Effective mass and mass renormalization of nonrelativistic QED

Fumio Hiroshima[*†]

November 29, 2003

Abstract

The effective mass $m_{\rm eff}$ of the nonrelativistic QED is considered. $m_{\rm eff}$ is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass $m_{\rm eff} = m_{\rm eff}(e^2, \Lambda, \kappa, m)$ is a function of bear mass $m > 0$, ultraviolet cutoff $\Lambda > 0$, infrared cutoff $\kappa > 0$, and the square of charge e of an electron. Introduce a scaling $m \to m(\Lambda) = (b\Lambda)^\beta$, $\beta < 0$. Then asymptotics behavior of $m_{\rm eff}$ as $\Lambda \to \infty$ is studied.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn.[1] We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}) \right],$$

where $\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1,2\})$ with $\otimes_s^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}$. The inner product in $\mathcal{F}$ is denoted by $(\cdot,\cdot)$ and the Fock vacuum by Ω. On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k,j)^* a(k,j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1,2\}), \qquad (1.1)$$

*Department of Mathematics and Physics, Setsunan University, 572-8508, Osaka, Japan. email: hiroshima@mpg.setsunan.ac.jp

†This work is partially supported by Grant-in-Aid for Science Reserch C 1554019 from MEXT.

1 Zentrum Mathematik and Physik Department, TU München, D-80290, München, Germany. email: spohn@ma.tum.de

where $a(f)$ and $a^*(f) = a(\bar{f})^*$ are densely defined and satisfy the CCR

$$[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3 \times \{1,2\})},$$
$$[a(f), a(g)] = 0,$$
$$[a^*(f), a^*(g)] = 0.$$

The free Hamiltonian of $\mathcal{F}$ is read as

$$H_{\mathrm{f}} = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \tag{1.2}$$

where the dispersion relation is given by

$$\omega(k) = |k|.$$

The free Hamiltonian H_{f} acts as

$$H_{\mathrm{f}}\Omega = 0,$$
$$H_{\mathrm{f}} a^*(f_1) \cdots a^*(f_n)\Omega = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n)\Omega.$$

The Pauli-Fierz Hamiltonian H is defined as a self-adjoint operator acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx$$

by

$$H = \frac{1}{2m}(p_x \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}},$$

where m and e denote the mass and charge of electron, respectively,

$$p_x = \left(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3}\right)$$

and V an external potential. The quantized radiation field $A_{\hat{\varphi}}$ is defined by

$$A_{\hat{\varphi}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} (a(f_x) + a^*(\bar{f}_x)) dx, \tag{1.3}$$

where

$$f_x(k, j) = \frac{1}{\sqrt{\omega}} \hat{\varphi}(k) e(k, j) e^{ikx}, \tag{1.4}$$

$e(k, 1), e(k, 2), k/|k|$ form a right-handed dreibain, and $\hat{\varphi}$ is a form factor. $A_{\hat{\varphi}}$ acts for $\Psi \in \mathcal{H}$ as

$$(A_{\hat{\varphi}}\Psi)(x) = (a(f_x) + a^*(\bar{f}_x))\Psi(x), \quad x \in \mathbb{R}^3.$$

Theorem 1.1 *Assume that $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega}, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and V is relatively bounded with respect to $-\Delta$ with a relative bound < 1. Then, for arbitrary values of e, H is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_{\mathrm{f}})$ and bounded from below.*

Proof: See Hiroshima [3, 4]. □

1.2 Effective mass

The momentum of the photon field is given by

$$P_{\mathrm{f}} = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk \tag{1.5}$$

and the total moment by

$$P_{\mathrm{total}} = p_x \otimes 1 + 1 \otimes P_{\mathrm{f}}.$$

Let as assume that

$$V \equiv 0.$$

Then we see that

$$[H, P_{\mathrm{total}\,\mu}] = 0, \quad \mu = 1,2,3.$$

Hence H and $\mathcal{H}$ can be decomposable with respect to $\mathrm{Spec}(P_{\mathrm{total}}) = \mathbb{R}^3$, i.e.,

$$\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(p) dp,$$
$$H = \int_{\mathbb{R}^3}^{\oplus} H(p) dp.$$

