Renormalization Group Flow of the Hierarchical Two–Dimensional Coulomb Gas

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Abstract

In this lecture we examine a nonlinear parabolic differencial equation associated with the renormalization group transformation of the hierarchical two–dimensional Coulomb gas. We review some of the results recently published in [GM]. The solution of the initial value problem is shown to converge, as $t \to \infty$, to one of the countably infinite equilibrium solutions. The j–th nontrivial equilibrium solution bifurcates from the trivial solution at $\alpha = 2/j^2$, $j = 1, 2, \ldots$, where α is a parameter related to the inverse temperature. We here describe these equilibrium solutions and present their local stability analysis for all $\alpha > 0$. Our results ruled out the existence of an intermediate phase between the plasma and the Kosterlitz–Thouless phase, at least in the hierarchical model considered.

1 Introduction

We consider the quasilinear parabolic differential equation

$$u_t - \alpha(u_{xx} - u_x^2) - 2u = 0 \tag{1.1}$$

on $\mathbb{R}_+ \times (-\pi, \pi)$ with $\alpha > 0$, $u(t, 0) = 0$ and periodic boundary conditions.

The following has been proven in [GM].

1. The initial value problem is well defined in a appropriated function space $\mathcal{B}$ and the solution exists and is unique for all $t > 0$;

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2. As $t \to \infty$, the solution converges in $\mathcal{B}$ to one of the infinitely many (equilibrium) solutions ϕ of
$$\alpha\left(\phi'' - (\phi')^2\right) + 2\phi = 0$$
with $\phi(-\pi) = \phi(\pi)$ and $\phi'(-\pi) = \phi'(\pi)$;

3. For $\alpha > 2$, $\phi_0 \equiv 0$ is the (globally) asymptotically stable solution of PDE;

4. For $\alpha < 2$ such that $2/\left(k+1\right)^2 \leq \alpha < 2/k^2$ holds for some $k \in \mathbb{N}_+$, there exist $2k$ non–trivial equilibria solutions $\phi_1^{\pm}, \ldots, \phi_k^{\pm}$;

5. For $j \geq 1$, $\phi_j^{\pm}$ have a $(j-1)$ –dimensional unstable manifold $\mathcal{M}_j \subset \mathcal{B}$ so $\phi_j^{\pm}$ are more stable than $\phi_{j'}^{\pm}$ if $j < j'$. Moreover, there exists a dense set of initial conditions in $\mathcal{B}$ such that $\phi_1^{\pm}$ (ϕ_1^{-} is not physically admissible) are the non–trivial stable solution for all $\alpha < 2$.

Chaffe–Infant's geometric analysis [CI] of a class of semilinear parabolic PDE, whose prototype is
$$u_t - \alpha\left(u_{xx} - u^3\right) - 2u = 0\,,$$
with $u(t,0) = u(t,\pi) = 0$ (see e.g. [H]), is thus extended to equation (1.1). In the present lecture we address only itens 1, 4 and the local stability analysis.

The above scenario can be state as follows: there exist a sufficient large ball $\mathcal{B}_0 \subset \mathcal{B}$ about the origin such that, if $u(t,\mathcal{B}_0)$ denotes the set of points reached at time t starting from any initial function in $\mathcal{B}_0$, then the invariant set $\bigcap_{t\geq 0} u(t,\mathcal{B}_0)$ coincide with the k–dimensional unstable manifold $\mathcal{M}_k$ provided $2/(k+1)^2 \leq \alpha < 2/k^2$.

The initial value problem above describes the renormalization group (RG) flow of the effective potential in the two–dimensional hierarchical Coulomb system and the stationary solutions $\left\{\phi_j^{+}\right\}$, the fixed points of RG, contain informations on its critical phenomena.

Gallavotti and Nicoló[GN] have conjectured a sequence of "intermediate" phase transitions from the plasma phase ($\alpha \leq \alpha_1 = 1$) to the multipole phase ($\alpha \geq \alpha_\infty = 2$) with some partial screening taking place when the inverse temperature $\alpha = \beta/4\pi$, decreases from 2 to 1.

The Kosterlitz–Thouless phase (multipole phase) was established by Fröhlich–Spencer[FS] and extended up to $\beta = 8\pi$ by Marchetti and Klein[MK]. Debey screening (plasma phase) was only proved for sufficiently small $\beta << 4\pi$[BF]. The excursion on the region $[4\pi, 8\pi]$ has begun with the work by Benfatto, Gallavotti and Nicoló[BGN] on the ultraviolet collapses of neutral clusters in the Yukawa gas. Although a conclusive answer to Gallavotti–Nicoló's conjecture seems unprovable to appear sooner, the scenario of an intermediate phase has been contested by Fisher *et al* [FLL] based on Debye–Hückel–Bjerrum theory and by Dimock and Hurd[DH] who have reinterpreted the ultraviolet collapses in the Yukawa gas.

The Kosterlitz–Thouless phase is manifested in the hierarchical model as a bifurcation from the trivial solution[MP]. Our results rule out the existence of further phase transitions since no other bifurcation occurs from the stable solution.

