$Q = -\frac{2\nu-1}{2}$ with $\mathbb{N} \ni \nu \geq 2$] *If $B_{g,\mu} \gg 1$, then there always exists a ground state $|\Psi_{grd}\rangle$ with $Q = -\frac{2\nu-1}{2}$ for some $\mathbb{N} \ni \nu \geq 2$ different from both $|p)$ and $|\mathbf{n}\rangle$. Moreover, $|p)$ and $|\mathbf{n}\rangle$ (also $|\mathbf{n}'\rangle$ if it exists) become excited states of H_π, and the essential spectra of H_{π^-} is given as $\sigma_{ess}(H_{\pi^-}) = [E_{grd} + m, \infty)$.*
We note here that if the hypotheses in the above statement satisfies, then the ground state, Ψ_{grd}, always appears with $Q = -\frac{2\nu-1}{2}$ for some $\mathbb{N} \ni \nu \geq 2$. In this sense, the appearance of Ψ_{grd} is stable. We conjecture that the ν in Q gets large as $B_{g,\mu} \gg 1$ grows. Although we cannot prove that yet, we can see the tendency in the Jaynes-Cummings model [Mi, §6.4] for our model as we show in the next section.

2 Transition of Ground State of Lee Model

In this section, we use $\omega(k) = \sqrt{k^2 + \mu^2}$ defined in \$ 1.1 for the sake of simplicity though we can treat more general $\omega(k)$ with certain mathematical conditions. For each $\mu \geq 0$ we take $\rho(k)$ so that $\rho(\omega(k))/\sqrt{\omega(k)}$ gets independent of $\mu \geq 0$. So, we set $\lambda(k) := \rho(\omega(k))/\sqrt{2\omega(k)}$ independent of $\mu \geq 0$, and we assume that $\lambda \in L^2(\mathbb{R}^d)$, real-valued and continuous.

We here employ special annihilation and creation operators for $\psi_V, \psi_V^\dagger$, and $\psi_N, \psi_N^\dagger$, namely we define them by Pauli's spin-flip matrices. Let state space $\mathcal{F}$ for H be the Hilbert space given by $\mathcal{F} := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{F}_b$, where $\mathcal{F}_b$ is a boson Fock space over $L^2(\mathbb{R}^d)$. For operators A, B on $\mathbb{C}^2$ and C acting on $\mathcal{F}_b$, we denote $A \otimes B \otimes C$ acting on $\mathcal{F}$ by just ABC with abbreviation. Then, we set

$$\psi_V = \psi_N^\dagger = \sigma_- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi_V^\dagger = \psi_N = \sigma_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$

where $\sigma_\pm$ are Pauli's spin-flip matrices. So, the Hamiltonian H in this section has the following form:

$$H = H_0 + gH_I,$$

where

$$H_0 = m_V \psi_V^\dagger \psi_V + m_N \psi_N^\dagger \psi_N + \int_{\mathbb{R}^d} d^d k \; \omega(k) \alpha_\theta^\dagger(k) \alpha_\theta(k) \tag{2.2}$$

$$H_I = \int_{\mathbb{R}^d} d^d k \; \lambda(k) \left(\psi_V^\dagger \psi_N \alpha_\theta(k) + \psi_V \psi_N^\dagger \alpha_\theta^\dagger(k) \right). \tag{2.3}$$

So, $\lambda(k)$ is a real-valued ultraviolet cutoff function independent of μ. We note here the following: we set $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $A = m_V \psi_V^\dagger \psi_V + m_N \psi_N^\dagger \psi_V$, $B_1 = \left(\psi_V^\dagger \psi_N + \psi_V \psi_N^\dagger\right)/\sqrt{2}$, $B_2 = i\left(\psi_V^\dagger \psi_N - \psi_V \psi_N^\dagger\right)/\sqrt{2}$; $\lambda_1 = \lambda$, and $\lambda_2 = i\lambda$. Then, we know that the special Lee model is one example of the generalized spin-boson (GSB) model which we defined in [AH97].

The point eigenvalues $\sigma_p(H_0)$ of H_0 are $0, m_V, m_N, m_V + m_N$, i.e., $\sigma_p(H_0) = \{0, m_V, m_N, m_V + m_N\}$. The essential spectrum $\sigma_{ess}(H_0)$ is $[0, \infty)$, i.e., $\sigma_{ess}(H_0) = [0, \infty)$, where $\sigma_{ess}(T)$ for a Hamiltonian T is the set of spectrum (energies) of T except simple or finitely degenerate discrete eigenvalues.