Note that

$$e^{-ix\otimes P_{\mathrm{f}}} P_{\mathrm{total}} e^{ix\otimes P_{\mathrm{f}}} = p_x,$$
$$e^{-ix\otimes P_{\mathrm{f}}} H e^{ix\otimes P_{\mathrm{f}}} = \frac{1}{2m}(p_x \otimes 1 - 1 \otimes P_{\mathrm{f}} - e 1 \otimes A_{\hat{\varphi}}(0)) + 1 \otimes H_{\mathrm{f}},$$

where

$$A_{\hat{\varphi}}(0) = \frac{1}{\sqrt{2}}(a(f_0) + a(\bar{f}_0)).$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\mathcal{H}(p) \cong \mathcal{F},$$
$$H(p) \cong \frac{1}{2m}(p - P_{\mathrm{f}} - e A_{\hat{\varphi}}(0)) + H_{\mathrm{f}},$$

Let

$$E_{m,\Lambda}(p) = \inf \mathrm{Spec}(H(p)). \tag{1.6}$$

Let us assume sharp ultraviolet cutoff Λ and infrared cutoff κ, which means

$$\hat{\varphi}(k) = \begin{cases} 0 & \text{for } |k| < \kappa, \\ (2\pi)^{-3/2} & \text{for } \kappa \le |k| \le \Lambda, \\ 0 & \text{for } |k| > \Lambda. \end{cases} \tag{1.7}$$

Lemma 1.2 *There exists constants p_* and e_* such that for*

$$(p,e) \in \mathcal{O} = \{(p,e) \in \mathbb{R}^3 \times \mathbb{R} \,|\, |p| < p_*, |e| < e^*\},$$

$H(p)$ has a ground state $\psi_{\mathrm{g}}(p)$ and it is unique. Moreover $\psi_{\mathrm{g}}(p) = \psi_{\mathrm{g}}(p,e)$ is strongly analytic and $E_{m,\Lambda}(p) = E_{m,\Lambda}(p,e)$ analytic with respect to $(p,e) \in \mathcal{O}$.

Proof: See Hiroshima and Spohn [6, 7]. □

In what follows we assume that $(p, e) \in \mathcal{O}$.

Definition 1.3 *The effective mass $m_{\rm eff} = m_{\rm eff}(e^2, \Lambda, \kappa, m)$ is defined by*

$$\frac{1}{m_{\rm eff}} = \frac{1}{3}\Delta_p E(p, e)\lceil_{p=0}. \tag{1.8}$$

1.3 Mass renormalization

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \to \infty, \quad m \to 0 \tag{1.9}$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. To achive this, as a first step we want to find constants

$$\beta < 0, \quad 0 < b$$

such that

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = m_{\rm ph}, \tag{1.10}$$

where $m_{\rm ph}$ is a given constant. Actually $m_{\rm ph}$ is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff Λ goes to infinity.

We will see later that $m_{\rm eff}/m$ is a function of e^2, Λ/m and κ/m. Let

$$\frac{m_{\rm eff}}{m} = f(e^2, \Lambda/m, \kappa/m), \tag{1.11}$$

where $f(0, \Lambda/m, \kappa/m) = 1$ holds. An analysis of (1.10) can be reduce to investigate the asymptotic behavior of f as $\Lambda \to \infty$. Namely we want to find constants

$$0 \le \gamma < 1, \quad 0 < b_0$$

such that

$$\lim_{\Lambda\to\infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0. \tag{1.12}$$

If we succeed to find constants γ and b_0 such as in (1.12) then by

$$m_{\rm eff}(e^2, \Lambda, \kappa, m) = m f(e^2, \Lambda/m, \kappa/m),$$

we have

$$m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = (b\Lambda)^\beta f(e^2, \Lambda/(b\Lambda)^\beta, \kappa/b^\beta) \approx b_0(b\Lambda)^\beta(\Lambda/(b\Lambda)^\beta)^\gamma. \tag{1.13}$$

Taking

$$\beta = \frac{-\gamma}{1-\gamma} < 0, \quad b = 1/b_1^{1/\gamma},$$

we see that by (1.13)

