Absence of Phase Transitions in two-dimensional $O(N)$ Spin Models and Anderson Localization

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Though spontaneous mass generations in 2D non-Abelian sigma models and quark confinement in 4 dimensional (4D) non-Abelian lattice gauge theories are widely believed [1], they remained unsolved in the last century.

Most of mathematical physicists believe that non-abelian systems ($SO(N)$ spin models, lattice gauge theories) do not exhibit phase transitions in lower dimensions. But this is too optimistic. See [2] for another type of spin models (of non-linear interraction) which exhibit phase transitions of first order. Our main result is for the original Heisenberg model ($SO(N)$ spin model of linear interaction):

Theorem *The following bound holds for $\beta_c(N)$:*

$$\beta_c(N) \geq c_1 e^{c_2 N} \tag{1}$$

where c_1 and c_2 are positive constants.

This is proved by our new analysis [3] based on the argument [4] of the Anderson localization type. This will be improved to

$$\beta_c(N) = \infty, \quad N \geq 3 \tag{2}$$

by some additional calculations which will be published in the near future.

The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by

$$< f >\equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - N\beta) d\phi_i \tag{3}$$

Here $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $< 1 >= 1$, and

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|_1=1} \phi(x)\phi(y), \tag{4}$$

Using $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ [5] where $\mathrm{Im}\, a_i = -(\nu + m^2/2)$, $\quad \mathrm{Re}\, a_i = (1/\sqrt{N})\psi_i$ and $m > 0$, we have

$$Z_\Lambda \quad = \quad c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{5}$$

where $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian,

$$F(\psi) \quad = \quad \det(1 + i\kappa G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j], \quad \kappa \equiv \frac{2}{\sqrt{N}} \tag{6}$$

and $G = (m^2 - \Delta)^{-1}$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where $\tilde{G}(p) = [m^2 + 2\sum(1 - \cos p_i)]^{-1}$. Then this is possible if $\nu \le 2$, and $m^2 \sim 32e^{-4\pi\beta}$ for $\nu = 2$ as $\beta \to \infty$. Thus for $\nu = 2$,

$$F(\psi) = {\det}_3{}^{-N/2}(1 + i\kappa G\psi)\exp[- < \psi, G^{\circ 2}\psi >], \qquad (7)$$

$${\det}_3(1 + A) \equiv \det[(1 + A)e^{-A+A^2/2}] \qquad (8)$$

where $G^{\circ 2}(x, y) = G(x, y)^2$ so that $\mathrm{Tr}(G\psi)^2 =< \psi, G^{\circ 2}\psi >$.

To show the analyticity of the free energy, we polymer-expand the determinant $F(\psi)$ and establish the convergence. This is not so easy since $G(x, y) \sim e^{-m|x-y|}$ yields long-range interactions of $\psi(x)$. The following facts are essential to overcome this difficulty:

Facts 1. We note that $[G^{\circ 2}]^{-1} \sim (m^2 - \Delta)/2\beta$ for large β. In fact, the Fourier transform of $G^2(x, y)$ is about $(p^2 + m^2)^{-1}\log(1 + p^2/m^2)$, which comes from the infrared divergences in the theory. Therefore the Gaussian measure $\exp[- < \psi, G^{\circ 2}\psi >]\prod d\psi$ has short-range correlations (of order $O(1)$) though $G^{\circ 2}(x, y)$ has long-range correlations of order $m^{-1} \sim e^{2\pi\beta}$.

Fact 2. The determinant $\det_3^{-N/2}(1 + i\kappa G\psi)$ is factorized and decomposed by the Feshbach-Krein tranformation of matrices.

Fact 3. In the factorization, we see that the Green's function $G_\psi(x, y) \equiv [-\Delta + m^2 + i\kappa\psi]^{-1}(x, y)$ plays important roles. Though $G_0(x, y) \equiv [-\Delta + m^2]^{-1}(x, y)$ exhibits very long correlations, $G_\psi(x, y)$ has rather short correlations by the (complex) "impurities" $i\psi$. This is the famous Anderson localization, and improve the convergences.

These facts enable us to establish the convergence.

References

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