Renormalization for Near-Parabolic Fixed Points of Holomorphic Maps

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Consider a fixed point z_0 of a holomorphic function $f(z)$. We say z_0 is *parabolic* if its multiplier $\lambda = f'(z_0)$ is a root of unity. Here we consider the case $\lambda = 1$ and it is non-degenerate, i.e., $f''(z_0) \neq 0$. By a Möbius transformation sending z_0 to infinity, $f(z)$ is conjugate to $\check{f}(z) = z + 1 + O(1/z)$. There exist univalent maps $\Phi_\pm : \{z;\ \pm \operatorname{Re} z > L\} \to \mathbb{C}$ for sufficiently large L such that

$$\Phi_\pm(\check{f}(z)) = \Phi_\pm(z) + 1 \tag{1}$$

where both sides are defined. We call Φ_+ (resp. Φ_-) *attracting* (resp. *repelling*) *Fatou coordinate* for $\check{f}(z)$ (or $f(z)$). We can extend the domain of definition of Fatou coordinates (or its inverses) by the functional equation (1), and then $\Phi_\pm$ are defined on $W_\pm = \{z;\ \pm \operatorname{Im} z > M + |\operatorname{Re} z|\}$ for sufficiently large M. Therefore, $E_f = \Phi_+ \circ \Phi_-^{-1}$ is well-defined on $\Phi_-(W_\pm)$. Since $E_f(z+1) = E_f(z) + 1$, it can be extended to $\{z;\ |\operatorname{Im} z| > L'\}$ for sufficiently large L'. It is called a *horn map*. Since $\Phi_+(z) + c_+$ and $\Phi_-(z) + c_-$ ($c_\pm \in \mathbb{C}$) are also Fatou coordinates for $\check{f}(z)$, we can replace $\Phi_\pm(z)$ by $\Phi_\pm(z) + c_\pm$ for some $c_\pm$ so that $E_f(z)$ is normalized as $E_f(z) = z + o(1)$ as $\operatorname{Im} z \to +\infty$.

Define $\Pi : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by $\Pi(z) = e^{2\pi i z}$. Then we can define a map $\hat{E}_f : \{0 < |z| < e^{-2\pi L'}\} \cup \{|z| > e^{2\pi L'}\} \to \mathbb{C}^*$ satisfying $\hat{E}_f \circ \Pi = \Pi \circ E_f$. We can extend $\hat{E}_f$ holomorphically at 0 and ∞. They are fixed points of $\hat{E}_f$ and by the normalization above, 0 is a parabolic fixed point for $\hat{E}_f$ of multiplier 1.

When we perturb $f(z)$ in an appropriate direction, z_0 bifurcates to two fixed points and return maps near each of the fixed points can be defined (cf. Yoccoz renormalization for a holomorphic germ of an indifferent fixed point). Consider the case $f(z) = e^{2\pi i \alpha} z + O(z^2)$ is an perturbation of $f_0(z) = z + z^2 + O(z^3)$ and assume $\alpha \neq 0$, $|\arg \alpha| < \pi/4$. For f sufficiently close to f_0, we can still define Fatou coordinates and the horn map E_f. Then the return map around a fixed point 0 can be written as $\operatorname{Return}(f)(z) = E_f(z) - 1/\alpha$. Taking a semiconjugacy by Π, we obtain a map $\widehat{\operatorname{Return}}(f)(z) = e^{-2\pi i/\alpha} \hat{E}_f(z)$.

Since $E_f(z)$ converges to $E_{f_0}(z)$ locally uniformly on an appropriate domain, horn maps play an important role in studying bifurcations at parabolic fixed points (e.g. linearizability at a fixed point, existence of a Julia set of positive measure, satellite renormalizations,...). Furthermore, if $\alpha = 1/(n - \alpha_1)$ for $n \in \mathbb{Z}$ sufficiently large, we have $(\widehat{\operatorname{Return}}(f))'(0) = \alpha_1$. If α_1 is also small, $|\arg \alpha_1| < \pi/4$ and $(\widehat{\operatorname{Return}}(f))''(0) \neq 0$, then we can again consider the return map $\widehat{\operatorname{Return}}^2(f)(z)$ for $\widehat{\operatorname{Return}}(f)(z)$.

Hence, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which has the continued fraction of the form

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots}}, \qquad a_i > \exists N \gg 1 \tag{2}$$

and appropriate $f_0(z) = z + z^2 + O(z^3)$, we can consider a sequence of return maps $e^{2\pi i\alpha_n} f_n(z) = \widehat{\mathrm{Return}}(e^{2\pi i\alpha_{n-1}} f_{n-1})$, where $\alpha_0 = \alpha$ and $\alpha_n \equiv -1/\alpha_{n-1} \mod \mathbb{Z}$. To obtain such an infinite sequence $\{f_n\}$, we must have some a priori estimate for f_0. Let us denote $\mathcal{R}_\alpha f = e^{-2\pi i/\alpha}\widehat{\mathrm{Return}}(e^{2\pi i\alpha} f)$ if it is defined.