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2,\Lambda,\kappa\Lambda^\beta,(b\Lambda)^\beta) = \lim_{\Lambda\to\infty} b_0 \left(\frac{\Lambda}{b_1^{1/\gamma}}\right)^\beta \left(\frac{\Lambda}{(\Lambda/(b_1)^{1/\gamma})^\beta}\right)^\gamma = b_0 b_1,$$

where b_1 is a parameter, which is adjusted such as

$$b_0 b_1 = m_{\rm ph}.$$

Hence we will be able to establish (1.10). It is easily seen that

$$f(e^2,\Lambda/m,\kappa/m) = 1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2}) + O(\alpha^2),$$

where $\alpha = e^2/4\pi$, which suggests

$$f(e^2,\Lambda/m,\kappa/m) \approx (\Lambda/m)^{8\alpha/3\pi},$$

for sufficiently small α and large Λ, and therefore

$$\gamma = 8\alpha/3\pi.$$

One may assume that

$$f(e^2,\Lambda/m,\kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi)+\alpha^2 b}$$

for sufficiently small α with some constant b. Then by expading $m_{\rm eff}/m$ to order α^2 one may expect that

$$f(e^2,\Lambda/m,\kappa/m) \approx 1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda}{m})+\frac{1}{2}\alpha^2\left(\frac{8}{3\pi}\log(\frac{\Lambda}{m})\right)^2+b\alpha^2\log(\frac{\Lambda}{m})+O(\alpha^3) \tag{1.14}$$

for sufficiently small α and large Λ. It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant $C>0$ such that

$$f(e^2,\Lambda/m,\kappa/m) = 1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2}) + \alpha^2 C\sqrt{\Lambda/m} + O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of viwe by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in α of the effective mass of the Pauli-Fierz Hamiltonian with spin.

2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m} :(p - P_{\rm f} - eA_{\hat{\varphi}}(0))^2: + H_{\rm f}$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m} :(p - P_{\rm f} - eA_{\hat{\varphi}}(0))^2: + H_{\rm f}.$$

Furthermore for notational convenience we write A and $E(p)$ for $A_{\hat{\varphi}}(0)$ and $E_{m,\Lambda}(p)$, respectively.

2.1 Formulae

Lemma 2.1 *We have*

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}\sum_{\mu=1,2,3} \frac{(\psi_{\rm g}(0), (P_{\rm f} + eA)_\mu (H(0) - E(0))^{-1} (P_{\rm f} + eA)_\mu \psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}.$$

Proof: It is seen that $E(p,e) = E(p,-e) = E(-p,e)$. Then

$$\frac{\partial}{\partial p_\mu} E(p,e)\Big\lceil_{p_\mu=0} = 0, \quad \mu = 1,2,3, \tag{2.1}$$

follows. Moreover it is seen that $E(p,e)$ is a function of e^2 and

$$\frac{d^{2m-1}}{de^{2m-1}} E(p,e)\Big\lceil_{e=0} = 0. \tag{2.2}$$

In this proof, $f'(p)_\mu$ means the strong derivative of $f(p)$ with respect to p_μ. Since

$$H(p)\psi_{\rm g}(p) = E(p)\psi_{\rm g}(p),$$

we have

$$H'(p)_\mu \psi_{\rm g}(p) + H(p)\psi_{\rm g}'(p)_\mu = E'(p)_\mu \psi_{\rm g}(p) + E(p)\psi_{\rm g}'(p)_\mu \tag{2.3}$$

and

$$\begin{aligned} &H''(p)_\mu \psi_{\rm g}(p) + 2H'(p)_\mu \psi_{\rm g}'(p)_\mu + H(p)\psi_{\rm g}''(p)_\mu \\ &= E''(p)_\mu \psi_{\rm g}(p) + 2E'(p)_\mu \psi_{\rm g}'(p)_\mu + E(p)\psi_{\rm g}''(p)_\mu. \end{aligned} \tag{2.4}$$

By (2.1) it follows that $E'(0)_\mu = 0$, and by (2.3) with $p = 0$,

$$\begin{aligned} &(P_{\rm f} + eA)_\mu \psi_{\rm g}(0) \in D((H(0) - E(0))^{-1}), \\ &\psi_{\rm g}'(0)_\mu = (H(0) - E(0))^{-1}(P_{\rm f} + eA)_\mu \psi_{\rm g}(0). \end{aligned}$$

Then we have by (2.3) and (2.4),

$$\begin{aligned}\frac{m}{m_{\rm eff}} &= \frac{1}{3}\sum_{\mu=1,2,3}\frac{(\psi_{\rm g}(0),E''(0)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0),\psi_{\rm g}(0))}\\ &= 1-\frac{2}{3}\sum_{\mu=1,2,3}\frac{((P_{\rm f}+eA)_\mu\psi_{\rm g}(0),(H(0)-E(0))^{-1}(P_{\rm f}+eA)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0),\psi_{\rm g}(0))}\end{aligned}$$