2 The RG flow equation

The equilibrium Gibbs measure $\mu_\Lambda : \mathbb{Z}^\Lambda \longrightarrow \mathbb{R}_+$ of a hierarchical Coulomb system in $\Lambda \subset \mathbb{Z}^2$ is given by

$$\mu_\Lambda(q) := \frac{1}{\Xi_\Lambda} F(q)\, e^{-\beta E(q)}$$

where β is the inverse temperature,

$$E(q) = \frac{1}{2} \sum_{x,y\in\Lambda} q(x)\, V(x,y)\, q(y)$$

is the energy of a configuration q,

$$V(x,y) = -\frac{1}{2\pi} \ln d_h(x,y)$$

is the hierarchical Coulomb potential,

$$F(q) = \prod_{x\in\Lambda} \lambda(q(x))$$

is an "a priori" weight and

$$\Xi_\Lambda = \sum_{q\in\mathbb{Z}^\Lambda} F(q)\, e^{-\beta E(q)}$$

is the grand partition function.

In the hierarchical model, the Euclidean distance $|x-y|$ is replaced by the hierarchical distance

$$d_h(x,y) := L^{N(x,y)}$$

where

$$N(x,y) := \inf \left\{ N \in \mathbb{N}_+ : \left[\frac{x}{L^N}\right] = \left[\frac{y}{L^N}\right] \right\} ,$$

$L > 1$ is an integer and $[z] \in \mathbb{Z}^2$ has components the integer part of the components of $z \in \mathbb{R}^2$.

Let $\Lambda = \Lambda_N = [-L^N, L^N - L^{N-1}]^2 \cap \mathbb{Z}^2$, $N > 1$, and define for each configuration $q \in \mathbb{Z}^\Lambda$ the block configuration $q^1 : \Lambda_{N-1} \longrightarrow \mathbb{Z}$

$$q^1(x) = \sum_{\substack{0\le y_i<L \\ i=1,2}} q(Lx+y)\,.$$

The renormalization group transformation $\mathcal{R}$ acts on the space of Gibbs measures

$$\begin{aligned} \mu^1_{\Lambda_{N-1}}(q^1) &= [\mathcal{R}\mu_{\Lambda_N}](q^1) = \sum_{\substack{q\in\mathbb{Z}^{\Lambda_N}: \\ q^1 \text{fixed}}} \mu_{\Lambda_N}(q) \\ &= \frac{1}{\Xi^1_{\Lambda_{N-1}}} F^1(q^1)\, e^{-\beta E(q1)} \end{aligned}$$

where

$$F^1(q^1) = \prod_{x \in \Lambda_{N-1}} \lambda^1(q^1(x))$$

with

$$\lambda^1(p) = L^{-\alpha p^2} (\underbrace{\lambda \star \lambda \star \cdots \star \lambda}_{L^2-\text{times}})(p) \tag{2.2}$$

with $\alpha = \beta/4\pi$ and $(\lambda \star \varrho)(p) = \sum_{q \in \mathbb{Z}} \lambda(p-q)\, \varrho(q)$. Note that $\Xi_{\Lambda_N}(\lambda) = \Xi_{\Lambda_{N-1}}(\lambda^1)$.

Applying the convolution theorem and Poisson formula to equation (2.2), give

$$\widetilde{\lambda}^1(\varphi) = \widetilde{r\lambda}(\varphi) = \left(\nu * \widetilde{\lambda}^{L^2}\right)(\varphi)$$

where $\widetilde{\lambda}(\varphi) = \displaystyle\sum_{q \in \mathbb{Z}} \lambda(q)\, e^{iq\varphi}$ and

$$(\nu * f)(\varphi) = L^{\alpha \ln L \left(d^2/d\varphi^2\right)} f(\varphi)$$

is a convolution by a Gaussian measure with mean zero and variance $\beta \ln L/(2\pi)$.

For $t := n \ln L$, let us define

$$u(t,x) = -\ln \widetilde{\lambda^n}(x)$$

with $\lambda^n = r^n \lambda$. Taking the limit $L \downarrow 1$ together with $n \to \infty$ maintaining t fixed, we have

$$u_t = \alpha \left(u_{xx} - u_x^2\right) + 2u \ .$$

3 Existence, uniqueness and continuous dependence

To avoid the appearance of zero modes upon linearization, we differentiate the PDE (1.1) with respect to x and consider the equation for $v = u_x$,

$$v_t - \alpha\,(v_{xx} - 2v\, v_x) - 2v = 0$$

with $v\,(t,-\pi) = v\,(t,\pi)$ and $v_x\,(t,-\pi) = v_x\,(t,\pi)$, in the subspace of odd functions and initial value $v(0,\cdot) = v_0$. Note the equation preserves this subspace.

The standard initial condition $u_0(x) = z\,(1 - \cos x)$, corresponding to the standard gas with particle activity z, satisfies $u(0) = u_0'(\pi) = u_0'(-\pi) = 0$. Note the condition $u\,(s,0) = 0$ is already imposed for all s if $u(s,x) = \displaystyle\int_0^x v(s,y)\, dy$.

The boundary and initial value problem can be written as an ordinary differential equation

$$\frac{dz}{dt} + Az = F(z) \tag{3.3}$$

in a Banach space $\mathcal{B}$ where

$$Az = -\alpha z'' - 2z \qquad \text{and} \qquad F(z) = -2\alpha z_x z\,,$$

with initial value $z(0) = z_0$.

