# A Path Integral Preliminary Approach to the FKG Inequality for Yukawa$_2$ Quantum Field Theory*

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1. By the method used in our previous paper [I1], we construct a countably additive path space measure for the 2-D Euclidean Dirac equation in the polar coordinates to give a path integral representation to its Green's function (For a brief survey, see [I2]). This is a report of trying a preliminary approach with use of the result to give an alternative proof of the FKG inequality for Yukawa$_2$ quantum field theory obtained by Battle–Rosen [BR], though not yet incomplete.

G.A.Battle and L.Rosen used Vekua–Bers theory of generalized analytic functions to show the FKG inequality for $Y_2$ QFT. The $Y_2$ measure is formally given by

$$\nu := \frac{1}{Z} e^{W(\phi)} \prod_{x\in\mathbf{R}^2} d\phi(x)$$

$$W(x) := \frac{1}{2}(\phi, (-\Delta + m_b^2)\phi) + \operatorname{Tr} K - \frac{1}{2} : \operatorname{Tr} K^* K : + \operatorname{Tr} \ln(1-K)K,$$

with $Z$ is a normalized constant, where

$$K(x,y) := S(x,y)\phi(y)\chi_\Lambda(y), \ \ \phi: \text{ Boson field (mass} : m_b),$$
$$\chi_\Lambda : \text{indicator function of a square } \Lambda \subset \mathbb{R}^2,$$

and

$$S(x,y) := (-\beta\partial_x + m_f)^{-1}\Gamma, \quad \beta\partial_x = \beta_0\partial_0 + \beta_1\partial_1, x = (x_0, x_1),$$
$$\beta_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \beta_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3,$$

with $m_f \geq 0$ the Fermi mass. They considered the two models

$$a)\ \Gamma := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \ \ \text{(scalar } Y_2), \quad b)\ \Gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \ \ \text{(pseudo-scalar } Y_2).$$

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Then FKG enequality (like $\langle fg\rangle \geq \langle f\rangle\langle g\rangle$) holds: $\dfrac{\delta^2 W}{\delta\phi(x)\delta\phi(y)} \geq 0,\ x \neq y.$

By some heuristic arguments, this is equivalent to showing

$$\mathrm{tr}\, S'(x,y)S'(y,x) \leq 0, \quad x \neq y.$$

where $S' := (1-K)^{-1}S$ is the Green's function (vanishing at $\infty$) for 2D- Euclidean Dirac equation

$$[\Gamma^{-1}(-\beta\partial_x + m_f) - \phi(x)\chi_\Lambda(x)]S'(x,y) = \delta(x-y).$$

Battle and Rosen proved the above inequality for $m_f \geq 0$ in the case a) and for $m_f = 0$ in the case b).

So, the first thing to do is to construct this Green's function.

In [I1], we constructed a countably additive path space measure to give a path integral representent for the Green's function for $3D$-Dirac equation in the radial coordinate.

The aim of this talk is to give a preliminary approach to ask whether this method can apply to get the Green's function for the above $2D$-Euclidean Dirac equation to show the desired inequality.

Put the 2D-Euclidian operator $L^2(\mathbb{R}^2)^2 \equiv L^2(\mathbb{R}^2)\otimes\mathbb{C}^2$ as:

$$\begin{aligned} T_\Gamma :&= \Gamma^{-1}(-\beta\partial_x + m_f) - V(x), \quad V(x) := \phi(x)\chi_\Lambda(x), \\ &= \Gamma^{-1}\Big[-\sigma_1\frac{\partial}{\partial x_0} - \sigma_3\frac{\partial}{\partial x_1} + m_f\Big] - V(x), \quad x = (x_0, x_1) \in \mathbb{R}^2, \\ &\beta = (\beta_0, \beta_1),\ \beta_0 = \sigma_1,\ \beta_1 = \sigma_3. \end{aligned}$$

They considered the two models: a) scalar $Y_2$ model: $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

b) pseudoscalar $Y_2$ model: $\Gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In this note let us consider only a) the scalar $Y_2$ model.

