Anderson localization for a random flux model

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Abstract

This is the review of the work [7]. We prove Anderson localization on the bottom of the spectrum for the discrete Schrödinger operator with certain random magnetic field. Our strategy of the proof is to show Lifschitz tail estimate and Wegner estimate on the bottom of the spectrum from which Anderson localization follows via the multiscale analysis.

1 Introduction

It is Anderson [1] who first discussed that certain disorder may cause materials to have insulating property. From the mathematical point of view, it is expressed as the presence of localized states of Hamiltonians with random potentials where it has pure point spectrum with exponentially decaying eigenfunctions, which was first proved by [4] for one-dimensional case, and by [3] for multi-dimensional case with an simplification by [9].

The purpose of this paper is to study the spectral properties of the Hamiltonians with random magnetic flux with no random potential. Such models originated in a theory of the quantum Hall effect and are actively studied in physics community recently, mainly on nature of states near the middle of the spectrum band. However, there seems to be an agreement of the presence of the localized states on the spectral bottom which is our main topic.

Our Hamiltonian is the Schrödinger operator on the two dimensional lattice given by

$$(Hu)(x) := \sum_{|x-y|=1} \left(u(x) - e^{iA(x,y)} u(y) \right), \quad u \in l^2(\mathbf{Z}^d)$$

where $A : \mathbf{Z}^2 \times \mathbf{Z}^2 \to \mathbf{T} \cong [-\pi, \pi)$ is the vector potential satisfying $A(x, y) = -A(y, x)$, $x, y \in \mathbf{Z}^2$. Let $\mathcal{F}$ be the set of plaquettes on $\mathbf{Z}^2$. For $f = \{x_1, x_2, x_3, x_4\} \in \mathcal{F}$ (indexed counterclockwise), $B(f) := \sum_{j=0}^{3} A(x_j, x_{j+1}) \in \mathbf{T}$ is called the magnetic flux through f. It is well known that the spectral properties of H are determined only on $\{B(f)\}_{f \in \mathcal{F}}$ and independent of the choice of $\{A(x, y)\}_{x,y \in \mathbf{Z}^2}$ such that $dA = B$.

We consider the case in which $\{B(f)\}_{f \in \mathcal{F}}$ are random variables which locally distribute as plus-minus pairs given as follows.

$$A_\omega(x, y) := \begin{cases} B_\omega(2n, m), & (x = (2n+1, m), y = (2n+1, m+1)) \\ -B_\omega(2n, m), & (x = (2n+1, m+1), y = (2n+1, m)) \\ 0, & \text{otherwise} \end{cases}$$

where $\{B(2n, m)\}_{n,m \in \mathbf{Z}}$ are independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that the common distribution has a density g satisfying (i) supp $g \in \mathbf{T} \setminus (-c, c)$ for some $0 < c < \pi$, (ii) $\pm c \in$ supp g, and (iii) g is Lipshitz continuous.

It then follows that

$$\sigma(H) = \left[4(1 - \cos \frac{c}{4}), 4(1 + \cos \frac{c}{4})\right], \quad \text{a.s.}$$

Let $E_0 := \inf \sigma(H) = 4(1 - \cos \frac{c}{4})$. Our main result is the following.

Theorem 1.1 *There exists $E_l > E_0$ such that spectrum of H in $[E_0, E_l]$ are dense pure point with exponentially decaying eigenfunctions.*

The essential ingredient of the proof is the Lifschitz tail and Wegner estimate. Then the exponential decay of the eigenfunctions follows by the use of the multiscale analysis [3, 9]. To be precise, we introduce the integrated density of states. Let $\Lambda_L := [-L, L]^2 \cap \mathbf{Z}^2$ be a finite box of size $2L$ and let H_L be the Hamiltonian on Λ_L defined by restricting H on Λ_L in certain sense to be defined in section 2. The integrated density of states is given by

$$k(E) := \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \sharp \{\text{eigenvalues of } H_L \leq E\}$$

which is known to be non-random almost surely. The following theorem gives an upper bound on $k(E)$ which is called the Lifshitz tail.

Theorem 1.2

$$\limsup_{E\downarrow E_0} \frac{\log(-\log k(E))}{\log(E-E_0)} \le -1.$$

Theorem 1.2 roughly says $k(E) \cong e^{-(E-E_0)^{-1}}$ as $E \downarrow E_0$ which contrasts with the power law growth of $k(E)$ in usual non-random cases. The naive picture of that is, because of the randomness, most states go up so that the density of states near the bottom of spectrum becomes exponentially thin. The Wegner estimate concerns the probability of finding eigenvalues of H_L in a fixed small interval.

Theorem 1.3 *There exists constants $E_0' > E_0, C > 0$ such that*

$$\mathbf{P}\left\{d(E,\sigma(H_L)) \le \epsilon\right\} \le C|\Lambda_L|\epsilon$$

for any $E_0 < E < E_0'$, $L > 0$ and $\epsilon > 0$.

