Application of Renormalization Group Method to Kinetic Equations: roles of initial condition

Y. Hatta[1] and T. Kunihiro[2]

[1] Department of Physics, Kyoto University, Kyoto 606-8502, Japan
[2] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

The so-called renormalization group (RG) method is applied to derive Boltzmann equation in classical mechanics and Fokker-Planck equation from the respective microscopic equations. Utilizing the formulation of the RG method which elucidates the important role played by the choice of the initial conditions, the general structure and the underlying assumptions in the derivation of kinetic equations in the RG method is clarified.

1 Introduction

In [1], the reduction-theoretical aspect of the so called renormalization group (RG) method [2] and the improved formulation given in [19] were reformulated mathematically with the notion of invariant manifolds familiar in the theory of dynamical systems [6]. The perturbative RG method can be used to construct invariant manifolds successively as the initial values of evolution equations using the Wilsonian RG [16, 20, 21]; the would-be integral constants, which have one-to-one correspondence with the initial values, in the unperturbed solution, constitute natural coordinates of the invariant manifold. It was also shown that the RG equation determines the slow motion of the would-be integral constants on the invariant manifold of the dynamical system, hence a reduction of evolution equation is achieved.

We apply the RG [16] method to derive and reduce kinetic equations to a slower dynamics [2, 18, 19, 1]: the Boltzmann equation is derived from the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy[3] and the Fokker-Planck equation is derived from the Langevin equation. It seems that the basic notions to implement the reduction are given by (1) the coarse-grained time-derivative and (2) the choice of the initial conditions in solving the microscopic equations, respectively:
(1) In an attempt to characterize hydro-dynamical processes microscopically, Mori pointed out that time derivatives appearing in equations which define transport coefficients are "the average" of time derivatives describing microscopic dynamics [10]. His definition of the macroscopic derivative of an observable F is

$$\frac{\delta}{\delta t}\langle F\rangle(t) \equiv \frac{1}{\tau}\{\langle F\rangle(t+\tau) - \langle F\rangle(t)\} = \frac{1}{\tau}\int_0^\tau ds \frac{d}{ds}\langle F\rangle(t+s), \tag{1.1}$$

where τ is some time scale between microscopic (mean free) time and macroscopic (relaxation) time. An important point is that τ is finite. The idea of the coarse-graining of time in kinetic and transport equations were first given by Kirkwood[11]; see also [12] for a rigorous formulation.

(2) The importance of the choice of the initial condition in the derivation of kinetic equation is noticed and emphasized in the literature[4, 5, 13, 14, 15]. For instance, Kawasaki clarifies in an excellent monograph[15] that the initial value of the microscopic distribution function before averaging must be given solely in terms of the averaged distribution to obtain a closed equation for the distribution function of macroscopic slow variables, which is equivalent to the construction of an invariant manifold mentioned above. He also clarifies that by this initial condition, the dominating class of states ("typical states") are selected which leads to an increase of the entropy, while exceptional states from which the entropy would increase could be unstabilized to the dominating typical states by a mechanism producing chaotic behavior.

We shall show that the straightforward application of the RG method as formulated in [19, 1] naturally leads to the choice for the initial value of the microscopic distribution function at an arbitrary time t_0 to be on the averaged distribution, which is an implementation of (2) in the RG method, thereby leads to time-irreversible equations even from a time-reversible equation. The averaged distribution function may be thought as an integral constant of the solution of microscopic evolution equation. The RG equation gives the slow dynamics of the would-be initial constant, which is actually the kinetic equation governing the averaged distribution function. It will be further shown that the averaging as given above automatically gives rise to a coarse-graining of the time-derivative, which is expressed with the initial time t_0. This shows that the initial time t_0 has a macroscopic nature in contrast to the time t appearing in the microscopic equations, which is an implementation of (1) in our method.

It should be noticed here that the RG equation has been already applied to kinetic equations in [22]: In [22], it was ascertained that Boltzmann equation is a renormalization group equation on the basis of the work by M.S. Green[23] on the uniform system which shows that the perturbative solution for the BBGKY hierarchy exhibit a secular term, and a sketch was given to derive the Fokker-Planck equation from a simple Langevin equation noticing again an appearance of a secular term. On the basis of these two examples, they claimed that all other kinetic equations are also RG equations. These models dealt in [22] will be retreated in our formulation, and implicit assumptions in their treatment will be made explicit so that the roles of the initial conditions and the scale transformation of the time-derivative will become clear for the RG method to lead to kinetic or transport equations.

In section 2, we shall deal with the Langevin equation and derive the Fokker-Planck equation as a typical problem of dynamical reduction leading to a kinetic equation in the RG method. We shall summarize the basic structure of the reduction given by the RG method. One will see that a similar definition to Eq.(1.1) of the macroscopic time-derivative naturally emerges in the RG method. In section 3, the Boltzmann equation is derived from the Liouville equation of the classical mechanics; we shall clarify the difference between the present method and the one by Bogoliubov.