Thus the lemma follows. □

Let

$$\psi_{\rm g}(0)=\sum_{n=0}^\infty\frac{e^n}{n!}\varphi_n,\quad E(0)=\sum_{n=0}^\infty\frac{e^{2n}}{(2n)!}E_{2n}.$$

Note that

$$\varphi_{2m}\in\bigoplus_{m=0}^\infty\mathcal{F}^{(2m)},\quad \varphi_{2m+1}\in\bigoplus_{m=0}^\infty\mathcal{F}^{(2m+1)}.$$

We want to get the explicit form of φ_n. Let

$$\begin{aligned}&\mathcal{F}_{\rm fin}=\{\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}|\Psi^{(m)}=0 \text{ for } m\geq\ell \text{ with some } \ell\},\\ &\mathcal{F}_0=\Big\{\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}_{\rm fin}\Big|\ \text{(i)}\ \Psi^{(0)}=0,\\ &\qquad \text{(ii)}\ {\rm supp}_{(k_1,...,k_n)\in\mathbb{R}^{3n}}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n)\not\ni\{(0,...,0)\}\Big\}.\end{aligned}$$

Lemma 2.2 *We see that* $\mathcal{F}_0\subset D(H_0^{-1})$.

Proof: Let $\Psi=\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}_0$. Since

$$\begin{aligned}&(H_0\Psi)^{(n)}(k_1,...,k_n,j_1,...,j_n)\\ &=\left[\frac{1}{2}(k_1+\cdots+k_n)^2+\sum_{j=1}^n\omega(k_j)\right]\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n),\end{aligned}$$

we see that

$$\begin{aligned}&(H_0^{-1}\Psi)^{(n)}(k_1,...,k_n,j_1,...,j_n)\\ &=\left[\frac{1}{2}(k_1+\cdots+k_n)^2+\sum_{j=1}^n\omega(k_j)\right]^{-1}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n).\end{aligned}$$

Since ${\rm supp}_{(k_1,...,k_n)\in\mathbb{R}^{3n}}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n)\not\ni\{(0,...,0)\}$, we obtain that

$$\|H_0^{-1}\Psi\|_{\mathcal{F}}^2=\sum_{n=1}^{\rm finite}\|(H_0^{-1}\Psi)^{(n)}\|_{\mathcal{F}^{(n)}}^2<\infty.$$

Then the lemma follows. □

We split $H(0)$ as

$$H(0)=H_0+eH_1+\frac{e^2}{2}H_2,$$

where

$$H_0 = \frac{1}{2}{P_{\mathrm{f}}}^2 + H_{\mathrm{f}},$$
$$H_1 = \frac{1}{2}(P_{\mathrm{f}} \cdot A + A \cdot P_{\mathrm{f}}) = P_{\mathrm{f}} \cdot A = A \cdot P_{\mathrm{f}},$$
$$H_2 = :A^2: .$$

Lemma 2.3 *We have* $E_0 = E_1 = E_2 = E_3 = 0$ *and*

$$\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -{H_0}^{-1} H_2 \Omega, \quad \varphi_3 = 3{H_0}^{-1} H_1 {H_0}^{-1} H_2 \Omega.$$

In particular $\varphi_2 \in \mathcal{F}^{(2)}$ *and* $\varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)}$.

Proof: Let us set $H(0)$, $E(0)$ and $\psi_{\mathrm{g}}(0)$ as H, E and ψ_{g}, respectively. It is obvious that $E_0 = 0$ and $\varphi_0 = a\Omega$ with arbitrary $a \in \mathbb{C}$, and by (2.2), $E_1 = E_3 = 0$. Set $a = 1$. We denote the strong derivative of $f = f(e)$ with respect to e by f'. We have

$$H'\psi_{\mathrm{g}} + H\psi_{\mathrm{g}}' = E'\psi_{\mathrm{g}} + E\psi_{\mathrm{g}}' \tag{2.5}$$

and

$$H''\psi_{\mathrm{g}} + 2H'\psi_{\mathrm{g}}' + H\psi_{\mathrm{g}}'' = E''\psi_{\mathrm{g}} + 2E'\psi_{\mathrm{g}}' + E\psi_{\mathrm{g}}''. \tag{2.6}$$

From (2.6) it follows that

$$(\psi_{\mathrm{g}}, H''\psi_{\mathrm{g}}) + (\psi_{\mathrm{g}}, 2H'\psi_{\mathrm{g}}') + (\psi_{\mathrm{g}}, H\psi_{\mathrm{g}}'') = E''(\psi_{\mathrm{g}}, \psi_{\mathrm{g}}) + (\psi_{\mathrm{g}}, 2E'\psi_{\mathrm{g}}') + (\psi_{\mathrm{g}}, E\psi_{\mathrm{g}}''). \tag{2.7}$$