The linear operator A is defined on the space $\mathcal{C}^2_{\rm o,p}$ of smooth odd and periodic real–valued functions in $[-\pi, \pi]$, with inner product $(f, g) := \displaystyle\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$, and since $(f, Ag) = (Af, g)$, it may be extended to a self–adjoint operator in $L^2_{\rm o,p}\,(-\pi, \pi)$. The domain $D(A)$ of A is

$$D(A) = \left\{f \in L^2_{\rm o,p}\,(-\pi, \pi) : Af \in L^2_{\rm o,p}\,(-\pi, \pi)\right\}$$

and the spectrum of A,

$$\sigma(A) = \left\{\lambda_n = \alpha n^2 - 2,\, n \in \mathbb{N}_+\right\}$$

consists of simple eigenvalues with corresponding eigenfunctions $\phi_n(x) = (1/\pi)^{1/2} \sin nx$.

Let A_1 denote a positive definite linear operator given by A if $\alpha > 2$ and $A + aI$ for some $a > 2 - \alpha$, otherwise.

The operator A generates an analytic semi–group $T(t) = e^{-tA}$. Given $\gamma \geq 0$, $A_1^{-\gamma}$ is a bounded operator (compact if $\gamma > 0$) with $A_1^{-1/2}\,(d/dx)$ and $(d/dx)\,A_1^{-1/2}$ bounded in the $L^2_{\rm o,p}$ norm. In addition, for $\gamma > 0$, A_1^{γ} is closely defined with the inclusion $D(A_1^{\gamma}) \subset D(A_1^{\tau})$ if $\gamma > \tau$.

It thus follows the basic estimate

$$\left\|A_1^{\gamma} e^{-tA_1}\right\| \leq \frac{C_\gamma}{t^\gamma}\, e^{-ct} \tag{3.4}$$

holds for $0 < \gamma < 1$, $t > 0$ where $C_\gamma = \displaystyle\sup_{n\in\mathbb{N}_+} \left|(t\lambda_n)^\gamma\; e^{-t\lambda_n}\right| \leq \left(\frac{\gamma}{e}\right)^\gamma$.

Following Picard's method, the integral equation

$$z(t) = e^{-tA} z_0 + \int_0^t e^{-(t-s)A}\, F\left(z(s)\right)\, ds \tag{3.5}$$

solves the initial value problem provided $F\left(z(s)\right)$ is shown to be locally Hölder continuous on the interval $0 \leq t < T$.

Let $\mathcal{B}^\gamma = D(A^\gamma)$, $\gamma \geq 0$, denote the Banach space with the graph norm

$$\|f\|_\gamma := \|A^\gamma f\|$$

$F : \mathcal{B}^\gamma \longrightarrow L^2_{\rm p,o}\,(-\pi, \pi)$ is said to be locally Lipschtzian if there exist $U \subset \mathcal{B}^\gamma$ and a finite constant L such that

$$\|F(z_1) - F(z_2)\| \leq L\, \|z_1 - z_2\|_\gamma \tag{3.6}$$

holds for any $z_1,\, z_2 \in U$.

Theorem 3.1 *The initial value problem has a unique solution $z(t)$ for all $t \in \mathbb{R}_+$ with $z(0) = z_0 \in \mathcal{B}^{1/2}$. In addition, if $\|z(t)\|_{1/2}$ is bounded as $t \to \infty$, the trajectories $\{z(t)\}_{t \geq 0}$ is in a compact set in $\mathcal{B}^{1/2}$.*

Proof. The proof is divided into four parts. First, $F(z(t))$ is shown to be Hölder continuous under Lipschtz condition establishing the equivalence between the integral equation the initial problem. Second, the Banach fixed point theorem is used to show the existence of a unique solution $z(t)$ for $0 \leq t \leq T$. Hence, using an extension of Gronwell lemma, the solution $z(t)$ is extended to all $t \in \mathbb{R}_+$by a compactness argument. Finally, assuming that $\|z(t)\|_{1/2}$ stays bounded for all $t > 0$, the proof is concluded by the domain inclusion.

Skipping *Part I* on Hölder continuity (see [GM]), we go to *Part II.*

Local existence. Let $V = \left\{z \in \mathcal{B}^{1/2} : \|z - z_0\| \leq \varepsilon\right\}$ be an ε–neighborhood and let L be the Lipschitz constant of F on V. We set $B = \|F(z_0)\|$ and let T be a positive number such that

$$\left\|\left(e^{-hA} - I\right) z_0\right\|_{1/2} \leq \frac{\varepsilon}{2} \tag{3.7}$$

with $0 \leq h \leq T$ and

$$C_{1/2}\,(B + L\varepsilon) \int_0^T s^{-1/2}\, e^{-cs}\, ds \leq \frac{\varepsilon}{2} \tag{3.8}$$

hold.

Let $\mathcal{S}$ denote the set of continuous functions $y : [t_0, t_0 + T] \longrightarrow \mathcal{B}^{1/2}$ such that $\|y(t) - z_0\| \leq \varepsilon$. Provided with the sup–norm

$$\|y\|_T := \sup_{t_0 \leq t \leq t_0 + T} \|y(t)\|_{1/2}$$

$\mathcal{S}$ is a complete metric space.