2 Reduction of Langevin equation to Fokker-Planck equation

In this section, the RG equation is applied to obtain the Fokker-Planck(FP) equation[9] from the stochastic Liouville equation [25] corresponding to Langevin equation[9]. The present derivation is thought to be a typical one for the reduction of evolution equations appearing in non-equilibrium physics[15]. We shall clarify that the initial values of the stochastic distribution function at arbitrary time t_0 are naturally chosen to be on the averaged distribution function for the RG equation derives the FP equation governing the averaged distribution function. We shall also notice that the time derivative in the RG equation which will be converted to the derivative in the FP equation is with respect to a macroscopic time, hence the coarse-graining of time is automatically built in in the present RG method.

2.1 From generic Langevin equation to Fokker-Planck equation

Let us consider the following generic Langevin equation with R_i $(i = 1, 2, ..., n)$ being stochastic variables;

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{h}(\boldsymbol{u}) + \hat{g}(\boldsymbol{u})\boldsymbol{R}, \tag{2.1}$$

where $\boldsymbol{u} = {}^t(u_1, u_2, ..., u_n)$, $\boldsymbol{h} = {}^t(h_1, h_2, ..., h_n)$, $\hat{g}$ a n times n matrix and $(\hat{g}(\boldsymbol{u})\boldsymbol{R})_i = \sum_j g_{ij}R_j$. We remark that the noise enters multiplicatively. Here we assume without loss of generality that the noise has the vanishing average,

$$\langle \boldsymbol{R}(t) \rangle = 0, \tag{2.2}$$

where $\langle \mathcal{O}(t) \rangle$ denotes the average of $\mathcal{O}(t)$ with respect to the noise $\boldsymbol{R}$. Let $f(\boldsymbol{u}, t)$ be the distribution function with $\boldsymbol{R}(t)$ given; the continuity equation reads

$$\frac{\partial f(\boldsymbol{u}, t)}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\boldsymbol{v} f(\boldsymbol{u}, t)) = 0, \tag{2.3}$$

where $\nabla_{\boldsymbol{u}} = \sum_i \partial/\partial u_i$ and $\boldsymbol{v} = d\boldsymbol{u}/dt$ is the velocity of $\boldsymbol{u}$, which is given in (2.1). Inserting (2.1) into (2.3), one has the Kubo's stochastic Liouville equation[25],

$$\frac{\partial f}{\partial t} = -\nabla_{\boldsymbol{u}} \cdot [(\boldsymbol{h} + \hat{g}\boldsymbol{R})f]. \tag{2.4}$$

Although it is a rather easy task to derive the FP equation in an exact way *if the noise is Gaussian*, it is formidably difficult if the noise is *non-Gaussian*[9]. The present approach is admittedly based on the perturbation theory and of approximate nature. Nevertheless, it will be found that the first order calculation suffices to derive the exact FP equation when the noise in Gaussian, and furthermore that the method is applicable even to non-Gaussian noises without difficulties.

The solution to (2.4) with the initial condition given at $t = t_0$ is formally given by [25]

$$\tilde{f}(\boldsymbol{u}, t; t_0) = T\exp[\int_{t_0}^{t} ds L(s)]\tilde{f}(\boldsymbol{u}, t_0; t_0), \tag{2.5}$$

where

$$L(s) = -\nabla_{\boldsymbol{u}} \cdot (\boldsymbol{h}(\boldsymbol{u}) + \hat{g}\boldsymbol{R}(s)), \tag{2.6}$$

and T denotes the time ordering operator. The initial distribution $\tilde{f}(\boldsymbol{u}, t_0; t_0)$ will be specified later and found to play a significant role in the present method.

Now we are interested in the averaged distribution function $\tilde{P}(\boldsymbol{u}, t; t_0)$ which is defined as an average of $f(\boldsymbol{u}, t; t_0)$ with respect to the noise $\boldsymbol{R}$, i.e.,

$$\tilde{P}(\boldsymbol{u}, t; t_0) = \langle T\exp[\int_{t_0}^{t} dsL(s)]f(\boldsymbol{u}, t_0)\rangle. \tag{2.7}$$

We take an interaction picture dividing the "Hamiltonian" L as follows;

$$L = L_0 + L_1, \tag{2.8}$$

$$L_0 = -\nabla_{\boldsymbol{u}} \cdot \boldsymbol{h}, \quad L_1 = -\nabla_{\boldsymbol{u}}\hat{g}\boldsymbol{R}. \tag{2.9}$$

We first define $U_0(t$fltonian(

is an "interaction Hamiltonian" in the "interaction picture".