Put $e = 0$ in (2.7). Then

$$(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega). \tag{2.8}$$

Since the left-hand side of (2.8) vanishes, we have $E_2 = 0$. From (2.5) with $e = 0$ and the fact $E_0 = E_1 = 0$, it follows that

$$H_1\Omega + H_0\varphi_1 = 0,$$

from which it holds that $H_0\varphi_1 = 0$. Since H_0 has the unique eigenvector Ω (the ground state) with eigenvalue zero, it follows that $\varphi_1 = b\Omega$ with some constant b. $\varphi_1 \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}$ which implies $b = 0$. Hence $\varphi_1 = 0$ follows. By (2.6) with $e = 0$, we have

$$H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.$$

Since $H_2\Omega \in \mathcal{F}_0$, we see that by Lemma 2.2, $H_2\Omega \in D(H_0^{-1})$. Thus we have $\varphi_2 = -{H_0}^{-1} H_2\Omega$. From the identity

$$H'''\psi_{\mathrm{g}} + 3H''\psi_{\mathrm{g}}' + 3H'\psi_{\mathrm{g}}'' + H\psi_{\mathrm{g}}''' = E'''\psi_{\mathrm{g}} + 3E''\psi_{\mathrm{g}}' + 3E'\psi_{\mathrm{g}}'' + E\psi_{\mathrm{g}}''' \tag{2.9}$$

it follows that at $e = 0$,

$$3H_1\varphi_2 + H_0\varphi_3 = 0.$$

Since $H_1\varphi_2 = -H_1{H_0}^{-1}H_2\Omega \in \mathcal{F}_0$, Lemma 2.2 ensures that $H_1\varphi_2 \in D({H_0}^{-1})$. Hence $\varphi_3 = -3{H_0}^{-1}H_1\varphi_2 = 3{H_0}^{-1}H_1{H_0}^{-1}H_2\Omega$. Then the lemma is proven. □

2.2 Order e^4

In this subsection we expand $m/m_{\rm eff}$ up to order e^4. We define A^- and A^+ by

$$A^- = \frac{1}{\sqrt{2}} a(f), \quad A^+ = \frac{1}{\sqrt{2}} a^*(f).$$

Then $A = A^+ + A^-$.

Lemma 2.4 *We have*

$$\begin{aligned}
\frac{m}{m_{\rm eff}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1}^{3} \left(\Omega, A_\mu H_0^{-1} A_\mu \Omega\right) \\
&-e^4 \frac{2}{3} \sum_{\mu=1}^{3} \Big\{ 2\left(\Psi_3^\mu, H_0^{-1} \Psi_1^\mu\right) + \left(\Psi_2^\mu, H_0^{-1} \Psi_2^\mu\right) - 2\left(\Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu\right) \\
&-\frac{1}{2}\left(\Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu\right) + \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu\right) \Big\} + O(e^6),
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\Psi_1^\mu &= A_\mu \Omega, \\
\Psi_2^\mu &= -\frac{1}{2} P_{{\rm f}\,\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, \\
\Psi_3^\mu &= \frac{1}{2} \left\{ -A_\mu H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{{\rm f}\,\mu} H_0^{-1} (P_{\rm f} \cdot A + A \cdot P_{\rm f}) H_0^{-1} (A^+ \cdot A^+) \Omega \right\}.
\end{aligned}$$

Proof: In Lemma 2.1 we have seen that

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\rm f} + eA)_\mu \psi_{\rm g}(0), (H(0) - E(0))^{-1} (P_{\rm f} + eA)_\mu \psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}. \tag{2.11}$$

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$\begin{aligned}
&(H(0) - E(0))^{-1} = H_0^{-1} - e H_0^{-1} H_1 H_0^{-1} \\
&+e^2 \left(-\frac{1}{2} H_0^{-1} H_2 H_0^{-1} + H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \right) + O(e^3).
\end{aligned} \tag{2.12}$$

Here we set

$$H_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$$

Note that

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$

In particular

$$\frac{1}{(\psi_{\rm g},\psi_{\rm g})} = 1 - e^4(\frac{1}{2}\varphi_2, \frac{1}{2}\varphi_2) - e^4(\Omega, \frac{1}{24}\varphi_4) + O(e^6) = 1 - e^4\frac{1}{4}(\varphi_2,\varphi_2) + O(e^6). \tag{2.13}$$