Defining $\Phi[y] : [t_0, t_0 + T] \longrightarrow \mathcal{B}^{1/2}$ for each $y \in \mathcal{S}$ by

$$\Phi[y](t) = e^{-(t-t_0)A} z_0 + \int_{t_0}^{t} e^{-(t-s)A}\, F\left(y(s)\right)\, ds\,,$$

we now show that, under the conditions (3.7) and (3.8), $\Phi : \mathcal{S} \longrightarrow \mathcal{S}$ is a strict contraction. Using

$$\|F(y(t))\| \leq \|F(y(t)) - F(z_0)\| + \|F(z_0)\| \leq L\,\|y(t) - z_0\|_{1/2} + B \leq L\varepsilon + B$$

and (3.4), we have

$$\begin{aligned} \|\Phi[y](t) - z_0\|_{1/2} &\leq \left\|\left(e^{-(t-t_0)A} - I\right) z_0\right\|_{1/2} + \int_{t_0}^{t_0+T} \left\|A^{1/2} e^{-(t-s)A}\right\|\, \|F(y(s))\|\, ds \\ &\leq \frac{\varepsilon}{2} + C_{1/2}\,(B + L\varepsilon) \int_0^T s^{-1/2}\, e^{-cs}\, ds \leq \varepsilon \end{aligned}$$

and since $\Phi[y]$ is continuous, $\Phi[y] \in \mathcal{S}$.

Analogously, from (3.6) and (3.8), for any $y, w \in \mathcal{S}$

$$
\begin{aligned}
\|\Phi[y](t) - \Phi[w](t)\|_{1/2} &\leq \int_{t_0}^{t_0+T} \left\|A^{1/2} e^{-(t-s)A}\right\| \, \|F(y(s)) - F(w(s))\| \, ds \\
&\leq C_{1/2} L \int_0^T s^{-1/2} \, e^{-cs} \, ds \, \|y - w\|_T \leq \frac{1}{2} \|y - w\|_T
\end{aligned}
$$

holds uniformly in $t \in [t_0, t_0 + T]$ concluding our claim.

By the contraction mapping theorem, Φ has a **unique** fixed point z in $\mathcal{S}$ which is the continuous solution of the integral equation (3.5) on $(t_0, t_0 + T)$ and, by *Part I*, is the solution of (3.3) in the same interval with $z(t_0) = z_0 \in \mathcal{B}^{1/2}$.

We shall briefly sketch *Part III (for details see [GM])*.

Global existence. One can define an **open maximal interval** $I_{\max} = (t_-, t_+)$ (containing the origin), where the solution $z(t)$ of (3.3) is uniquely given by patching together the solutions $z_j(t)$ on intervals I_j with $z_j(t_j) = z_{0,j}$. By construction, there is no solution to (3.3) on (t_0, t') if $t' > t_+$. Therefore, either $t_+ = \infty$, or else there exist a sequence $\{t_n\}_{n \in \mathbb{N}_+}$, with $t_n \to t_+$ as $n \to \infty$ such that $z(t_n)$ tend to the boundary ∂U of the compact set U where (3.6) holds.

It thus follows that, if t_+ is finite, the solution $z(t)$ blows–up at finite time. In what follows we show that $\|z(t)\|_{1/2}$ remains finite for all $t > t_0$ and this implies global existence of $z(t)$. Let us begin with the following generalization of the Gronwall inequality (for proof, see Lemma 7.1.1 in [H]).

Lemma 3.2 (Gronwall) *Let ξ and γ be numbers and let θ and ζ be non–negative continuous functions defined in a interval $I = (0, T)$ such that $\xi \geq 0$, $\gamma > 0$ and*

$$\zeta(t) \leq \theta(t) + \xi \int_0^t (t - \tau)^{\gamma - 1} \, \zeta(\tau) \, d\tau \, . \tag{3.9}$$

Then

$$\zeta(t) \leq \theta(t) + \int_0^t E'_\gamma(t - \tau) \, \theta(\tau) \, d\tau \tag{3.10}$$

holds for $t \in I$, where $E'_\gamma = dE_\gamma/dt$,

$$E_\gamma(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\gamma + 1)} \left(\xi \Gamma(\gamma) \, t^\gamma\right)^n$$

and $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function. In addition, if $\theta(t) \leq K$ for all $t \in I$, then

$$\zeta(t) \leq K \, E_\gamma(t) \, \leq K' \, e^{\xi \Gamma(\gamma) T} \tag{3.11}$$

holds for some finite constant K'.

Taking the graph norm of (3.5), we have in view of (3.4) and (3.11)

$$\begin{aligned}
\|z(t)\|_{1/2} &\leq \left\|e^{-(t-t_0)A} z_0\right\|_{1/2} + L \int_{t_0}^{t} \left\|A^{1/2} e^{-(t-s)A}\right\| \, \| z(s)\|_{1/2} \, ds \\
&\leq C \|z_0\|_{1/2} + L \int_{t_0}^{t} (t-s)^{-1/2} \, \| z(s)\|_{1/2} \, ds \qquad (3.12) \\
&\leq C \, \exp\left(LC_{1/2}\sqrt{\pi} t\right) \, \|z_0\|_{1/2} \, ,
\end{aligned}$$

which is finite for any $t \in \mathbb{R}_+$.