Thus we obtains the compact form of $\tilde{P}(\boldsymbol{u},t;t_0)$ as follows,

$$\begin{aligned}\tilde{P}(\boldsymbol{u},t;t_0) &= \langle U_0(t,t_0)\rho_1(\boldsymbol{u},t;t_0)\rangle, &(2.17)\\ &= U_0(t,t_0)\langle T\exp[\int_{t_0}^{t} ds\mathcal{L}_1(s;t_0)]\rangle P(\boldsymbol{u},t_0), &(2.18)\\ &\equiv U_0(t,t_0)S(t;t_0)P(\boldsymbol{u},t_0), &(2.19)\end{aligned}$$

where we have used the fact that $\rho_1(\boldsymbol{u},t=t_0,t_0)=P(\boldsymbol{u},t_0)$ and the abbreviation

$$S(t;t_0)\equiv\langle T\exp[\int_{t_0}^{t} ds\mathcal{L}_1(s;t_0)]\rangle.$$

The computation may be performed in a perturbative way:

$$\begin{aligned}S(t;t_0) &= 1+T\int_{t_0}^{t} ds\langle\mathcal{L}(s)\rangle+\frac{1}{2}T\int_{t_0}^{t} ds_1\int_{t_0}^{t} ds_2\langle\mathcal{L}(s_1)\mathcal{L}(s_2)\rangle+\ldots\\ &= 1+\frac{1}{2}T\int_{t_0}^{t} ds_1\int_{t_0}^{t} ds_2\Gamma(s_1,s_2)+\ldots\end{aligned} \tag{2.20}$$

where we have put

$$\Gamma(s_1,s_2)\equiv\langle\mathcal{L}_1(s_1)\mathcal{L}_1(s_2)\rangle. \tag{2.21}$$

If the noise is stationary, which we shall assume from now, $\Gamma(s_1,s_2)$ will be a function of the difference s_1-s_2; furthermore, owing to the time-reversible invariance of the microscopic law, $\Gamma(s_1,s_2)$ becomes a function of the absolute value $|s_1-s_2|$, i.e., $\Gamma(s_1,s_2)=\Gamma(|s_1-s_2|)$. Then one has for $t>t_0$,

$$S(t;t_0)=1+(t-t_0)G(t-t_0)+\cdots, \tag{2.22}$$

where we have put for $t>0$

$$G(t)=\int_0^t ds\Gamma(s). \tag{2.23}$$

If we stop at the second order approximation, we have

$$\tilde{P}(\boldsymbol{u},t;t_0)=U(t;t_0)[1+(t-t_0)G(t-t_0)]P(\boldsymbol{u},t_0). \tag{2.24}$$

Notice the appearance of the secular term which indicates that the above formula is only valid for t around t_0.

Now we apply the RG equation to (2.24) which reads

$$\partial\tilde{P}(\boldsymbol{u},t;t_0)/\partial t_0|_{t_0=t}=0,$$

which leads to

$$\partial_{t_0}U_0(t,t_0)|_{t_0=t}P(\boldsymbol{u},t)+\partial_t P(\boldsymbol{u},t)-G(0)P(\boldsymbol{u},t)=0,$$

where use has been made that $U_0(t_0, t_0) = 1, \forall t_0$. Noticing that $\partial_{t_0} U_0(t, t_0)|_{t_0=t} = -L_0 = -\nabla_{\boldsymbol{u}} \cdot \boldsymbol{h}$, we arrive at the Fokker-Planck equation,

$$\partial_t P(\boldsymbol{u}, t) = -\nabla_{\boldsymbol{u}} \cdot \boldsymbol{h} P(\boldsymbol{u}, t) + G(0) P(\boldsymbol{u}, t). \tag{2.25}$$

The concrete form of $G(0)$ depends on the character of the noise $\boldsymbol{R}(t)$. This is one of the main results of this section.

To see that (2.25) is the desired equation, let us evaluate $G(0)$ for a simple Gaussian noise given by

$$\langle R_i(t) R_j(t') \rangle = 2\delta_{ij} D_i \delta(t - t'). \tag{2.26}$$

For this case, one has

$$\Gamma(s) = U_0^{-1} \partial_i g_{ij} \partial_k g_{kl} 2 D_j \delta_{jl} \delta(s),$$

where $\partial_i = \partial / \partial u_i$. Then $G(t)$ is evaluated as follows;

$$\begin{aligned} G(t) &\equiv \int_0^t ds \Gamma(s) = \frac{1}{2} U_0^{-1} \partial_i g_{ij} \partial_k g_{kl} 2 D_j \delta_{jl}, \\ &= G(0). \end{aligned} \tag{2.27}$$

Here we have used the identity $\theta(0) = 1/2$, in accordance with the Stratonovich scheme[9]. Notice that $G(t)$ in this case is independent of t.