2. Since $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have by the polar coordinates $x_0 = r\cos\theta,\ x_1 = r\sin\theta\,(0 \leq r < \infty,\ 0 \leq \theta < 2\pi)$,

$$T_\Gamma = -C(\theta)\frac{\partial}{\partial r} - \frac{1}{r}D(\theta)\frac{\partial}{\partial\theta} + m_f - V,$$

where

$$\begin{aligned} C(\theta) &:= \sigma_1\cos\theta + \sigma_3\sin\theta = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}, \\ D(\theta) &:= -(\sigma_1\sin\theta - \sigma_3\cos\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}. \end{aligned}$$

We write $\mathbb{R}_+ = (0, \infty)$ and $\overline{\mathbb{R}_+} = [0, \infty)$.

Making the unitary tansformation

$$U(\theta) := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\sin\theta} & \frac{\cos\theta}{\sqrt{1+\sin\theta}} \\ -\frac{\cos\theta}{\sqrt{1+\sin\theta}} & \sqrt{1+\sin\theta} \end{pmatrix},$$

we have

$$U(\theta)T_\Gamma U(\theta)^{-1} = \Big[ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} + \frac{1}{2r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \Big] + m_f - V$$

in $L^2(\mathbb{R}^2)^2 = L^2(\overline{\mathbb{R}_+} \times [0, 2\pi)\,;\, rdrd\theta)^2$.

We make one more unitary transformation $W$ of the $rdr$-measure space to the $dr$-measure space:

$$W : L^2(\mathbb{R})^2 \equiv L^2(\overline{\mathbb{R}_+} \times [0, 2\pi)\,;\, rdrd\theta)^2 \ni f \mapsto r^{1/2} f \in L^2(\overline{\mathbb{R}_+} \times [0, 2\pi)\,;\, drd\theta)^2$$

to get

$$WU(\theta)T_\Gamma U(\theta)^{-1}W^{-1} = \Big[ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} - \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \Big] + m_f - V.$$

Then we multiply $r^{1/2}$ from the left and the right and then multiply the factor $i$ to put

$$\begin{aligned} H_{sc}(rV) &:= ir^{1/2}WU(\theta)T_\Gamma U(\theta)^{-1}W^{-1}r^{1/2} \\ &= \Big[ -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \Big] + i(m_f - V)r. \end{aligned}$$

Since the operator $-i\frac{\partial}{\partial\theta}$ is a selfadjoint operator in $L^2([0, 2\pi)\,;\, d\theta)$ having as the spectrum consisting of only the eigenvalues $\{k\}_{k\in\mathbb{Z}}$ with eigenfunctions $\{\frac{e^{ik\theta}}{\sqrt{2\pi}}\}_{k\in\mathbb{Z}}$, our $L^2$ space $L^2(\overline{\mathbb{R}_+} \times [0, 2\pi)\,;\, drd\theta)^2$ admits the direct sum decomposition:

$$L^2(\overline{\mathbb{R}_+} \times [0, 2\pi)\,;\, drd\theta)^2 = \sum_{k\in\mathbb{Z}} \oplus \Big( L^2(\overline{\mathbb{R}_+}\,;\, dr)^2 \otimes \Big[ \frac{e^{ik\theta}}{\sqrt{2\pi}} \Big] \Big).$$

Then we have

$$\begin{aligned} H_{sc}(rV) &= \sum_{k\in\mathbb{Z}} \oplus H_{sc}(k), \\ H_{sc}(k) &:= \Big[ -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big] + i(m_f - V)r. \end{aligned}$$

We want to find a path integral representation for the Green's function for this operator having a singularity at $r = 0$.