Moreover we have

$$\begin{aligned}(P_{\rm f}+eA)_\mu\psi_{\rm g}(0) &= eA_\mu\Omega + e^2(\frac{1}{2}P_{{\rm f}\,\mu}\varphi_2) + e^3(\frac{1}{2}A_\mu\varphi_2 + \frac{1}{6}P_{{\rm f}\,\mu}\varphi_3) + O(e^4)\\ &= e\Psi_1^\mu + e^2\Psi_2^\mu + e^3\Psi_3^\mu + O(e^4). \end{aligned}\tag{2.14}$$

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows. □

For each $k\in\mathbb{R}^3$ let us define the projection $Q(k)$ on $\mathbb{R}^3$ by

$$Q(k) = \sum_{j=1,2}|e_j(k)\rangle\langle e_j(k)|.$$

We set

$$\hat\varphi_j = \hat\varphi(k_j),\quad \omega_j=\omega(k_j),\quad Q(k_j)=Q_j,\quad j=1,2.$$

Let

$$\begin{aligned}\frac{1}{F_j} &= \frac{1}{r_j^2/2+r_j},\quad j=1,2,\\ \frac{1}{F_{12}} &= \frac{1}{(r_1^2+2r_1r_2X+r_2^2)/2+r_1+r_2},\quad r_1,r_2\geq 0,\quad -1\leq X\leq 1.\end{aligned}$$

Lemma 2.5 *We have*

$$\frac{m}{m_{\rm eff}} = 1-\alpha a_1(\Lambda/m,\kappa/m) - \alpha^2 a_2(\Lambda/m,\kappa/m) + O(\alpha^3),$$

where

$$a_1(\Lambda/m,\kappa/m) = \frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right) \tag{2.15}$$

and

$$\begin{aligned}& a_2(\Lambda/m,\kappa/m)\\ &= \frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\int_{-1}^1 {\rm d}X\int_{\kappa/m}^{\Lambda/m}{\rm d}r_1\int_{\kappa/m}^{\Lambda/m}{\rm d}r_2 \pi r_1r_2\times\\ &\times\Bigg\{-\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\frac{1}{F_{12}}(1+X^2) + \left(\frac{1}{F_{12}}\right)^3\frac{r_1^2+2r_1r_2X+r_2^2}{2}(1+X^2)\\ &+\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2 r_1r_2X(-1+X^2) - \frac{1}{F_1}\frac{1}{F_2}(1+X^2)\\ &+\left(\frac{r_1^2}{F_1^2}+\frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}}(1-X^2) + \frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}}r_1r_2X(-1+X^2)\Bigg\}.\end{aligned}\tag{2.16}$$

Proof: Note that

$$\begin{aligned} a_1(\Lambda,\kappa) &= \frac{2}{3}(\sqrt{4\pi})^2(A_\mu^+\Omega, H_0^{-1}A_\mu^+\Omega) \\ &= \frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right). \end{aligned}$$

Thus (2.15) follows. To see $a_2(\Lambda,\kappa)$ we exactly compute the five terms on the right-hand side of (2.10) separately. Let

$$\begin{aligned} \frac{1}{E_j} &= \frac{1}{|k_j|^2/2+\omega_j}, \quad j=1,2, \\ \frac{1}{E_{12}} &= \frac{1}{|k_1+k_2|^2/2+\omega_1+\omega_2}. \end{aligned}$$

(1) We have

$$\begin{aligned} &2\left(\Psi_3^\mu, H_0^{-1}\Psi_1^\mu\right) = \left(\Omega, -(A^-\cdot A^-)H_0^{-1}A_\mu H_0^{-1}A_\mu^+\Omega\right) \\ &+\frac{1}{2}\left(\Omega,(A^-\cdot A^-)H_0^{-1}(P_{\rm f}\cdot A+A\cdot P_{\rm f})H_0^{-1}P_{{\rm f}\mu}H_0^{-1}A_\mu^+\Omega\right). \\ &= -\iint {\rm d}k_1^3{\rm d}k_2^3\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}(\frac{1}{E_1}+\frac{1}{E_2}){\rm tr}(Q_1Q_2). \end{aligned} \tag{2.17}$$