Theorem 1.3 roughly says eigenvalues of H_L "typically" arrange with distance of the order of $\frac{1}{|\Lambda_L|}$ each other.

Having established Theorem 1.2, 1.3, the multiscale analysis proceeds as follows. Let $G_L(E;x,y) =< \delta_x, (H_L-E)^{-1}\delta_y >$ be the Green function of H_L. Theorem 1.2 says for $L_0 \gg 1$, there are no eigenvalues of H_{L_0} near E_0 so that $G_{L_0}(E;x,y)$ decays exponentially with probability close to 1. On the other hand, Theorem 1.3 says $G_L(E;x,y)$ decays exponentially with certain probability and thus, together with the resolvent equation, we prove exponential decay of G_L for larger and larger boxes Λ_L inductively.

For the Hamiltonian with random flux, Theorem 1.2 follows, roughly speaking, by the diamagnetic inequality which says, if magnetic flux goes up, so does the lowest eigenvalue of H_L. The argument of proof follows that in [6] with a improvement to adjust it to the case in which $c > 0$. To prove Theorem 1.3, it is sufficient to have an estimate such as $\frac{\partial E}{\partial |B(f)|} =< u_E, \frac{\partial H_L}{\partial |B_f|} u_E >> 0$, where E, u_E are the eigenvalue and eigenvector of H_L respectively; because this inequality says that once we change $|B(f)|$ on a plaquette, then eigenvalues moves rapidly enough so that the probability to find eigenvalues near E becomes small. If H were the free Laplacian plus the random potential, this inequality would follow from the fact that, when the value of the potential at one site goes up, so do the eigenvalues of H_L. However, in the case of random magnetic flux, some eigenvalues go up but others may go down under the variation of $|B(f)|$. Moreover, in contrast to the case of random potential, u_E may change *globally*, which can be regarded as a kind of Aharonov-Bohm effect.

Our strategy is to decompose H_L to the sum of that on each plaquettes. Eigenstates of H_L near the bottom of the spectrum should "close" to the superpositions of those on each plaquettes, so that the assumption $c > 0$ leads us to the statement $\frac{\partial E}{\partial |B(f)|} > 0$. By our assumption that magnetic fluxes are locally in plus-minus pairs, the Aharonov-Bohm effect mentioned in the preceding paragraph can be controlled in our case.

In the next section, we briefly sketch the proof of Theorem 1.2, 1.3. Details are given in [7]. Some extentions to other lattices, such as triangular or hexagonal lattice as well as their line graphs are discussed in [8], where the theory discussed in [10, 11] is used.

2 Sketch of Proofs

The precise definition of the local Hamiltonian H_L on Λ_L is

$$< u, H_L u > := \sum_{|x-y|=1, x,y\in\Lambda_L} \left| u(x) - e^{iA(x,y)} u(y) \right|^2 .$$

Let

$$0 < \alpha < 1 - \frac{1}{\sqrt{2}}, \quad \gamma := \frac{1}{4}\left(1 - \frac{1}{\sqrt{2}} - \alpha\right),$$
$$\beta(t) := \min\left\{1 - \cos\frac{t}{4}, \alpha\right\}, \quad W_B(x) := \sum_{x\in f} \beta(B(f)).$$

The key lemma for the proof of Theorem 1.2 is :

Lemma 2.1 $< u, H_L u > \geq < u, W_B u > + \gamma < |u|, H_{0,L}|u| >$.

$H_{0,L}$ is the free local Hamiltonian which is defined by setting $A = 0$ in the definition of H_L. Once we have Lemma 2.1, Theorem 1.2 is proved in the following step. Let E be the eigenvalue of H_L sufficiently close to E_0. Then Lemma 2.1 says there exists $E' \sim E_0$ which is the eigenvalue of

$\gamma H_{0,L} + W_B$. Then Temple's inequality says that there should be sufficiently many plaquettes $f \in \mathcal{F}$ such that $B(f)$ is close to $\pm c$ whose probability is exponentially small by the large deviation principle.

Proof. We decompose $< u, H_L u >$ into the forms of plaquettes in Λ_L as follows.

$$
\begin{aligned}
< u, H_L u > &= \sum_{|x-y|=1} \left| u(x) - e^{iA(x,y)} u(y) \right|^2 \\
&= \sum_{f \in \mathcal{F}} \sum_{(x,y) \in f} \frac{1}{2} \left| u(x) - e^{iA(x,y)} u(y) \right|^2 \\
&=: \sum_{f \in \mathcal{F}} < u_f, H_f u_f >
\end{aligned}
$$

where $u_f = u|_f$. We pick $f \in \mathcal{F}$ arbitrary and label $f = \{1, 2, 3, 4\}$ counter-clockwise. We adjust the gauge such that the vector potential is constant:

$$
< u_f, H_f u_f > = < \tilde{u}_f, \tilde{H}_f \tilde{u}_f > := \frac{1}{2} \sum_{j=0}^{3} |\tilde{u}_f(j) - e^{i\frac{B}{4}} \tilde{u}_f(j+1)|^2.
$$