By the way, we can decompose H into the direct sum of H_1 and H_2,

$$H = H_1 \oplus H_2, \tag{2.4}$$

where

$$\begin{aligned}
H_1 &:= m_V \psi_V^\dagger \psi_V \psi_N \psi_N^\dagger + m_N \psi_V \psi_V^\dagger \psi_N^\dagger \psi_N + \left(\psi_V^\dagger \psi_V \psi_N \psi_N^\dagger + \psi_V \psi_V^\dagger \psi_N^\dagger \psi_N\right) H_\theta + H_I \qquad (2.5) \\
H_2 &:= m_V \psi_V^\dagger \psi_V \psi_N^\dagger \psi_N + m_N \psi_V^\dagger \psi_V \psi_N^\dagger \psi_N + \left(\psi_V^\dagger \psi_V \psi_N^\dagger \psi_N + \psi_V \psi_V^\dagger \psi_N \psi_N^\dagger\right) H_\theta, \qquad (2.6) \\
H_\theta &:= \int_{\mathbb{R}^d} d^d k \ \omega(k) \alpha_\theta^\dagger(k) \alpha_\theta(k). \qquad (2.7)
\end{aligned}$$

Then, since the infimum of the energy of H_2 is zero, the ground state of H_1 becomes that of H. Namely, to investigate the ground state of H we have only to study the ground state of H_1. We denote the state space which H_j acts on by $\mathcal{F}_j$ for $j = 1, 2$. Then, $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$. Then, H_1 on $\mathcal{F}_1$ is unitary equivalent to the Weisskopf-Wigner model argued in §1.3 and given by [Hi, (3.11)] ($\alpha, \varepsilon_0^+, \varepsilon_1^+$ in [Hi, (3.11)] are our $g/2, m_V, m_N$ respectively). Therefore, we can understand that the regular renormalized mass m_{V_c} satisfying

$$m_{V_c} = m_V + g^2 \int_{\mathbb{R}^d} d^d k \ \frac{|\rho(\omega(k))|^2}{2\omega(k)} \ \frac{1}{m_{V_c} - (m_N + \omega(k))} \tag{2.8}$$

represents the higher order revision for H_1 in Weisskopf-Wigner theory. Thus, m_{V_c} does not have the same order in the coupling length as g^2 following from the regular perturbation theory. We note here that $m_{V_c} \sim g\sqrt{\int_{\mathbb{R}^3} d^3k \ |\rho(\omega(k))|^2/(2\omega(k))}$ as $g \to \infty$, and the term with the order g vanishes in the regular perturbation theory as remarked in (1.52).

By applying the argument in §1.3 for the case of $\mu > 0$, and by applying [Sk, Theorem 3.1] for the cases of $\mu = 0$, in the same way as [Hi, Proposition 2.1], we know that if $\lambda(k)$ has some proper mathematical conditions, then *the normal renormalized mass m_{V_c} is the ground state energy of H for such small $|g|$ with fixed $\mu \geq 0$ as $d_\mu(g) < 0$, and moreover, other excited state energies are $0, m_N, m_V + m_N$ only.* More precisely, in the case of $\mu = 0$, let

$$g_{nor} := \left\{ 2\int_{\mathbb{R}^d} d^d k \ |\Lambda(k)|^2 \right\}^{-1}, \qquad \Lambda(k) := \frac{\partial \lambda(k)}{\partial |k|} + (d-1)\frac{\lambda(k)}{2|k|}. \tag{2.9}$$

If $d_\mu(g) < 0$ and $\omega^{-1}\lambda \in L^2(\mathbb{R}^d)$, then the total number of bound states of H_1 defined by (2.5) is just 2 for $|g| < g_{nor}$. Thus, the ground state energy of H is the normal renormalized mass m_{V_c} for $|g| < g_{nor}$ with $\mu = 0$.

By applying the argument §1.3 to the direct sum decomposition (2.4) to our special Lee model, for sufficiently large $\mathcal{B}_{g,\mu} \gg 1$, we can prove mathematically the existence of the ground state different from $|\mathbf{V}\rangle$ so that the ground state energy is less than the energy of $|\mathbf{V}\rangle$. Of course, it is not strange state such as Källén and Pauli showed in [KP], namely our ground state lies in the standard Hilbert space $\mathcal{F}$.

