Fourier Transformation of 2D O(N) Spin Model and Anderson Localization

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(September 08,2005)

1: Introduction

It is a longstanding problem to prove or disprove non-existence of phase transitions in 4D lattice gauge thoeries or in 2D sigma models. In the latter model, the model is transformed in to a (complex valued) random potential system [1, 2]. We expect that this problem can be solved from this point of view.

Scaling $\phi \in S^{N-1}$ by $(N\beta)^{1/2}$, we put $\phi_x \in (\beta N)^{1/2} S^{N-1}$. Using $\delta(\phi_x^2 - \beta N) = \int \exp[i\psi_x(\phi_x^2 - \beta N)]d\psi_x$, we start with the expression for the two-point function:

$$\begin{aligned} < \phi_0 \phi_\zeta > &= \frac{1}{Z} \int \phi_0 \phi_\zeta \exp[- < \phi, (-\Delta)\phi >] \prod \delta(\phi_x^2 - \beta N) d^N \phi_x \\ &= \frac{1}{Z} \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, \zeta) F(\psi) \prod d\psi_x \end{aligned}$$

where Δ is the Laplacian defined on the lattice space Z^2 $((\Delta)_{xy} = -4\delta_{x,y} + \delta_{|x-y|,1})$, $\kappa = 2/\sqrt{N}$ and

$$\begin{aligned} F(\psi) &= \det{}^{-N/2}(1 + i\kappa G\psi) \exp[i\sqrt{N}\beta \sum \psi_x], \\ G(x, y) &= \frac{1}{-\Delta + m^2}(x, y) \end{aligned}$$

We choose $m > 0$ so that $G(0) = \beta$ and then

$$F(\psi) = \det{}_2^{-N/2}(1 + i\kappa G\psi) = \det{}_3^{-N/2}(1 + i\kappa G\psi) \exp[-\mathrm{Tr}(G\psi)^2]$$

2: Anderson Localization means mass generation

We investiagte this model as a variation of Anderson's tight binding Hamiltonian[2]. The distribution of ψ is

$$\begin{aligned} F(\psi) &= \det{}_3^{-N/2}(1 + i\kappa G\psi) \exp[- < \psi, G^{\circ 2}\psi >], \\ \mathrm{Tr(G\psi)}^2 &= \sum_{x,y} \psi(x)\psi(y) G^2(x, y) =< \psi, G^{\circ 2}\psi > \end{aligned}$$

We set

$$G^{(ave)}(x, y) \equiv \frac{1}{Z} \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(x, y) F(\psi) \prod d\psi_\zeta$$

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and we would like to prove the following conjecture
Conjecture:

$$G^{(ave)}(x,y) < ce^{-\alpha|x-y|}, \quad \alpha > 0$$

for all $\beta > 0$ and $N >> 2$.

Our present result is several steps before the goal. Assume that the interaction is restricted to a finite rectangular region $\Lambda \subset Z^2$ and study the limit $\Lambda \to Z^2$. We decompose Λ into many small squares $\Lambda = \cup_{i=1}^{n} \Delta_i$ and apply the Feshbach formula to obtain $\det^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda)$ in the following form:

$$\left[\prod_{i=1}^{n-1} \det{}^{-N/2}\left(1 + W(\Delta_i, \Lambda_i)\right)\right] \prod_{i=1}^{n} \det{}^{-N/2}\left(1 + i\kappa G_{\Delta_i} \psi_{\Delta_i}\right) \tag{1}$$

where $\kappa = 2/\sqrt{N}$, $\Lambda_k = \cup_{i=k+1}^{n} \Delta_i$, $G_\Delta = \chi_\Delta G \chi_\Delta$, ${}_\Lambda G_\Delta = \chi_\Lambda G \chi_\Delta$ and

$$W(\Delta_i, \Lambda_i) \quad = \quad -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}} G_{\Delta_i,\Lambda_i} \psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i}\psi_{\Lambda_i}} G_{\Lambda_i,\Delta_i} \psi_{\Delta_i}$$

$[G_\Lambda]^{-1}$ is a Laplacian with free boundary conditions at $\partial\Lambda_i$. To apply cluster expansion to prove the long-standing conjecture, we will have to show that $W(\Delta, \Lambda)$ are small. So this work is the first step in this direction.

We prove that $([G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i})^{-1}$ behaves as massive Green's functions which decrease fast (localization). The measure restricted to each block is

$$d\mu_\Delta = \det{}_3^{-N/2}(1 + \frac{2i}{\sqrt{N}} G_\Delta \psi_\Delta) \exp[-(\psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta)] \prod_{x \in \Delta} d\psi(x) \tag{2}$$

which is almost equal to the Gaussian measure $\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)] \prod_{x\in\Delta} d\psi(x)$.

In this talk, we show that our *Conjecture* is proved if we use the localized measure $\prod d\mu_\Delta$ is used for $F(\psi) \prod d\psi_x$, [3, 4]. We also show that $W(\Delta, \Lambda)$ are rather small to such an extent that the polymer-expansion is applicable.

References

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