(2) We have

$$\begin{aligned} &\left(\Psi_2^\mu, H_0^{-1}\Psi_2^\mu\right) \\ &= \left(\frac{1}{2}\right)^2\left(P_{{\rm f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}P_{{\rm f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega\right) \\ &= \left(\frac{1}{2}\right)^2\iint {\rm d}k_1^3{\rm d}k_2^3\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^3|k_1+k_2|^2 2{\rm tr}(Q_1Q_2). \end{aligned} \tag{2.18}$$

(3) We have

$$\begin{aligned} &-2\left(\Psi_2^\mu, H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right) \\ &= \frac{1}{2}\left(P_{{\rm f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}(P_{\rm f}\cdot A+A\cdot P_{\rm f})H_0^{-1}A_\mu^+\Omega\right) \\ &= \iint {\rm d}k_1^3{\rm d}k_2^3\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^2(\frac{1}{E_1}+\frac{1}{E_2})(k_2,Q_1Q_2k_1). \end{aligned} \tag{2.19}$$

(4) We have

$$\begin{aligned} &-\frac{1}{2}\left(\Psi_1^\mu, H_0^{-1}H_2H_0^{-1}\Psi_1^\mu\right) \\ &= -\frac{1}{2}\left(A_\mu^+\Omega, H_0^{-1}((A^+\cdot A^+)+2(A^+\cdot A^-)+(A^-\cdot A^-))H_0^{-1}A_\mu^+\Omega\right) \\ &= -\iint {\rm d}k_1^3{\rm d}k_2^3\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_1}\frac{1}{E_2}{\rm tr}(Q_1Q_2). \end{aligned} \tag{2.20}$$

(5) We have

$$\begin{aligned}
&\left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu\right) \\
&= \left(\frac{1}{2}\right)^2 \left(A_\mu^+\Omega, H_0^{-1}(P_\mathrm{f}\cdot A + A\cdot P_\mathrm{f}) H_0^{-1}(P_\mathrm{f}\cdot A + A\cdot P_\mathrm{f}) H_0^{-1} A_\mu^+\Omega\right) \\
&= \iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\left\{\left(\frac{1}{E_1}\right)^2 (k_1, Q_2 k_1) + \left(\frac{1}{E_2}\right)^2 (k_2, Q_1 k_2)\right\} \\
&+ \iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\frac{1}{E_1}\frac{1}{E_2}(k_2, Q_1 Q_2 k_1). \qquad (2.21)
\end{aligned}$$

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts

$$\begin{aligned}
&\mathrm{tr}[Q_1 Q_2] = 1 + (\hat k_1, \hat k_2)^2, \\
&(k_1, Q_2 Q_1 k_2) = (k_1, k_2)((\hat k_1, \hat k_2)^2 - 1), \\
&(k_1, Q_2 k_1) = |k_1|^2 (1 - (\hat k_1, \hat k_2)^2).
\end{aligned}$$

Thus the proof is complete. □

3 Main theorem

The main theorem is as follows.

Theorem 3.1 *There exist strictly positive constants* $C_{\min}$ *and* $C_{\max}$ *such that*

$$C_{\min} \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

Proof: We show an outline of a proof. See Hiroshima and Spohn [7] for details. By (2.16) we can see that

$$a_2(\Lambda, \kappa) = \frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\sum_{j=1}^6 b_j(\Lambda/m), \qquad (3.1)$$

where

$$\begin{aligned}
b_1(\Lambda/m) &= -\int (1+X^2)\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\frac{1}{F_{12}}, \\
b_2(\Lambda/m) &= \int (1+X^2)\left(\frac{1}{F_{12}}\right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2}, \\
b_3(\Lambda/m) &= \int X(-1+X^2) r_1 r_2 \left(\frac{1}{F_1}+\frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2, \\
b_4(\Lambda/m) &= -\int (1+X^2)\frac{1}{F_1}\frac{1}{F_2}, \\
b_5(\Lambda/m) &= \int (1-X^2)\left(\frac{r_1^2}{F_1^2}+\frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}}, \\
b_6(\Lambda/m) &= \int X(-1+X^2) r_1 r_2 \frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}},
\end{aligned}$$

where

$$\int = \int_{-1}^{1} \mathrm{d}X \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_1 \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_2 \pi r_1 r_2.$$

