Renormalization: Three ways of *re*-normalization

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In this talk I will touch upon theoretical problems of real space renormalization applied to classically interacting (i.e., non-quantum) lattice spin systems. These issues present themselves when a renormalization map is attempted to be implemented in a mathematically rigorous way. Though users are most interested in what they call renormalized interaction, this can be obtained as arising from the diagram

$$\begin{array}{ccc} \Phi & \longrightarrow & \varrho \\ \downarrow & & \downarrow \\ \Phi' & \longleftarrow & \varrho' \end{array}$$

Here Φ is the interaction to start with, ϱ is a Gibbs measure for this interaction, ϱ' is the image of ϱ under the chosen renormalization map, and Φ' is the interaction reconstructed from ϱ', i.e. what might be considered to be the renormalized interaction. More ambitiously, to obtain a possible fixed point of the renormalization transformation, an infinite sequence of such diagrams must be looked at. As the diagram indicates, renormalization maps are rigorously defined between probability measures describing equilibrium states, and the question is asked if this induces a map between interactions. As I will point out, there are difficult mathematical problems about the diagram not necessarily being commutative.

First let us briefly describe the ingredients of the diagram. The interacting spin system will be realized on an infinite lattice $\mathcal{L}$ (such as $\mathbb{Z}^d$ etc) on each of whose points a variable called spin is placed. Denote the one-site spin state space by S, assumed here to be a finite set. The configuration space is then $\Omega = S^{\mathcal{L}}$. By endowing this set with its Borel structure $\mathcal{F}$, and introducing the product measure χ by multiplying the counting measure on S over the lattice points, the space $(\Omega, \mathcal{F}, \chi)$ will describe the non-interacting system. Interactions between spins are introduced by a measurable function $\Phi : \mathcal{P}(\mathcal{L}) \times \Omega \to \mathbb{R}$, $(\Lambda, \omega) \mapsto \Phi_\Lambda(\omega)$, called potential, where $\mathcal{P}(\mathcal{L})$ denotes the set of all finite subsets of the lattice. For convenience we assume here that Φ_Λ are translation invariant. The energy associated with a configuration $\omega_\Lambda \times \xi_{\Lambda^c}$ (i.e., one agreeing with ω on $\Lambda \in \mathcal{P}(\mathcal{L})$ and with ξ on $\Lambda^c = \mathcal{L} \setminus \Lambda$) is given by the Hamiltonian

$$\mathcal{H}_\Lambda^\Phi(\omega|\xi) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_{X\cap\Lambda} \times \xi_{X \cap \Lambda^c}) = \sum_{X \subset \Lambda} \Phi_X(\omega) + \sum_{\substack{X \subset \Lambda \\ Y \subset \Lambda^c}} \Phi_{X \cup Y}(\omega_X \times \xi_Y). \tag{1}$$

Since the range of the interaction may be infinite, the sum above may diverge; to circumvent this we require that the interaction energy of each spin with all others is uniformly bounded:

$$\sum_{\substack{X \in \mathcal{P}(\mathcal{L}) \\ X \ni 0}} \|\Phi_X\|_\infty < c, \tag{2}$$

for some $c > 0$. This condition gives rise to a norm and a Banach space $\mathcal{B}(\Omega)$ of potentials.

The states of the system are described by Gibbs measures. A Gibbs measure for potential $\Phi \in \mathcal{B}(\Omega)$ and reference measure χ is a probability measure ϱ^Φ on $(\Omega, \mathcal{F}, \chi)$ if a version of the Λ-indexed family of its conditional probabilities $\varrho^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c})$ satisfies the DLR equations

$$\varrho^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) = e^{-\left(\mathcal{H}^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) - \mathcal{H}^\Phi_\Lambda(\tau_\Lambda|\xi_{\Lambda^c})\right)} \varrho^\Phi_\Lambda(\tau_\Lambda|\xi_{\Lambda^c}) \tag{3}$$

for all $\Lambda \in \mathcal{P}(\mathcal{L})$, and $\omega, \tau, \xi \in \Omega$. Since S is a finite set, compactness arguments guarantee that at least one Gibbs measure exists. The possibility of multiple Gibbs measures for a given potential (selected by different "boundary conditions" ξ in the above notation) is of great interest for it corresponds to situations when a first-order phase transition occurs. Gibbs measures are natural models of thermodynamic equilibrium states since for $\mathcal{B}$-potentials they minimize free energy. There is a procedure of inverting (3) and reconstruct the potential Φ (modulo from our point of view minor details) when first ϱ is given.

A renormalization transformation is a probability kernel between in general two distinct probability spaces mapping one probability measure into another, i.e. $T : S^{\mathcal{L}} \times \mathcal{F} \to S'^{\mathcal{L}'} \times \mathcal{F}'$,

$$\varrho'(d\omega) = \int_{S^{\mathcal{L}}} T(\xi, d\omega)\varrho(d\xi). \tag{4}$$

Usually these are taken to be block-spin transformations in the sense that the lattice is partitioned into blocks and T is a product of kernels defined on blocks of "internal" spins ξ:

$$T(\xi, d\omega) = \prod_{x \in \mathcal{L}'} \hat{T}(\xi_{B(x)}, d\omega_x) \tag{5}$$

where $B(x) \in \mathcal{P}(\mathcal{L})$ is a block associated with site x in a specific way.

Now let us turn back to the original diagram. The input to the renormalization procedure are $\mathcal{L}$, S, $\Phi \in \mathcal{B}(\Omega)$, ϱ^Φ (i.e., one specific Gibbs measure of the possibly many for the given potential), S', $\mathcal{L}'$, and T. The output is ϱ'. The basic problem encountered is that, as many examples show, it is possible that there is no $\Phi' \in \mathcal{B}(S'^{\mathcal{L}'})$ such that ϱ' is a Gibbs measure for Φ', i.e., ϱ' may be non-Gibbsian. If this happens, there is no obvious way how to get a renormalized potential by reconstructing the interaction from the image of ϱ^Φ.

One immediate answer might be to enlarge the space $\mathcal{B}$ of potentials, however, this would introduce a number of "unphysical" features related with the image spin system. Also, one might say that it is the fixed point behaviour of the renormalization transformation that is ultimately relevant, and if after the first step the measure is non-Gibbsian it does not in itself rule out that the fixed point behaves in a sensible way. This argument may stand, but there is little known rigorously about fixed points at the moment.

Our present day understanding is that if we want to define renormalization transformations so that they are internal operations within a class of probability spaces, we can proceed by three possible ways. An important advantage of these approaches is that they do not throw away what is generally useful in the mathematical modelling of such thermodynamic systems, in particular the physical picture stays unchanged. One approach is to enlarge the concept of Gibbs measure. This can be done in two ways: either by introducing so called almost Gibbsian measures (which are non-Gibbsian only for a subset of measure zero of configurations) or weakly Gibbsian measures (whose potentials are not uniformly but only almost surely pointwise summable). A third way is to compose an ill-behaving renormalization transformation with a decimation to obtain a well-behaved image measure.

In my talk I will first explain the mechanisms of losing Gibbsianness under a renormalization step, and illustrate through examples the three ways of making renormalization transformations internal operations.