Namely, *if $\mathcal{B}_{g,\mu} \gg 1$, then there exists a ground state different from $|\mathbf{V}\rangle$, and $|\mathbf{V}\rangle$ becomes an excited state of H.*

We have

$$\begin{aligned}
\sigma_p(H) &= \left\{ E_{grd}, m_{V_c}, 0, m_N, m_V + m_N \right\} \\
&\quad \textit{with } E_{grd} < m_{V_c} < 0 < m_N < m_V + m_N, \qquad (2.10) \\
&\min\{m_V, m_N\} - \mathcal{B}_{g,\mu} \leq E_{grd} \leq \frac{m_V + m_N}{2} - \frac{1}{4}\mathcal{B}_{g,\mu} \qquad (2.11)
\end{aligned}$$

for $\mathcal{B}_{g,\mu} \gg 1$. Moreover, by [Ar00], we have

$$\sigma_{ess}(H) = [E_{grd} + \mu, \infty) \tag{2.12}$$

for $\mathcal{B}_{g,\mu} \gg 1$.

Therefore, H for $\mathcal{B}_{g,\mu} \ll 1$ and H for $\mathcal{B}_{g,\mu} \gg 1$ are different physics respectively, which gives a transition of ground state in the same way as H in §1.3. Moreover, by (2.11) the non-perturbative ground state energy recovers the order of the square in the coupling length when the Lee model is outside the region of the regular perturbation theory. In order to get such a ground state $|\Psi_{grd}\rangle$, it is important that we balance the coupling length with the infrared singularity $\mathcal{I}_\mu$ such that (1.57) holds. Namely, even if $|g|$ (resp. $\mathcal{I}_\mu$) is large, the very small $\mathcal{I}_\mu$ (resp. $|g|$) breaks (1.57), which is not enough to get $|\Psi_{grd}\rangle$. For the appearance of $|\Psi_{grd}$, we have to add not only the coupling length but also the infrared singularity IR_μ into sufficient condition.

In the case of $\mu = 0$ with (1.57), we cannot prove the self-adjointness for $\mu = 0$ by the Kato-Rellich theorem [RS2, Theorem X.12], so we cannot apply the regular perturbation theory to this case. Of course, we cannot employ the regular perturbation theory in the case $|g|\quad 1$ with the fixed $\mu \geq 0$. Therefore, $|\Psi_{grd}$ appears in the case beyond the perturbation theory.

As to the extra bound states with *non-standard* resonance, the recent Billionnet's works [Bi98, Bi01] are interesting and worth noting. As remarked in [AH00, Remark 6.4] and [Hi, Remark 2.6], if we add an extra condition to $\lambda(k)$ in the same way as the case of the α decay, then an extra eigenvalues appears in $[m_{_N} + \mu\,,\ \infty\,)$, and it is different from E_{grd}, m_{V_c}, 0, and $m_{_V} + m_N$. Billionnet showed in [Bi98, Bi01] that the reason why such eigenvalues appear is not for the result of the preceding complex eigenvalue of resonance turning into real eigenvalues when the coupling is continuously increased. We indeed knew that E_{grd}, m_{V_c}, 0, and $m_{_V} + m_N$ are stable for $\mathcal{B}_{g,\mu}\quad 1$ in our Lee model. Moreover, Billionnet clarified in [Bi01] the way of appearance through the *non-standard* resonance.

Once we have the ground state $|\Psi_{grd}$ different from $|\mathbf{V}$, by applying [AH01, Theorem 3.1] in the same way as [AH01, Theorem 4.5], we obtain that there exist $(g_{_0}\,,\ \mu_{_0})$ in $\mathcal{A} := \{(g\,,\ \mu)\ |\, g \quad \mathbb{R}$ and μ with $-\infty \leq d_\mu(g) < 0\ \}$ such that $H\ _{g=g_0\,,\mu=\mu_0}$ which is H with $g = g_{_0}$ and $\mu = \mu_{_0}$ has a degenerate ground states. And, there exist $(g_{_1}\,,\ \mu_{_1})$ in $\mathcal{A}$ such that $\inf\left\{\sigma(H\ _{g=g_1\,,\mu=\mu_1}) \setminus \{E_{grd}\}\right\} < \inf\sigma_{ess}(H\ _{g=g_1\,,\mu=\mu_1})$.