Inserting $G(0)$ thus obtained into (2.25), one has the familiar form of the Fokker-Planck equation for the multiplicative Gaussian noise,

$$\partial_t P(\boldsymbol{u}, t) = -\nabla_{\boldsymbol{u}} \cdot \boldsymbol{h} P(\boldsymbol{u}, t) + D_j \partial_i g_{ij} \partial_k g_{kj} P(\boldsymbol{u}, t). \tag{2.28}$$

This shows that the initial distribution $P(\boldsymbol{u}, t_0)$ satisfies the Fokker-Planck equation and justifies the identification of the initial distribution with the averaged one made in Eq. (2.14).

2.2 Discussion

Firstly, it is noteworthy that we have been naturally led to identify the initial values of the microscopic distribution function $f(\boldsymbol{u}, t_0, t_0)$ before averaging with the averaged value $P(\boldsymbol{u}, t_0)$ at an arbitrary initial time $t = t_0$. As mentioned in §1, the necessity to take such an initial condition to achieve reduction of evolution equation was advocated by Bogoliubov[5] and others[14, 15] including Boltzmann[4]. Secondly, this means that the nature of the initial time t_0 in the RG method is completely different from that of the time t in the stochastic equation (microscopic equation); t_0 represents the coarse-grained time describing the variation of the averaged quantity, and the derivative ∂_{t_0} in the RG equation is a macroscopic time-derivative. Again as mentioned in §1, this coarse-graining of time was also noted by others [11, 10, 12] in different approaches.

This automatic averaging and the appearance of the macroscopic time-derivative given in the RG method may be generically understood as a generalization of the scheme given in §2 of [1]: First discretize the variable $\boldsymbol{u} \to \boldsymbol{u}_i$ and write as $P(\boldsymbol{u}, t)(\boldsymbol{u}_i, t) = X_i(t)$ and use a vector notation $\boldsymbol{X} = (X_1, X_2, ...)$. Thus the discretized stochastic Liouville equation

with the initial value $\boldsymbol{X}(t_0)$ at an arbitrary time t_0 may be solved perturbatively, and the solution is denoted as $\tilde{\boldsymbol{X}}(t; t_0, \boldsymbol{X}(t_0))$, which satisfies the initial condition

$$\tilde{\boldsymbol{X}}(t; t_0, \boldsymbol{X}(t_0)) = \boldsymbol{X}(t_0). \tag{2.29}$$

We could solve the same equation with the initial condition given at a shifted initial time $t = t_0 + \Delta t$;

$$\tilde{\boldsymbol{X}}(t; t_0 + \Delta t, \boldsymbol{X}(t_0 + \Delta t)) = \boldsymbol{X}(t_0 + \Delta t). \tag{2.30}$$

We suppose that the time difference Δt is macroscopically small but microscopically so large that it may be taken as infinity. For the time t between t_0 and $t_0 + \Delta t$, i.e., $t_0 < t < t_0 + \Delta t$, the perturbation should be valid. If $t - t_0$ and $\Delta t \to \infty$ in the microscopic scale, we may anticipate that the system is relaxed to the averaged trajectory $\boldsymbol{X}(t)$ and have

$$\tilde{\boldsymbol{X}}(t; t_0 + \Delta t, \boldsymbol{X}(t_0 + \Delta t)) \simeq \tilde{\boldsymbol{X}}(t; t_0, \boldsymbol{X}(t_0)),$$

which implies that the macroscopic time derivative $\delta/\delta t_0$ vanishes,

$$0 = \frac{\delta \tilde{\boldsymbol{X}}}{\delta t_0} \equiv \frac{\tilde{\boldsymbol{X}}(t; t_0 + \Delta t, \boldsymbol{X}(t_0 + \Delta t)) - \tilde{\boldsymbol{X}}(t; t_0, \boldsymbol{X}(t_0))}{\Delta t_0}, \tag{2.31}$$

$$= \frac{\partial \tilde{\boldsymbol{X}}}{\partial t_0}|_{t_0 = t} + \frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \cdot \frac{d\boldsymbol{X}}{dt_0}. \tag{2.32}$$

Notice that in the macroscopic scale, the equality $t_0 \simeq t \simeq t_0 + \Delta t$ should be taken for granted. This is the RG equation underlying the derivation of the Fokker-Planck equation and also other transport equations including kinetic equations as will be shown in the next section.

3 Reduction of BBGKY hierarchy to Boltzmann equation

As is well known, Bogoliubov first derived the Boltzmann equation from the BBGKY hierarchy in his classic paper[5]. His derivation starts from an ansatz that the many particle distribution function depends on time only through the one-particle distribution function and uses a special perturbative expansion method. His approach is actually an application and generalization of the asymptotic theory by Krylov and Bogoliubov (KB) successful to non-linear oscillators[8]. In this section, we apply the RG method to derive the Boltzmann equation. We do not use that ansatz and start from the *naive* perturbation theory. We will see how the ansatz given by KB can be incorporated in the RG method. The importance of the initial condition again emerges. This implies that the appearance of a secular term[22] does not constitutes the final story for the derivation of the Boltzmann equation.

