Liouville Operator Approach to Symplecticity-Preserving RG Method

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abstract: We present a method to construct symplecticity-preserving renormalization group maps by using the Liouville operator, and obtain correctly reduced symplectic maps describing their long-time behavior even when a resonant island chain appears.

There has been a long history to study an asymptotic solution of Hamiltonian flows by means of singular perturbation methods such as the averaging method and the method of multiple time-scales. A Hamiltonian flow can be reduced to a symplectic discrete map called the Poincaré map, which has the lower dimension than the original flow and is, therefore, extensively studied [LL91]. However, neither the averaging method nor the method of multiple time-scales may be immediately applicable to symplectic maps.

The perturbative renormalization group (RG) method developed recently may be a useful tool to tackle asymptotic behaviours of discrete maps as well as flows. The original RG method is an asymptotic singular perturbation technique developed for diffential equations [CGO96]. The RG method is reformulated on the basis of a naive renormalization transformation and the Lie group [GMN99]. The application of the RG method to some non-symplectic discrete systems has been attempted in the framework of the envelope method [KM98].

However, the extension of the RG method to discrete symplectic systems is not trivial because the symplectic struture is easily broken in naive RG equations (maps) as shown in Ref.[GN01], while the application of the RG method to Hamiltonian flows does not cause such a problem as the broken symplectic symmetry except a special case [YN98]. To recover the symmetry breaking, not only the paper [GN01] has introduced a process of the "exponentiation" which gives us analytical expressions of some physical quantities [GN01, Tze01], but also another paper [GNY02] has used a symplectic integrator by taking the time-continuous limit of the time-step parameter. Furthermore, S.I. Tzenov et al have showed that the exponentiation procedure can successfully be applied in the Hénon map [TD03]. However this procedure may seem somewhat artifical, and resonant islands have never been studied by the RG method.

The main purpose of our talk is to present a general RG procedure to preserve the symplectic structure in RG maps and to obtain correctly reduced symplectic RG maps.

In order to show our result, we consider the following nearly integrable 2-dimensional symplectic mapping $(x^n, y^n) \mapsto (x^{n+1}, y^{n+1})$:

$$x^{n+1} = x^n + y^{n+1}, \qquad y^{n+1} = y^n - ax^n + 2\varepsilon J(x^n)^3, \tag{1}$$

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where ε is the small parameter, J and a are the real parameters. A chaotic motion occurs when ε is finite. The naive perturbation $x^n = x^{(0)n} + \varepsilon x^{(1)n} + \cdots$ yields

$$x^{(0)n} = A\mathrm{e}^{i\theta n} + \text{c.c.}, \quad x^{(1)n} = \frac{-3i|A|^2 AJ}{\sin\theta} n\mathrm{e}^{i\theta n} + \frac{JA^3}{\cos 3\theta - \cos\theta}\mathrm{e}^{3i\theta n} + \text{c.c.}$$

where $A \in \boldsymbol{C}$ is the integration constant, θ is defined by $\cos\theta \equiv (1 - a/2)$ and c.c. denotes the conplex conjugate of the preceding terms. Here the secular behavior which violates the naive perturbation result is appeared. Now we concentrate on the case when $\cos\theta = \cos 3\theta$ is nearly satisfied[2]. The expansion $\theta = \frac{\pi}{2} + \varepsilon\theta^{(1)} + \cdots$, and *the Liouville operator approach to symplecticity-preserving RG method* give the reduced symplectic map (see. Fig.[1]),

$$A_1^{n+1} = A_1^n + 4J\varepsilon\Big\{(A_2^n)^3 - \theta^{(1)} A_2^n\Big\}, \quad A_2^{n+1} = A_2^n + \varepsilon\Big\{-4J(A_1^n)^3 + \theta^{(1)} A_1^n\Big\}. \qquad (2)$$

Here A_1^n and A_2^n are real, and canonical variables. The higher order corrections to the reduced map can systematically be obtained by this method[GN03]. We have been able to obtain a correctly reduced symplectic map even when a resonant island appears.

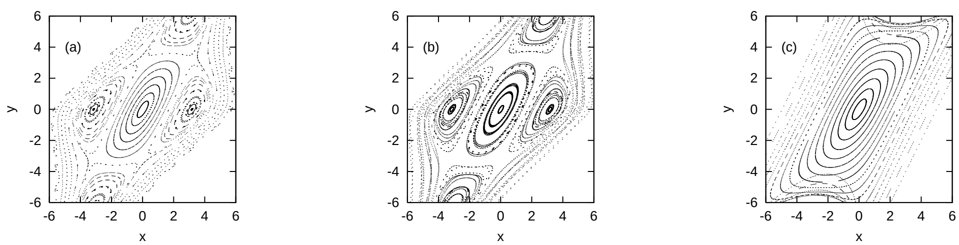

Figure 1: Phase portraits of the 2-dimensional symplectic map model when the parameters are $\varepsilon = 0.01, J = 1.0$, and $\theta^{(1)} = 10.0$; (a) the original map [Eq. (1)], (b) the Liouville operator approach to the RG method up to $\mathcal{O}(\varepsilon)$ [x, y are constructed by Eq. (2)], (c) the exponentiated RG method up to $\mathcal{O}(\varepsilon)$[This is calculated by the exponentiation method].

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[2]This is because it is important to study the nonlinear map near a resonance. Furthermore, the case of non resonance has been resolved by menas of the exponentiation method [GN01].