For each fixed $k \in \mathbb{Z}$, put the free part of $H_{sc}(k)$ to be equal to

$$H_0(k) := -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is an operator in $L^2(\overline{\mathbb{R}_+}\,; dr)^2$. We can show that $H_0(k)$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}_+)^2$, which is a non-trivial result. Therefore the Cauchy problem for it :

$$\frac{\partial}{\partial t}\psi(r,t) = -iH_0(k)\psi(r,t), \quad t \in \mathbb{R},$$
$$\psi(r,0) = g(r), \quad t = 0,$$

is $L^2$ well-posed. In other words, we can solve it in the space $L^2(\overline{\mathbb{R}_+}\,; dr)^2$.

Crucial is that this Cauchy problem is even $L^\infty$ well-posed. Namely, we have the following lemma.

**Lemma**. There exists a unique solution $\psi(r,t) = (e^{-itH_0(k)}g)(r)$ which satisfies

$$\|\psi(\cdot,t)\|_\infty = \|e^{-itH_0(k)}g\|_\infty \le e^{|t|(|k|+1/2)}\|g\|_\infty.$$

By the method in [I1] based on this lemma, we can construct a $2 \times 2$-matrix-distribution-valued countably additive path space measure $\mu_{t,0}^k$ on the space $C([0,t] \to \overline{\mathbb{R}_+})$ of the continuous paths $R : [0,t] \to \overline{\mathbb{R}_+}$ which represents the solution of the above Cauchy problem: for every pair of $f$ and $g$ in $C_0^\infty(\mathbb{R}_+)^2$,

$$\begin{aligned}(f, \psi(\cdot,t)) &= \int_0^\infty {}^t\overline{f(r)}(e^{-itH_{sc}(k)}g)(r)\,dr = \int_0^\infty \int_0^\infty {}^t\overline{f(r)}e^{-itH_{sc}(k)}(r,\rho)g(\rho)\,dr d\rho \\ &= \int_{C([0,t]\to\overline{\mathbb{R}_+})} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}^k(R)g(R(0))\rangle e^{\int_0^t (m_f - V(R(s))R(s)ds}.\end{aligned}$$

Hence, supposing that we can get the inverse of the operator $H_{sc}(k)$ as $H_{sc}(k)^{-1} = i\int_0^\infty e^{-itH_{sc}(k)}\,dt$ by the Laplace transform, we have the following path integral representation for its Green's function, which is a little formally expressed, suppressing the use of test funtions:

$$\begin{aligned}&H_{sc}(k)^{-1}(r,\rho) \\ &= i\int_0^\infty dt \int_{C([0,t]\to\overline{\mathbb{R}_+}),R(0)=\rho,R(t)=r} r^{1/2}\rho^{1/2}e^{\int_0^t (m_f - V(R(s))R(s)ds}d\mu_{t,0}^k(R).\end{aligned}$$

3. We have

$$T_\Gamma^{-1} = ir^{1/2}WU(\theta)H_{sc}(rV)^{-1}U(\theta)^{-1}W^{-1}r^{-1/2}.$$

Here, if we use the polar coordinates for $x = (x_0, x_1)$, $y = (y_0, y_1) \in \mathbb{R}^2$

$$x_0 = r\cos\theta, \quad x_1 = r\sin\theta \ (0 \le r < \infty, \ 0 \le \theta < 2\pi),$$
$$y_0 = r'\cos\theta', \ y_1 = r'\sin\theta' \ (0 \le r' < \infty, \ 0 \le \theta' < 2\pi),$$