Let $\rho_\Lambda(\cdot,\cdot) : [0,\infty) \times [-1,1] \to \mathbb{R}$ be defined by

$$\rho_\Lambda = \rho_\Lambda(r,X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,$$

where

$$\Delta = \Lambda^2(1-X^2) + 2\Lambda(1-X) - 1. \tag{3.2}$$

Then we can show that there exist constants C_1, C_2, C_3 and C_4 such that for sufficiently large $\Lambda > 0$,

$$\begin{aligned}
&(1)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)} \le C_1 \frac{1}{\Lambda},\\
&(2)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 \le C_2 \frac{1}{\Lambda^{5/2}},\\
&(3)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)} \frac{1}{r+2} \le C_3 \frac{\log\Lambda}{\Lambda^2},\\
&(4)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 (1-X^2) \le C_4 \frac{1}{\Lambda^3}.
\end{aligned}$$

Using (1)–(4) we can prove that there exists a constant $C > 0$ such that

$$\begin{aligned}
&|b_j(\Lambda/m)| \le C[\log(\Lambda/m)]^2, \quad j = 1,4,\\
&|b_2(\Lambda/m)| \le C(\Lambda/m)^{1/2},\\
&|b_j(\Lambda/m)| \le C\log(\Lambda/m), \quad j = 3,5,6.
\end{aligned}$$

Hence there exists a constant $C_{\max}$ such that

$$\lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

Next we can show that there exists a positive constant $\xi > 0$ such that

$$\lim_{\Lambda\to\infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,$$

which implies that there exists a constan ξ' such that

$$\xi' \le \lim_{\Lambda\to\infty} \frac{b_2(\Lambda/m)}{\sqrt{\Lambda/m}}.$$

Thus we have

$$C_{\min} \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

$\square$

Remark 3.2 *Theorem 3.1 may suggests $\gamma \geq 1/2$ uniformly in e but $e \neq 0$.*

Remark 3.3 *(1) $a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m}$ converges to a nonnegative constant as $\Lambda \to \infty$. (2) By (3.1), we can define $a_2(\Lambda/m, 0)$ since $b_j(\Lambda/m)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda/m, 0)$ also satisfies Theorem 3.1. (3) In the case of $\kappa = 0$, Chen [1] established that $H(0)$ has a ground state $\psi_{\rm g}(0)$ but does not for $H(p)$ with $p \neq 0$.*

4 Concluding remarks

The Pauli-Fierz Hamitonian with the dipole approximation, $H_{\rm dip}$, is defined by H with $A_{\hat\varphi}$ replaced by $1 \otimes A_{\hat\varphi}(0)$, i.e.,

$$H_{\rm dip} = \frac{1}{2m}(p \otimes 1 - e1 \otimes A_{\hat\varphi}(0))^2 + V \otimes 1 + 1 \otimes H_{\rm f}.$$

Set $V \equiv 0$. Note that

$$[H_{\rm dip}, P_{\rm total}] \neq 0.$$

It is established in [5] that there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that

$$UH_{\rm dip}U^{-1} = -\frac{1}{2(m+\delta m)}\Delta \otimes 1 + 1 \otimes H_{\rm f} + e^2 G,$$

where

$$\delta m = m + e^2 \frac{2}{3} \|\hat\varphi/\omega\|^2,$$
$$G = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2 \|\hat\varphi/(t^2+\omega^2)\|^2}{m + (2e^2/3)\|\hat\varphi/\sqrt{t^2+\omega^2}\|^2} dt.$$

Hence

$$[UH_{\rm dip}U^{-1}, P_{\rm total}] = 0.$$

Then we can define the effective mass $m_{\rm eff}$ for $UH_{\rm dip}U^{-1}$, and which is

$$m_{\rm eff}/m = 1 + \alpha\frac{4}{3\pi}(\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for $H_{\rm dip}$ is not available.

References

[1] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp-arc 01-301, preprint, 2001.

[2] C. Hainzl and R. Seiringer, Mass Renormalization and Energy Level Shift in Non-Relativistic QED, math-ph/0205044, preprint, 2002.

[3] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[4] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré*, **3** (2002), 171–201.

[5] F. Hiroshima and H. Spohn, Enhanced binding through coupling to a quantum field, *Ann. Henri Poincaré* **2** (2001), 1159–1187.

[6] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[7] F. Hiroshima and H. Spohn, Mass renormalization in nonrelativistic QED, arXiv:math-ph0310043, preprint, 2003.

[8] E. Lieb and M. Loss, A bound on binding energies and mass renormalization in models of quantum electrodynamics, *J. Stat. Phys.* **108**, 1057–1069 (2002).

[9] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[10] H. Spohn, Effective mass of the polaron: A functional integral approach, *Ann. Phys.* **175** (1987), 278–318.