The conservation law (1.6) on the total number of V-particles and θ-particles decomposes the state space $\mathcal{F}$ into the direct sum of some sectors as follows:

For (1.6), we define the number operator $N_{_{V\theta}}$ by

$$N_{_{V\theta}} := \psi^\dagger_{_V}\psi_{_V} + N_\theta \tag{2.13}$$

where N_θ denotes the number operator of θ-particle, i.e.,

$$N_\theta := \int_{\mathsf{R}^d} d^d k\ \ \alpha^\dagger_{_\theta}(k)\alpha_{_\theta}(k). \tag{2.14}$$

Then, the conservation law (1.6) is reflected in the relation,

$$[H\,,\ N_{_{V\theta}}] = 0. \tag{2.15}$$

We denote the orthogonal projection onto the ℓ-θ-particle space in $\mathcal{F}_b$ by $P^{(\ell)}_{_\theta}$ for each $\ell \quad \mathbf{N}$. Then, we get $N_{_\theta} = \sum_{\ell=0} \ell P^{(\ell)}_{_\theta}$. Then, the spectral resolution of $N_{_{V\theta}}$ is given by

$$N_{_{V\theta}} = \sum_{\ell=0} \ell P_\ell. \tag{2.16}$$

Here we set $P^{(-1)}_{_\theta} \equiv 0$.

We set $\mathcal{F}_j(\ell) := P_\ell \mathcal{F}_j$ for $j = 1, 2$ and $\ell = 0, 1, 2, \cdots$. Then,

$$\mathcal{F}_1 = \bigoplus_{\ell=0}^{\infty} \mathcal{F}_1(\ell). \tag{2.17}$$

We denote the vacuum by $|0$. Since $|V\ = \psi^\dagger_{_V}|0$ and $|N\ = \psi^\dagger_{_N}|0$, we have

$$|\mathbf{V}\quad = \quad Z_2^{1/2}\left\{\psi^\dagger_{_V}|0\ + g_0 \int_{\mathsf{R}^d} d^d k\ \frac{\lambda^2\,(k)}{m_c - m_{_N} - \omega\,(k)}\ \psi^\dagger_N \alpha^\dagger_{_\theta}(k)|0\ \right\} \tag{2.18}$$

by (1.8). Therefore, it is clear that $|\mathbf{V}$ is an eigenstate of H_1 with

$$|\mathbf{V}\quad \mathcal{F}_1(1). \tag{2.19}$$

Since our ground state Ψ_{grd} still lives in the standard state space $\mathcal{F}$, Ψ_{grd} has to belong to one of the sectors $\mathcal{F}_1(\ell)$'s ($\ell \quad \{0\} \quad \mathbf{N}$). Of course, Ψ_g has the positive norm because it is in the standard state space, which is a difference from the Källén and Pauli's state $|\mathbf{V}_{KP}$. Moreover, their state $|\mathbf{V}_{KP}$ belongs to the sector $\mathcal{F}_1(1)$, but our Ψ_{grd} does not belong to $\mathcal{F}_1(1)$. Because, we can prove in the same way as [Hi, Lemma 2.1(c)] that $|\Psi_{grd}\ = c|\mathbf{V}$ for some complex constant c contradicts the fact that the the ground state energy is less than $m_{_{V_c}}$. Therefore, $|\Psi_{grd}$ is different from $|\mathbf{V}_{KP}$.

3 Superradiant Ground State(?)

In the previous section, we proved mathematically that the ground state $|\Psi_{grd}$ another from $|\mathbf{V}$ appears, and $|\mathbf{V}$ becomes an excited state of H. We are interested in the physical reason of the appearance of such a ground state $|\Psi_{grd}$.

We have to note that the ground state of H does not have the form of (1.23) in spite of (1.5), (1.6), and (1.26). Moreover, since E_{grd} is much lower than m_{V_c}, V-particle has to emit so many θ-particles. But considering (1.5), (1.6), and (1.26), such a emission is not observable in the reaction (1.4), namely, the emission is prohibited from breaking the conservation laws (1.5) and (1.6). Moreover, remember that the infrared singularity (1.57) plays an important role for the existence of Ψ_{grd} with fixed the coupling length g, and that for each concrete model the individual coupling constant is fixed. Thus, such θ-particles may be like soft photons. Although we cannot give concretely a form of the ground state, we can consider the Jaynes-Cummings model [Mi, §6.4] in the light of quantum optics, the case of θ-particle for the mode with k (i.e., one mode), at very low energy (or mass).