3.1 Derivation of the Boltzmann equation

Let us consider a system of N identical classical particles enclosed in a volume V. We shall adopt the notation of [26]; the i-th particle's phase space coordinate is represented by $x_i = (\boldsymbol{r}_i, \boldsymbol{p}_i)$. The Hamiltonian of the system reads

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} U(|\boldsymbol{r}_i - \boldsymbol{r}_j|). \tag{3.1}$$

We suppose that the potential U depends only on the relative distance of two particles and that its range d is much shorter than the mean free path l. The N-particle distribution function $f_N(x_1, \cdots, x_N, t)$ is normalized as

$$\int f_N(x_1, \cdots, x_N, t) \frac{\prod_{i=1}^{N} d\boldsymbol{r}_i d\boldsymbol{p}_i}{N!} = 1. \tag{3.2}$$

We define the s-particle distribution function by

$$f_s(x_1, \cdots x_s, t) = \int f_N(x_1, \cdots, x_N, t) \frac{dx_{s+1} \cdots dx_N}{(N-s)!}. \tag{3.3}$$

Then the normalization condition for f_s becomes

$$\int f_s(x_1, \cdots, x_s, t) dx_1 \cdots dx_s = \frac{N!}{(N-s!)} \simeq N^s, \tag{3.4}$$

from which we see that f_s is of s-th order in the particle density $n = \frac{N}{V}$. We assume that $n \ll 1$.

The kinetic equation for f_s is obtained by integrating the Liouville equation $\frac{d}{dt} f_N = 0$ over $x_{s+1}, \cdots, x_N$. Equations for f_1 and f_2 read

$$\frac{d}{dt} f_1(x_1, t) = (\frac{\partial}{\partial t} + iL_1^0) f_1(x_1, t) = -\int dx_2 L'_{12} f_2(x_1, x_2, t), \tag{3.5}$$

$$\frac{d}{dt} f_2(x_1, x_2, t) = (\frac{\partial}{\partial t} + iL_{12}) f_2(x_1, x_2, t) = -\int dx_3 (iL'_{13} + iL'_{23}) f_3(x_1, x_2, x_3, t) \tag{3.6}$$

where

$$L_{12} = L_1^0 + L_2^0 + L'_{12},$$

$$L_i^0 = -i \frac{\boldsymbol{p}_i}{m} \cdot \frac{\partial}{\partial \boldsymbol{r}_i}, \qquad L'_{ij} = i \frac{\partial U(\boldsymbol{r}_i - \boldsymbol{r}_j)}{\partial \boldsymbol{r}_j} \cdot (\frac{\partial}{\partial \boldsymbol{p}_j} - \frac{\partial}{\partial \boldsymbol{p}_i}). \tag{3.7}$$

These are the first two equations of the BBGKY hierarchy which is a series of equations relating the evolution of f_s to f_{s+1}. Our goal is to derive an equation (or equations) which captures the essence of the system's dynamics described by the BBGKY hierarchy. In the language of the the theory of dynamical systems[6], we wish to construct a low-dimensional invariant manifold in the (practically) infinite-dimensional functional space spanned by $\{f_s\}$ and derive the reduced equations of motion on it.

Whereas the Liouville equation or, equivalently, the BBGKY hierarchy describes microscopic collisions between particles in detail, what interests us is the macroscopic variation of the system caused by the accumulation of many collisions. More concretely, we wish to know the variation of the system over the space-time scale much longer than the collision radius and the collision time and much shorter than the mean free path and the mean free time. Such scale is called the *mesoscale*. The derivatives appearing in (3.5) and (3.6) are, so to speak, microscopic derivatives, while those appearing in kinetic equations are macroscopic derivatives. We must take into account their difference when deriving kinetic equations.

Following [1], suppose that we have found the solution to the BBGKY hierarchy $\{f_s(,t)\}$ up to an arbitrary time t_0. With the initial condition $\{f_s(,t_0)\}$ we try to solve (3.5) and (3.6) by the perturbative expansion in the density (virial expansion) to obtain a solution $\tilde{f}_s(t;t_0)$ around $t \sim t_0$. Recalling that f_s is of s-th order in the density, we expand as follows.

$$\begin{aligned}
\tilde{f}_1(x_1,t) &= \tilde{f}_1^0(x_1,t) + \tilde{f}_1^1(x_1,t) + \tilde{f}_1^2(x_1,t) + \cdots, && (3.8)\\
\tilde{f}_2(x_1,x_2,t) &= \tilde{f}_2^0(x_1,x_2,t) + \tilde{f}_2^1(x_1,x_2,t) + \cdots, && (3.9)\\
\tilde{f}_3(x_1,x_2,x_3,t) &= \tilde{f}_3^0(x_1,x_2,x_3,t) + \cdots, && (3.10)
\end{aligned}$$

where $\tilde{f}_i^j(x_1,\cdots,x_i,t)$ is of $(i+j)$-th order in the density. Substituting the above expansion in (3.5) and (3.6), we get