Compact trajectories. Since $\mathcal{B}^{\gamma} \subset \mathcal{B}^{1/2}$ has compact inclusion if $1/2 < \gamma < 1$ [H], it suffices to show that $\|z(t)\|_{\gamma}$ remains bounded as $t \to \infty$. The hypothesis $\|z(t)\|_{1/2} < \infty$ combined with (3.6) implies the existence of $C' < \infty$ such that, analogously as in (3.12),

$$\begin{aligned}
\|z(t)\|_{\gamma} &\leq \left\|e^{-tA} z_0\right\|_{\gamma} + \int_{0}^{t} \left\|A^{\gamma} e^{-(t-s)A}\right\| \, \|F\left(z(s)\right)\| \, ds \\
&\leq C_{\gamma-1/2} \, t^{1/2-\gamma} \, e^{-ct} \, \|z_0\|_{1/2} + C' \, C_{\gamma} \int_{0}^{t} (t-s)^{-\gamma} \, e^{-c(t-s)} \, ds \, ,
\end{aligned}$$

which is bounded for $t > 0$ provided $c > 0$ (i.e. $\inf_{\lambda} \sigma(A) > 0$). Although the spectrum of A is not positive if $\beta \leq 8\pi$, we shall see in Section 5 that A in the integral equation (3.5) can be replaced by a positive linear operator L.

This concludes the proof of Theorem 3.1.

□

We may also consider the dependence of z with respect to the parameter α. The next statement is a corollary of the above analysis.

Theorem 3.3 *The solution $z(t) : \mathbb{R}_+ \times \mathcal{B}^{1/2} \longrightarrow \mathcal{B}^{1/2}$ to the initial value problem as a function of the bifurcation parameter α and the initial value z_0 is continuous.*

4 Equilibrium Solutions

The equilibrium ordinary differential equation

$$\alpha \left(\psi'' - 2\psi\psi'\right) + 2\psi = 0 \qquad (4.13)$$

with periodic conditions $\psi(-\pi) = \psi(\pi)$ and $\psi'(-\pi) = \psi'(\pi)$, can be written as

$$\left\{ \begin{array}{lcl} w' &=& 2p\left(w - \alpha^{-1}\right) \\ p' &=& w \, , \end{array} \right. \qquad (4.14)$$

by setting $p = \psi$ and $w = \psi'$.

We give a qualitative and quantitative description of the solutions in the phase space $\mathbb{R}^2$ and study their implications for the equilibrium solutions.

Theorem 4.4 *The equilibrium equation has two distinct regimes separated by $\alpha = 2$. For $\alpha \geq 2$, $\psi_0 \equiv 0$ is the unique solution. For $\alpha < 2$ such that $2/(k+1)^2 \leq \alpha < 2/k^2$ holds for some $k \in \mathbb{N}_+$, there exist $2k$ non–trivial solutions ψ_j^+, ψ_j^- , $j = 1, \ldots, k$, with fundamental period $2\pi/j$ and $\psi_j^-(x) = \psi_j^+(x+\pi)$. Moreover, each pair of non–trivial solutions are bifurcating branches from the trivial solution ψ_0 at $\alpha_j = 2/j^2$ with $\lim_{\alpha \uparrow \alpha_j} \psi_j^\pm = 0$.*

In the phase space, these solutions $\left(\psi_j', \psi_j\right)$, are closed orbits around $(0,0)$ whose distance from the origin increases monotonically as α decreases. Numerical computations indicate that these orbits approach rapidly to the open orbit $\{(\alpha^{-1}, \alpha^{-1}x)\,,\, x \in \mathbb{R}\}$ from the left as $\alpha \to 0$.

The vector field $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$(w,p) \longrightarrow f(w,p) = \left(2p(w-\alpha^{-1}), w\right) ,$$

defines a smooth autonomous dynamical system. It thus follows from Piccard's theorem that there exist a unique solution $(w(x), p(x))$ of this system, globally defined in $\mathbb{R}^2$, with $(w(0), p(0)) = (w_0, p_0)$. As a consequence, the phase space $\mathbb{R}^2$ is foliated by non–overlapping orbits

$$\gamma_P = \{(w(x), h(x)) : x \in \mathbb{R} \text{ and } P = (w(0), p(0))\}$$

which passes by $P = (w_0, p_0) \in \mathbb{R}^2$ at $x = 0$.

By the chain rule, the system can be written as

$$\frac{dp}{dw} = \frac{w}{2p\,(w-\alpha^{-1})} \tag{4.15}$$

provided $\alpha w \neq 1$. The trajectories γ_{w_0}, obtained by integrating (4.15) with initial point $P = (w_0, 0)$,

$$p^2 = w - w_0 + \alpha^{-1} \ln\left(\frac{1-\alpha w}{1-\alpha w_0}\right)$$

are portrayed in Figure 1.

Proof of Theorem 4.4. By fixing the period T of an orbit γ_{w_0} to be 2π, the label w_0 becomes dependent on the parameter α. Let $T = T(\alpha, w_0)$ denote the period of the dynamical system with initial value $(w_0, 0)$:

$$T = \int_{\gamma_{w_0}} dx = 2\int \frac{dp}{w} ,$$

We set

$$G_j = T - \frac{2\pi}{j}$$

and note that $G_j : \mathcal{D} = \{(\alpha, w_0) \in \mathbb{R}_+ \times \mathbb{R}_+ : \alpha w_0 \leq 1\} \longrightarrow \mathbb{R}$ is a continuous function of both variables satisfying