Let $\lambda_j = 1 - \cos\left(\frac{2\pi j + B}{4}\right), (j = 0, 1, 2, 3)$ be the eigenvalues of $\tilde{H}_f$ with Π_j orthogonal projection onto λ_j eigenspaces. Then

$$
\begin{aligned}
< u_f, H_f u_f > = < \tilde{u}_f, \tilde{H}_f \tilde{u}_f > &= \sum_{j=0}^{3} \lambda_j \|\Pi_j \tilde{u}_f\|^2 \\
&\geq \beta(B(f)) \|u_f\|^2 + 4\gamma \|(1 - \Pi_0) \tilde{u}_f\|^2. \qquad (2.1)
\end{aligned}
$$

Here we use the fact that $(1 - \Pi_0)\tilde{u}_f$ is orthogonal to the lowest eigenvector of $\tilde{H}_{0,f}$. Since $\Pi_0 \tilde{u}_f = \frac{1}{4} \sum_{j=1}^{3} \tilde{u}_f(j) =: u_0$,

$$
\|(1 - \Pi_0)\tilde{u}_f\|^2 = \sum_{j=0}^{3} |\tilde{u}_f(j) - u_0|^2 \geq \frac{1}{4} \sum_{j=0}^{3} \left| |\tilde{u}_f(j)| - |\tilde{u}_f(j+1)| \right|^2. \qquad (2.2)
$$

Substituting (2.2) into (2.1) and summing up w.r.t. $f \in \mathcal{F}$, we have the desired conclusion. □

The idea of proof of Theorem 1.3 is as follows. As in the proof of Lemma 2.1, $< u, Hu >= \sum_{f \in \mathcal{F}} \sum_{j=0}^{3} \lambda_j \|\Pi_j u_f\|$. Let $H_L u = Eu$, $\|u\| = 1$ and $E \sim$

E_0. Then for each $f \in \mathcal{F}$, u_f mostly "live" on Ran Π_0. Since $\lambda_0 = 1 - \cos\frac{B}{4}$, we may argue $< u_f, \frac{\partial H_f}{\partial |B(f)|} u_f > \sim \frac{\partial \lambda_0}{\partial |B(f)|} \geq \frac{1}{4}\sin\frac{c}{4} > 0$. The problem now is to clarify $\sim$ in the above computation.

Let v_j $(j = 0, 1, 2, 3)$ be normalized eigenvectors of H_f and

$$\alpha_f := |< v_0, u_f >|, \quad \beta_f := \left(\sum_{j=1}^{3} |< v_j, u_f >|^2 \right)^{\frac{1}{2}}, \quad b := |B(f)|.$$

The intuition discussed above implies $\sum_f \alpha_f^2 = 4 + o(1)$, $\sum_f \beta_f^2 = o(1)$ as $E \downarrow E_0$.

Lemma 2.2 *Let $H_L u = Eu$, $\|u\| = 1$, and $E \sim E_0$. Then*

$$\sum_f \alpha_f^2 \geq 4 - c_1(E - E_0), \quad \sum_f \beta_f^2 \leq c_2(E - E_0)$$

for some $c_1, c_2 < \infty$.

Lemma 2.3

$$< u_f, \frac{\partial H_f}{\partial |B(f)|} u_f > \geq \frac{1}{4}\sin\frac{c}{4}\alpha_f^2 - c_3\alpha_f\beta_f - c_4\beta_f^2$$

for some $c_3, c_4 < \infty$.

The key fact is that there is no quadratic contribution of α_f in the error term.

Proof. For simplicity, let $\lambda_j' := \frac{\partial \lambda_j}{\partial b}$, etc. Then

$$\begin{aligned} & < u_f, \frac{\partial H_f}{\partial b} u_f > \\ = & \sum_{j=0}^{3} \left(\lambda_j' |< u_f, v_j >|^2 + \lambda_j < u_f, v_j' >< v_j, u_f > + \lambda_j < u_f, v_j >< v_j', u_f > \right) \\ =: & \, I + II + III. \end{aligned}$$

By using $v_j' = \sum_k < v_k, v_j' > v_k$, $< v_j', v_k > = - < v_j, v_k' >$, we have

$$II + III = \sum_{j,k} (\lambda_j - \lambda_k) < v_k, v_j' >< u_f, v_k >< v_j, u_f > .$$

We notice the diagonal term vanishes, and arrive at the conclusion. □

By combining Lemma 2.2, 2.3, we have

$$\sum_f < u, \frac{\partial H_L}{\partial |B(f)|} u > \geq \sin \frac{c}{4} - c_5(E - E_0) - c_6\sqrt{E - E_0}$$

for some $c_5, c_6 < \infty$. For the rest of proof of Theorem 1.3, we refer [7]. We remark that some ideas in [5, 2] are used : integration by parts in the computation of the expectation values (and thus we need the Lipshitz continuity of g), and the use of the spectral shift function.

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