$$\frac{d}{dt}\tilde{f}_1^0(x_1,t) = 0, \tag{3.11}$$

$$\frac{d}{dt}\tilde{f}_2^0(x_1,x_2,t) = 0, \tag{3.12}$$

$$(\frac{\partial}{\partial t} + \frac{\boldsymbol{p}_1}{m}\cdot\frac{\partial}{\partial \boldsymbol{r}_1})\tilde{f}_1^1(x_1,t) = \int dx_2 \frac{\partial}{\partial \boldsymbol{r}_1}U(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\tilde{f}_2^0(x_1,x_2,t), \tag{3.13}$$

where we have dropped terms which result in the surface integral. We also expand the initial condition

$$\begin{aligned}
f_1(x_1,t_0) &= f_1^0(x_1,t_0) + f_1^1(x_1,t_0) + \cdots,\\
f_2(x_1,x_2,t_0) &= f_2^0(x_1,x_2,t_0) + \cdots. && (3.14)
\end{aligned}$$

Equation (3.11) and (3.12) are easily integrated:

$$\begin{aligned}
\tilde{f}_1^0(x_1,t) &= e^{-iL_1^0(t-t_0)}f_1^0(x_1,t_0),\\
\tilde{f}_2^0(x_1,x_2,t) &= e^{-iL_{12}(t-t_0)}f_2^0(x_1,x_2,t_0) = f_2^0(x_{10},x_{20},t_0), && (3.15)
\end{aligned}$$

where

$$x_{i0}(x_1,x_2,t,t_0), \quad i=1,2 \tag{3.16}$$

are positions and momenta of the particles 1 and 2 at time t_0 under the influence of the 2-body Hamiltonian

$$H^{(2)} \equiv \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + U(|\boldsymbol{r}_1-\boldsymbol{r}_2|). \tag{3.17}$$

The initial values $f_1^0(x_1,t_0)$ and $f_2^0(x_1,x_2,t_0)$ may be considered as the integration constants of the lowest-order equation. In the RG method as formulated in [19, 1], the integration constants will constitute the coordinates of the zeroth invariant manifold[6]. The decisive step of the present approach is to choose the initial condition as follows

$$f_2^0(x_1,x_2,t_0) = f_1^0(x_1,t_0)f_1^0(x_2,t_0), \tag{3.18}$$

irrespective of the distance between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The underlying picture of this choice is that the system is so dilute that the two particles at an arbitrary time t_0 are most probably located at distance much longer than the collision radius d, so that the correlation of the two particles is negligible and f_2 can be set to the product of one-particle distribution functions. We remark that a probabilistic nature enters at this point[27].

The integration of (3.13) from t_0 to t with $\frac{l}{v} \gg t-t_0$ (v is the average velocity), which implies that $t-t_0$ is small in the macroscopic scale, gives

$$\begin{aligned}\tilde{f}_1^1(x_1,t) &= e^{-iL_1^0(t-t_0)}f_1^1(x_1,t_0) \\ &\quad + \int_{t_0}^t dt' e^{-iL_1^0(t-t')}\int dx_2 \frac{\partial}{\partial \boldsymbol{r}_1}U(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\cdot\frac{\partial}{\partial \boldsymbol{p}_1}f_1^0(x'_{10},t_0)f_1^0(x'_{20},t_0),\end{aligned} \tag{3.19}$$

where we have used (3.15) and (3.18), and x'_{10} and x'_{20} are given by (3.16) with the replacement $t \to t'$. We remark that the condition ($\frac{l}{v} \gg t-t_0$) is also required for the expansion in the density to be valid [5]. In (3.19), only $\boldsymbol{r}_2$ for $|\boldsymbol{r}_1-\boldsymbol{r}_2| \le d$ contributes to the integral. In this region, we can write

$$\boldsymbol{r}'_{i0} \sim \boldsymbol{r}_i - \frac{\boldsymbol{p}_{i0}}{m}(t'-t_0). \tag{3.20}$$

for a microscopically large period $t'-t_0 \gg \frac{d}{v}$. Here we have neglected vectors whose magnitudes are of order d. Then the perturbative solution *in the mesoscopic regime* $\frac{l}{v} \gg t-t_0 \gg \frac{d}{v}$ is

$$\begin{aligned}\tilde{f}_1(x_1,t) &= \tilde{f}_1^0(x_1,t)+\tilde{f}_1^1(x_1,t) \\ &= e^{-iL_1^0(t-t_0)}f_1^0(x_1,t_0) + \int_{t_0}^t dt' e^{-iL_1^0(t-t')}\int dx_2 \frac{\partial}{\partial \boldsymbol{r}_1}U(|\boldsymbol{r}_1-\boldsymbol{r}_2|) \\ &\quad \cdot\frac{\partial}{\partial \boldsymbol{p}_1}f_1^0(\boldsymbol{r}_1-\frac{\boldsymbol{p}_{10}}{m}(t'-t_0),\boldsymbol{p}_{10},t_0)f_1^0(\boldsymbol{r}_2-\frac{\boldsymbol{p}_{20}}{m}(t'-t_0),\boldsymbol{p}_{20},t_0).\end{aligned} \tag{3.21}$$