$$G_j\left(2/j^2, 0\right) = 0.$$

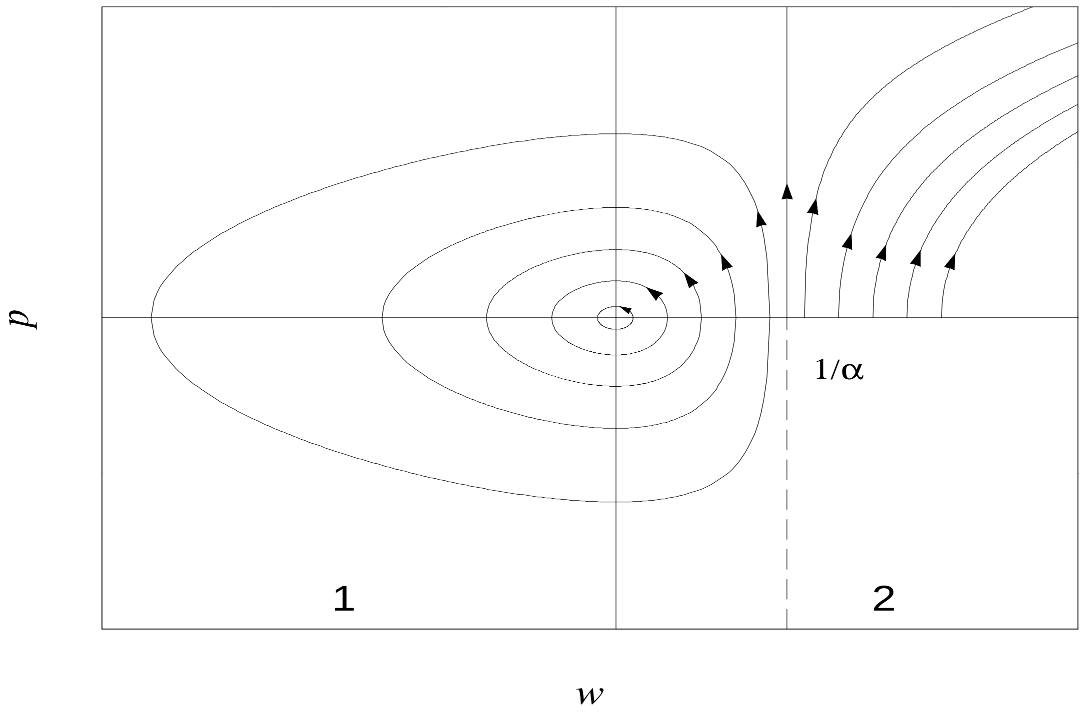

Figure 1: Trajectories of the dynamical system (4.14).

Note that the period T_L of an elliptic orbit of the linearized system at the origin ($f(w,p)$ replaced by $(2\alpha^{-1}p, w)$)

$$T_L = 4\int_0^{(\alpha/2)^{1/2}} \frac{dp}{\left(1-(2/\alpha)\,p^2\right)^{1/2}} = 2\pi\left(\frac{\alpha}{2}\right)^{1/2}$$

and $\lim_{w_0\to 0} T(\alpha, w_0) = T_L$.

Provided

$$\frac{\partial T}{\partial w_0} > 0 \tag{4.16}$$

holds for all $(\alpha, w_0) \in \mathcal{D}$, by the implicit function theorem, there exist a **unique** (strictly) monotone decreasing function $\widehat{w}_j : [0, 2/j^2] \longrightarrow \mathbb{R}_+$ with $\widehat{w}_j(2/j^2) = 0$ such that $G_j(\alpha, \widehat{w}_j(\alpha)) = 0$.

Note that (4.16) and

$$T(\alpha, w_0) = \alpha^{1/2} T(1, \alpha w_0)$$

(rescaling $x \to \overline{x} = x/\alpha^{1/2}$, $w \to \overline{w} = \alpha w$ and $p \to \overline{p} = \alpha^{1/2} p$) imply that T is an increasing function of both α and w_0 and explains the monotone behavior of $\widehat{w}_j$.

It thus follows that, if $\alpha < 2$, for each $j = 1, \ldots, k$ such that $2/\left(k+1\right)^2 \le \alpha < 2/k^2$ holds, a unique function $\widehat{w}_j$ such that $\widehat{w}_j(2/j^2) = 0$ exists. The non–trivial solutions $\psi_1^{\pm}, \ldots, \psi_k^{\pm}$ are the p–component of $\gamma_{\widehat{w}_j}$, $j = 1, \ldots, k$, which winds around the origin j–times: ψ_j^+ is 2π–periodic with fundamental period $2\pi/j$, $\left(\psi_j^+\right)'(0) > 0$ and satisfies $\psi_j^+(x+\pi) = \psi_j^-(x)$. If $\alpha \geq 2$, because $T(\alpha, w_0)$ is a strictly increasing function of w_0 and $T(\alpha, 0) \geq 2\pi$ there is no solution of $G_1(\alpha, w_0) = 0$.

This reduces the proof to the proof of inequality (4.16).

Let

$$q = \ln(1 - \alpha\, w)$$

be defined for $\alpha w < 1$. There is no loss of generality in taking $\alpha = 1$. The system is equivalent to the Hamiltonian system

$$\begin{cases} q' &= 2p \\ p' &= 1 - e^q, \end{cases}$$

whose energy function is given by

$$H(q, p) = p^2 + e^q - q - 1\,.$$

We denote by γ_E the orbits and note that there is a one–to–one correspondence between the two families of closed orbits $\{\gamma_{w_0},\ 0 \leq w_0 < 1\}$ and $\{\gamma_E,\ 0 \leq E < \infty\}$.