We consider the following Hamiltonian $H(k)$ for the system of V-particle, N-particle, and θ-particle for the mode with k:

$$\begin{aligned} H(k) \quad := \quad & m_V \psi_V^\dagger \psi_V + m_N \psi_N^\dagger \psi_N \\ & +\omega(k)\alpha_\theta^\dagger(k)\alpha_\theta(k) + g\lambda(k)\left(\psi_V^\dagger \psi_N \alpha_\theta(k) + \psi_V \psi_N^\dagger \alpha_\theta^\dagger(k)\right), \end{aligned} \tag{3.1}$$

where we employed (2.1) again. $\mathcal{F}_b^1(k)$ is the state space of one-mode θ-particle with k. Then, all eigenvalues $E_{n_k}^\pm$ $(n_k = 0, 1, 2, \cdots)$ of $H(k)$ are given as

$$E_{n_k}^\pm \quad = \quad \left(n_k + \frac{1}{2}\right)\omega(k) + \frac{m_V + m_N}{2} \pm \sqrt{\left(m_N + \omega(k) - m_V\right)^2 + 4g^2\lambda^2(k)\left(n_k + 1\right)}\,, \qquad n = 0, 1, 2, \cdots,$$

where n_k is the number of θ-particles for the mode with k. So, $\mathcal{F}_b(k)$ is spanned by $\{|n_k\}_{n_k=0}^\infty$. As we saw in (1.25), $m_N + \omega(k) - m_V$ becomes a shift of a frequency from the resonance, so $\sqrt{\left(m_N + \omega(k) - m_V\right)^2 + 4g^2\lambda^2(k)\left(n_k + 1\right)}$ in (3.2) is equal to a general quantized Rabi flopping frequency [MS, WM], which leads us to the spontaneous emission of the photon with a single mode. The eigenstate $|\Psi_{n_k}^\pm$ for $E_{n_k}^\pm$ is in $\mathcal{F}_1(n_k + 1)$. Therefore, for instance, we set $\lambda(k) \equiv 1$ now for the sake of simplicity, and

$$g = 10^L,\ \ \omega(k) = 10^{-L},\ \ n_k = 10^{4L-2} \quad \text{or} \quad g = 1,\ \ \omega(k) = 10^{-3L},\ \ n_k = 10^{4L-2}$$

for sufficiently large L $\mathbf{N}$. Then, we obtain

$$E_{10^{4L-2}}^- \geq -\mathcal{B}_{g,\mu}(k) = -\frac{g^2}{\omega(k)}$$

with

$$E_{10^{4L-2}}^- \quad -0.09 \times 10^{3L}, \qquad -\frac{g^2}{\omega(k)} \quad -10^{3L} \tag{3.2}$$

as $L \to \infty$. This means that the eigenvalue with the same order as $-g^2/\omega(k)$ is obtained as $n_k, |g|$ 1 $\omega(k)$ or n_k 1 $\omega(k)$ for fixed g. Moreover, the eigenstate with such an eigenvalue belongs to the sector, $\mathcal{F}_1(10^{4L-2} + 1)$, and switches to another sector with larger number of θ-particle as $L \to \infty$.

4 Conclusion

Following the renormalizable field theory by Lee, Källén and Pauli, in order to avoid a ghost, m_c, g_c, and ρ are restricted as we know from (1.14). On the other hand, m_c g, and ρ are chosen so that g_c can catch g_{obs}. For the large coupling, we cannot employ the perturbative way to get the renormalized constant and renormalized coupling constant any longer. Even in the case $|g|$ is not so large, if we have the case with the infrared singularity condition (1.57), then the effective mass is different from m_{V_c}. Moreover, in the case of $\mathcal{B}_{g,\mu}$ 1, we have to consider not only $Z_{V_c}^{ren}$ but also the renormalized constant for $|\Psi_{grd}$ to argue whether $|\Psi_{grd}$ is a ghost or not.

The interaction Hamiltonian H_I of the Lee model is given through the rotating wave approximation (RWA) or the Weisskopf-Wigner approximation. If we can check that the mathematical results of (2.10)-(2.12) are not realistic in physics, then we know the limit of the domain of $B_{g,\mu}$ and $d_\mu(g)$, i.e., the coupling constants g and the condition of the ultraviolet cutoff function $\lambda(k)$ of the Lee model, so the limit of RWA in a sense. But if we hope the results of (2.10)-(2.12) are realistic in physics, we have to prove the same results independent of RWA, and to develop the renormalizable field theory so that it can include the case such as $B_{g,\mu}$ 1.

Is the existence of $|\Psi_{grd}$ caused by the Rabi flopping? If it is correct, the phases may get harmonious by revival with very small $|k|$ after they are disordered by the Cummings collapse in the Rabi flopping. Namely, are there any relation between superradiant ground state by Preparata and the Rabi flopping?

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