Note that $\boldsymbol{p}_{i0} = \boldsymbol{p}'_{i0}$: The magnitudes of $\frac{l}{v}$ and $\frac{d}{v}$ are of course different for different systems. For a dilute gas system, typical values are $10^{-8} \sim 10^{-9}$s and $10^{-12} \sim 10^{-13}$s, respectively. The second term of the r.h.s. of (3.22) is the *secular term*. Indeed, it can be shown that in the spatially homogeneous case it is proportional to $t-t_0$ [23]. Accordingly, we have chosen $f_1^1(,t_0)$ to be zero following the prescription given in [1]. The RG equation reads

$$\frac{\partial}{\partial t_0}\tilde{f}_1(x_1,t)\Big|_{t=t_0} = 0, \tag{3.22}$$

$$\begin{aligned}\Rightarrow \quad \frac{\partial}{\partial t}f_1^0(x_1,t) &+ \frac{\boldsymbol{p}_1}{m}\cdot\frac{\partial}{\partial \mathbf{r}_1}f_1^0(x_1,t) \\ &= \int dx_2 \frac{\partial}{\partial \boldsymbol{r}_1}U(|\boldsymbol{r}_1-\mathbf{r}_2|)\cdot\frac{\partial}{\partial \boldsymbol{p}_1}f_1^0(\boldsymbol{r}_1,\boldsymbol{p}_{10},t)f_1^0(\boldsymbol{r}_1,\boldsymbol{p}_{20},t),\end{aligned} \tag{3.23}$$

In (3.22) we have imposed that $t = t_0$ although the expression (3.22) is valid for $t - t_0 \gg \frac{d}{v}$. This manipulation can be justified by the same logic given in the last part in §2 and will appear also in the case of field theory discussed in the following section: The t-derivative is the microscopic derivative and the t_0-derivative is the macroscopic one. Through the RG equation, we can *automatically* go over to the mesoscopic physics from the microscopic physics. Thus the mesoscopic nature of the Boltzmann equation is transparent in our approach.

(3.23) is the kinetic equation we have been seeking for. In the language of the RG method, it is the renormalization group equation describing the slow motion on the invariant manifold with the coordinate $f_1^0(x_1, t)$ [22]. To obtain the usual Boltzmann equation which contains the gain minus loss term, we have to manipulate the r.h.s. ignoring the spatial dependence. The result is

$$\frac{\partial}{\partial t} f_1^0(x_1, t) \quad + \quad \frac{\boldsymbol{p}_1}{m}$$

to elucidate the general structure of the reduction of the dynamical equations in the hierarchy of the evolution equations. We have noticed that the significance of the choice of the initial value on the attractive manifold which is also an invariant manifold[6] in deriving kinetic equations is fully recognized and emphasized by Bogoliubov[5], Lebowitz[14], Kubo[13] and Kawasaki[15], for instance. The notion of coarse-grained time derivative was also noticed by Mori[10] and others[11, 12]. Our point was that these basic ingredients naturally appear in the RG-theoretical derivation of kinetic equations when properly formulated so as to respect the role played by the initial condition as formulated in [19, 1]. This report is based on a part of [28], in which a detailed account of this report and applications to other kinetic equations including a quantum field theoretical model may be seen.

References

[1] S.-I. Ei, K. Fujii and T. Kunihiro, *Ann. Phys.* **280** (2000), 236.

[2] N. Goldenfeld, O. Martin and Y. Oono,*J. Sci. Comp.* **4**(1989),4; N. Goldenfeld, O. Martin, Y. Oono and F. Liu,*Phys. Rev. Lett.* **64** (1990), 1361; N. D. Goldenfeld, "Lectures on Phase Transitions and the Renormalization Group,"Addison-Wesley, Reading, Mass., 1992. L. Y. Chen, N. Goldenfeld, Y. Oono and G. Paquette, *Physica* **A 204**(1994)111. G. Paquette, L. Y. Chen, N. Goldenfeld and Y. Oono,*Phys. Rev. Lett.* **72**(1994)76; L. Y. Chen, N. Goldenfeld and Y. Oono,*Phys. Rev. Lett.***73**(1994)1311; L. Y. Chen, N. Goldenfeld and Y. Oono,*Phys. Rev.* **E 54** (1996), 376. Attempts to straighten the logic and to give a proper interpretation of what is done by Illinois group may be seen in [17]; see also [19]. Most of the references on the RG method published until 1999 may be found in [1].

[3] L.E. Reichl, *Modern Course in Statistical Physics 2nd ed.* (John Wiley and Sons, 1998).