Let $\widetilde{T} = \widetilde{T}(E)$ be the period of an orbit γ_E ,

$$\widetilde{T} = \int_{\gamma_E} dx = \int_{q_-}^{q_+} \frac{dq}{p}\,.$$

Using the energy conservation law, we have

$$p = p(q, E) = (E - v(q))^{1/2}\ ,$$

where the potential energy is given by

$$v(q) = e^q - q - 1\,,$$

and $q_\pm = q_\pm(E)$ are the positive and negative roots of equation $v(q) = E$.

Equation (4.16) holds if and only if $\dfrac{d\widetilde{T}}{dE} > 0$ holds uniformly in $E \in \mathbb{R}_+$. But this follows from the monotonicity criterion given by C. Chicone [C]:

Lemma 4.5 *Let $v \in \mathcal{C}^3(\mathbb{R})$ be a three–times differentiable function and let $F(q) = -v'(q)$ be the force acting at q. If v/F^2 is a convex function with*

$$\left(\frac{v}{F^2}\right)'' = \frac{6v\,(v'')^2 - 3\,(v')^2\,v'' - 2vv'v'''}{(v')^4} > 0\,, \qquad q \neq 0$$

then the period $\widetilde{T}$ is a monotone (strictly) increasing function of E.

This conclues the proof of Theorem 4.4.

□

Remark 4.6 *The value $\alpha = 2$ is a bifurcation point as one can see by linearizing the equation about $\psi \equiv 0$. The linear operator $L[0] = A$ in the subspace of odd 2π–periodic functions has eigenvalues and associate eigenfunctions as given before. Hence, if $\alpha > 2$, the eigenvalues are all postive and $\psi \equiv 0$ is locally stable. When $\alpha < 2$ (but close to 2) a single eigenvalue becomes negative and one can apply Crandall–Rabinowitz bifurcation theory to locally describe the stable solution which branches from the trivial one. Note that Crandall–Rabinowitz theory can also be applied in the neighborhood of $\alpha_j = 2/j^2$, $j > 1$, in the orthogonal complement of the span $\left\{\pi^{-1/2} \sin mx, m = 1, ..., j-1\right\}$ corresponding to the odd functions with fundamental period $T = 2\pi/j$.*

With this Theorem we have given a global characterization of the non–trivial stationary solutions.

Remark 4.7 *In the sine–Gordon representation, the effective potential $\phi(x) = \int_0^x \psi(y)\, dy = x^2/(2\alpha)$ at $\gamma_{\alpha^{-1}}$ corresponds the Debye–Hückel regime with Debye length α. Although this regime is not reached for all $\beta > 0$, it gets closed quite fast as $\beta = 4\pi\alpha$ approaches 0.*

5 Stability

Let $z(t; z_0)$ denote the solution of the initial value problem. It follows

$$S(t)z_0 = z(t; z_0)$$

defines a dynamical system on a closed subset $\mathcal{V} \subset D(A)$ of $\mathcal{B}^{1/2}$ with the topology induced by the graph norm $\|\cdot\|_{1/2}$. Note that $z(t; z_0)$ is continuous in both t and z_0 with $z(0; z_0) = z_0$ and satisfies the (nonlinear) semi–group property $S(t + \tau)z_0 = z(t; z(\tau; z_0)) = S(t)S(\tau)z_0$.

Local stability means that $z(t; z_0)$ is uniformly continuous in $\mathcal{V}$ for all $t \geq 0$. It is uniformly asymptotically stable if, in addition, $\lim_{t\to\infty} \|z(t; z_0) - z(t; z_1)\|_{1/2} = 0$.

Theorem 5.8 (Local Stability) *There exist a neighborhood $\mathcal{U} \in \mathcal{B}^{1/2}$ of origin such that, if $\alpha > 2$ and z_0 in $\mathcal{U}$, then $\psi_0 \equiv 0$ is stable, i.e., $\lim_{t\to\infty} \|z(t; z_0)\|_{1/2} = 0$. If $\alpha < 2$ is such that $2/(k+1)^2 \leq \alpha < 2/k^2$ holds, among all equilibrium solutions of (4.13), $\psi_0, \psi_j^{\pm}$, $j = 1, \ldots, k$, $\psi_1^{\pm}$ are the only asymptotically stables. So, there exist $\rho > 0$ such that if $\|z_0 - \psi\|_{1/2} \leq \rho$, then $\lim_{t\to\infty} \|z(t; z_0) - \psi\|_{1/2} = 0$ for $\psi = \psi_1^{\pm}$ and $\sup_{t>0} \|z(t; z_0) - \psi\|_{1/2} \geq \varepsilon > 0$ for $\psi \neq \psi_1^{\pm}$.*

Proof. Consider the equation

$$\frac{d\zeta}{dt} + L\zeta = F(\zeta)$$

for $\zeta = z - \psi$ where ψ is an equilibrium solution and

$$L\zeta = L[\psi]\,\zeta = -\alpha\zeta'' + 2\alpha\psi\zeta' - 2(1 - \alpha\psi')\,\zeta$$

is the linearization around ψ and F as before. Note $L = A$ if $\psi = \psi_0 = 0$.