Our aim here is to define a space of holomorphic maps which is invariant by $\mathcal{R}_\alpha$ and to obtain contraction property of $\mathcal{R}_\alpha$ on it. Since $\mathcal{R}_\alpha \to \mathcal{R}_0$ when $\alpha \to 0$, we first study the operator $\mathcal{R} = \mathcal{R}_0$. We call $f \rightsquigarrow \mathcal{R}(f)$ *parabolic renormalization.* Let

$$\mathcal{F}_0 = \left\{ f : U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in U_f\text{: open and connected in } \mathbb{C},\ f\text{: holomorphic in } U_f, \\ f(0) = 0, f'(0) = 1,\ f : U_f \setminus \{0\} \to \mathbb{C}^* \text{ is a branched covering} \\ \text{with a unique critical value, all critical points have local degree 2} \end{array} \right\}.$$

We can define $\mathcal{R}$ on $\mathcal{F}_0$ and the following theorem is known:

Theorem 1. (i) $\mathcal{R}(\mathcal{F}_0) \subset \mathcal{F}_0$.
(ii) *Let* $f_{\mathrm{Koebe}} = z/(1-z)^2$ *and* $f_\bigstar = \mathcal{R}(f_{\mathrm{Koebe}})$. *Then* $f_\bigstar$ *is defined on* $\mathbb{D}$ *and* $f_\bigstar \in \mathcal{F}_0$. *Any* $f \in \mathcal{R}(\mathcal{F}_0)$ *can be written as* $f = f_\bigstar \circ \phi^{-1}$ *where* $\phi : \mathbb{D} \to U_f$ *a conformal map with* $\phi(0) = 0$, $\phi'(0) = 1$.

Hence there exists a bijection between $\mathcal{R}(\mathcal{F}_0)$ and $\mathcal{S} = \{\phi : \mathbb{D} \to \mathbb{C}$; univalent, holomorphic, $\phi(0) = 0, \phi'(0) = 1\}$. We consider a topology on $\mathcal{R}(\mathcal{F}_0)$ which is induced from local uniform convergence topology on $\mathcal{S}$ by this bijection.

However, considering this space is not sufficient to study $\mathcal{R}_\alpha$ for $\alpha \neq 0$ (e.g. $\mathcal{R}_\alpha f$ have infinitely many critical values). Therefore, we need to relax covering property and we obtain the following:

Main Theorem 1. *Let* $P = z(1-z)^2$. *There exist simply connected domains* $V, V' \subset \mathbb{C}$ *such that* V *contains the fixed point* 0 *and the critical point* $-1/3$ *for* P, $V \Subset V'$, V *is a quasidisk and*

$$\mathcal{F}_1 = \left\{ f = P \circ \phi^{-1} \;\middle|\; \begin{array}{c} \phi : V \to \mathbb{C} \text{ is conformal, } \phi(0) = 0, \phi'(0) = 1, \\ \text{and has a quasiconformal extension to } \mathbb{C} \end{array} \right\}.$$

satisfies the following:

(i) $\mathcal{F}_0 \setminus \{$*quadratic polynomial*$\} \hookrightarrow \mathcal{F}_1$. *In particular,* $\mathcal{R}_0^n(z + z^2) \in \mathcal{F}_1$ $(n \geq 1)$.
(ii) $\mathcal{R}_0 f \in \mathcal{F}_1$ *for* $f \in \mathcal{F}_1$. *If* $\mathcal{R}_0 f = P \circ \psi^{-1}$, *then* ψ *can be extended conformally on* V'.
(iii) $f \rightsquigarrow \mathcal{R}_0$ *is "holomorphic".*
(iv) *There exists* $\alpha_0 > 0$ *such that* $\mathcal{R}_\alpha$ $(0 < \alpha < \alpha_0)$ *also satisfies* (ii) *and* (iii).

Main Theorem 2. *There exists a metric* d *on* $\mathcal{F}_1$ *such that* $\mathcal{R}_\alpha$ $(0 \leq \alpha < \alpha_0)$ *is a uniform contraction with respect to* d.

Corollay 2. *For* α *satisfying* (2) *and* $f = f_0 \in \mathcal{F}_1$, *the sequence of return maps* $\{e^{2\pi i\alpha_n} f_n = \widehat{\mathrm{Return}}^n(e^{2\pi i\alpha} f)\}$ $(\alpha_n \equiv -1/\alpha_{n-1} \mod \mathbb{Z}$ *and* $f_n = \mathcal{R}_{\alpha_{n-1}} f_{n-1}$ $(n = 1, 2, \dots))$ *is defined and* $f_n \in \mathcal{F}_1$. *It is also true for* $f(z) = z + z^2$.

Corollay 3. *For* α *in Corollary 2,* $g(z) = e^{2\pi i\alpha} z + z^2$ *does not have dense critical orbit in its Julia set.*