[4] L. Boltzmann, "Lectures on Gas Theory" University of California Press, Berkeley, 1964.

[5] N. N. Bogoliubov, in "Studies in Statistical Mechanics, vol. 1," edited by J. de Boer and G. E. Uhlenbeck, North-Holland, Amsteredam, 1962.

[6] See for example, J. Guckenheimer and P. Holmes, *Nonlinear Oscillators, Dynamical Systems, and Bifurcations of Vector Fields*,(Springer-Verlag, 1983);
S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, (Springer-Verlag, New York, 1990);
J. D. Crawford, *Rev. Mod. Phys.* **63** (1991), 991.

[7] S. Chapman and T.G. Cowling, "The Mathematical Theory of Non-Uniform Gases" (3rd ed.) Cambridge U.P., 1970.

[8] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, 1961.

[9] R. L. Stratonovich,*Topics in the Theory of Random Noise, vol. 1 and 2*, (Gordon and Breach, 1963). C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd ed. (Springer, 1985). H. Risken, *The Fokker-Planck Equation— Methods of Solution and Applications*, 2nd ed. (Springer, 1989). N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, rev. and enlarged ed. (North-Holland, 1992).

[10] H. Mori, J. Phys. Soc. Jpn. **11** (1956), 1029; Phys. Rev. **112** (1958), 1829; **115** (1959), 298.

[11] J. G. Kirkwood, *J. Chem. Phys.* **14** (1946), 180

[12] I. Ojima, J. Stat. Phys. **56**(1989), 203.

[13] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II Non-equilibrium Statistical Mechanics* (Springer-Verlag, 1985).

[14] J. L. Lebowitz, *Physica***A** (1993),1.

[15] K. Kawasaki, *Non-equilibrium and Phase Transition — Statistical Physics in Meso Scales*, chap. 7 (Asakura Shoten, 2000), in Japanese.

[16] E.C.G. Stueckelberg and A. Petermann, *Helv. Phys. Acta* **26**(1953), 499;
M. Gell-Mann and F. E. Low, *Phys. Rev.* **95** (1953), 1300.
K. Wilson, *Phys. Rev.* **D3**(1971), 1818, S.Weinberg, *in*" Asymptotic Realms of Physics" (A. H. Guth et al. Ed.), MIT Press, 1983.
As review articles, S.K. Ma, "Modern Theory of Critical Phenomena", W. A. Benjamin, New York, 1976. J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1989.
D. V. Shirkov, `hep-th/9602024`; `hep-th/9903073`.

[17] G.C. Paquette, *Physica* **A276** (2000), 122;
B. Mudavanhu and R.E. O'Malley, *Studies in Applied Mathematics***107**(2001), 63.

[18] J. Bricmont and A. Kupiainen,*Commun. Math. Phys.* **150** (1992), 193; J. Bricmont, A. Kupiainen and G. Lin, *Cooun. Pure. Appl. Math.* **47**(1994), 893; J. Bricmont and A. Kupiainen, `chao-dyn/9411015`.

[19] T. Kunihiro,*Prog. Theor. Phys.* **94**(1995), 503; (E) ibid., **95**(1996)835; *Jpn. J. Ind. Appl. Math.* **14** (1997), 51, *Prog. Theor. Phys.* **97**(1997),179; *Phys. Rev.* **D57** (1998), R2035; *Prog. Theor. Phys. Suppl.* **131** (1998), 459; see also `patt-sol/979003`.

[20] K.G.Wilson and M.E.Fisher,*Phys. ReV. Lett.* **28** (1972),240; K.G. Wilson, *Phys. Rev. Lett.* **28**(1972), 548; K.G. Wilson and J. Kogut,*Phys. Rep.* **12C** (1974), 75.

[21] F. Wegner and A. Houghton,*Phys. Rev.* **A8** (1973),401.

[22] O. Pashko and Y. Oono, *Int. J. Mod. Phys.* **B14** (2000), 555.

[23] M. S. Green, *J. Chem. Phys.* **25** (1956), 836.

[24] G. I. Barlenblatt, *Similarity, Self-Similarity, and Intermediate Asymptotics* (Consultants Bureau, N.Y., 1979).

[25] R. Kubo, *J. Math. Phys.***4**(1963), 174;
R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II Nonequilibrium Statistical Mechanics* (Springer-Verlag, 1985).

[26] D. N. Zubarev, V. G. Morozov and G. Ropke, *Statistical Mechanics of Nonequilibrium Processes Vol.1*,(Akademie Verlag, Berlin, 1996).

[27] E.M. Lifshitz and L.P. Pitaevskii, *Physical Kinetics* , (Butterworth-Heinemann, 1981)

R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics*, (Wiley, New York, 1975).
L.E. Reichl, *Modern Course in Statistical Physics, chap. 11*, 2nd ed. (John Wiley and Sons, 1998).

[28] Y. Hatta and T. Kunihiro, `hep-th/0108159`.