The local stability is consequence of the following two results.

Theorem 5.9 *If the spectrum $\sigma(L)$ lies in $\{\lambda \in \mathbb{R} : \lambda \geq c\}$ for some $c > 0$, then $\zeta = 0$ is the unique uniformly asymptotically stable solution. On the other hand, if $\sigma(L) \cap \{\lambda \in \mathbb{R} : \lambda < 0\} \neq \emptyset$, then $\zeta = 0$ is unstable.*

Theorem 5.10 *$\sigma(L) > 0$ whenever $\psi = \psi_0$ and $\alpha > 2$ or $\psi = \psi_1^{\pm}$ and $\alpha < 2$. If α is such that $2/(k+1)^2 \leq \alpha < 2/k^2$ holds for some $k \in \mathbb{N}_+$, then $\sigma(L) \cap \{\lambda \in \mathbb{R} : \lambda < 0\} \neq \emptyset$ for $\psi = \psi_0$ and $\psi = \psi_j^{\pm}$, $j = 2, \ldots, k$.*

Proof. For $\psi = \psi_0$ the proff with $\alpha \geq 0$ follows from the spectral computation of $L[\psi_0] = A$.

Let ψ be a nontrivial equilibrium solution and note that $\psi(0) = \psi(\pi) = 0$ by parity. ψ is asymptotically stable if $\sigma(L) > 0$ and unstable if $\sigma(L) \cap \{\lambda < 0\} \neq \emptyset$.

Let φ be the solution of

$$L[\psi]\varphi = 0$$

in the domain $0 < x < \pi$ satisfying

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi'(0) = 1.$$

By the comparison theorem[CL], ψ is asymptotically stable if $\varphi(x) > 0$ on $0 < x \leq \pi$ and unstable if $\varphi(x) < 0$ somewhere in $0 < x < \pi$.

To apply the comparison theorem a weight

$$p(x) := e^{-2\int_0^x \psi(y)\, dy}$$

is introduced in order to make L a self–adjoint operator:

$$p\, L[\psi]\zeta = -\alpha\,(p\,\zeta')' - 2p\,(1 - \alpha\psi')\,\zeta\,.$$

Note that $(L\zeta, \eta)_p = (\zeta, L\eta)_p$ for any odd periodic functions ζ and η of period 2π were $(f,g)_p := \displaystyle\int_{-\pi}^{\pi} f(x)\, g(x)\, p(x)\, dx$.

Let

$$\chi = c\,(-\alpha\psi'' + 4\psi)\ , \tag{5.17}$$

where $c > 0$ is chosen so that $\chi'(0) = 1$.

It follows from the equation $-\alpha\psi'' = 2\,(1 - \alpha\psi')\,\psi$,

$$\chi(0) = 0 \qquad\qquad \text{and} \qquad\qquad \chi > 0$$

whenever $\psi > 0$. In addition, we can verify

$$L[\psi]\chi = 8c\alpha^2\psi\,(\psi')^2 > 0\,.$$

If $\psi = \psi_1^+$, then $\chi > 0$ on $(0, \pi)$. By the comparison theorem, $\varphi > \psi \geq 0$ on $(0, \pi]$ which implies the stability of ψ_1^+ by the stability criterium.

For instability, we observe that ψ' satisfies

$$\begin{aligned} L[\psi]\psi' &= -\alpha\psi''' + 2\alpha\psi\psi'' - 2\left(1-\alpha\psi'\right)\psi' \\ &= \left(-\alpha\psi'' + 2\alpha\psi\psi' - 2\psi\right)' = 0\,, \end{aligned}$$

in view of equilibrium equation. Recall that $\psi = \psi_j^+$ with $j \geq 2$, has fundamental period $2\pi/j$ and satisfies $\psi(\pi/j) = \psi''(\pi/j) = 0$ by the odd parity and equilibrium equation. Since $\psi'(0) > 0$, this implies $\psi < 0$ on $(\pi/j, 2\pi/j)$ and the minimum of ψ is attained at $\underline{x} = \dfrac{3\pi}{2j}$. Since ψ' and φ satisfies the same self–adjoint equation $pL[\psi]\zeta = 0$, their Wronskian

$$\begin{aligned} W\left(\varphi, \psi'; x\right) &= \begin{vmatrix} \varphi & \psi' \\ -\alpha p\varphi' & -\alpha p\psi'' \end{vmatrix} \\ &= \alpha p\left(\varphi'\psi' - \varphi\psi''\right) = \alpha\psi'(0) > 0 \end{aligned}$$

is a non–vanishing constant (recall $p(0) = 1$, $\varphi(0) = 0$ and $\left(\psi_j^+\right)'(0) > 0$). As a consequence

$$W\left(\varphi, \psi'; \pi/j\right) = -\alpha p(\underline{x})\varphi\left(\underline{x}\right)\psi''\left(\underline{x}\right) > 0$$

implies $\varphi\left(\underline{x}\right) < 0$ because $\psi''\left(\underline{x}\right) > 0$. It thus follows from the stability criterium that ψ_j^+, $j = 2, \ldots, k$, are unstable since $\underline{x} \in (0, \pi)$ provided $j \geq 2$ and there exist $\overline{x} \in (0, \pi)$, $\overline{x} < \underline{x}$, such that $\varphi(\overline{x}) = 0$.

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