we may write the integral kernel of the operator $H_{sc}(rV)^{-1}$ as

$$
\begin{aligned}
&H_{sc}(rV)^{-1}(r,\theta\,;\,r',\theta')\\
&=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} H_{sc}(k)^{-1}(r,r')e^{-ik(\theta-\theta')}\\
&=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-ik(\theta-\theta')} i\int_{R(0)=r',\,R(t)=r} r^{1/2}{r'}^{1/2}e^{\int_0^\infty(m_f-V(R(s)))R(s)ds}d\mu_{t,0}^{(k)}(R).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathrm{tr}\left[T_\Gamma^{-1}(r,\theta\,;\,r',\theta')T_\Gamma^{-1}(r',\theta'\,;\,r,\theta)\right.\\
&=-\mathrm{tr}\left[r^{-1/2}WU(\theta)H_{sc}(rV)^{-1}(r,\theta\,;\,r',\theta')U(\theta')^{-1}W^{-1}r^{-1/2}\right.\\
&\quad\left.\times\, r^{-1/2}WU(\theta')H_{sc}(rV)^{-1}(r',\theta'\,;\,r,\theta)U(\theta)^{-1}W^{-1}r^{-1/2}\right]\\
&=-\mathrm{tr}\left[rr'H_{sc}(rV)^{-1}(r,\theta\,;\,r',\theta')H_{sc}(rV)^{-1}(r',\theta'\,;\,r,\theta)\right]\\
&=-rr'\,\mathrm{tr}\left[\Big(\sum_{k\in\mathbb{Z}}H_{sc}(k)^{-1}(r,r')\frac{e^{-ik(\theta-\theta')}}{2\pi}\Big)\Big(\sum_{\ell\in\mathbb{Z}}H_{sc}(\ell)^{-1}(r',r)\frac{e^{-i\ell(\theta'-\theta)}}{2\pi}\Big)\right]\\
&=-\frac{rr'}{(2\pi)^2}\mathrm{tr}\left[\sum_{k,\ell\in\mathbb{Z}}H_{sc}(k)^{-1}(r,r')H_{sc}(\ell)^{-1}(r',r)e^{-i(k-\ell)(\theta-\theta')}\right]\\
&=-\frac{rr'}{(2\pi)^2}\mathrm{tr}\sum_{k,\ell\in\mathbb{Z}}a_{k\ell}e^{-i(k-\ell)(\theta-\theta')}.
\end{aligned}
$$

Here we seem to have

$$
\begin{aligned}
a_{k\ell}:=&\,i\int_0^\infty e^{-itH_{sc}(k)}(r,r')dt\,(-i)\int_0^\infty e^{iuH_{sc}(\ell)}(r',r)du\\
=&\int_0^\infty dt\int_0^\infty du\\
&\times\int_{C([0,t]\to\overline{\mathbb{R}_+}),R_1(0)=r',R_1(t)=r} e^{\int_0^t(m_f-V(R_1(s))R_1(s)ds}d\mu_{t,0}^k(R_1)\\
&\quad\times\int_{C([0,u]\to\overline{\mathbb{R}_+}),R_2(0)=r',R_2(u)=r} e^{\int_u^0(m_f-V(R_2(s))R_2(s)ds}d\mu_{0,u}^\ell(R_2)\\
=&\int_0^\infty dt\int_0^\infty du\int_{C([0,t]\to\overline{\mathbb{R}_+}),R_1(0)=r',R_1(t)=r}\int_{C([0,u]\to\overline{\mathbb{R}_+}),R_2(0)=r',R_2(u)=r}\\
&\times e^{\int_0^t(m_f-V(R_1(s))R_1(s)ds-\int_0^u(m_f-V(R_2(s))R_2(s)ds}d\mu_{t,0}^k(R_1)d\,{}^t\mu_{u,0}^\ell(R_2),
\end{aligned}
$$

where ${}^t\mu_{u,0}^\ell$ is the transposed of the $2\times 2$-matrix-distribution valued-measure $\mu_{0,u}^\ell$.

Then the problem is to show in the case a) that

$$
\mathrm{tr}\sum_{k,\ell\in\mathbb{Z}}a_{k\ell}\,e^{-i(k-\ell)(\theta-\theta')}\geq 0.
$$

But our the argument is stopped here, and will be discussed elsewhere.

## References


[BR] Battle, G. A. and Rosen, L., *The FKG inequality for the Yukawa$_2$ quantum field theory*, J. Stat. Phys., **22**, no.2, 123–192 (1980).

[I1] Ichinose, T., *Path integral for the radial Dirac equation*, J. Math. Phys. **46**, 022103, 19 pages (2005).

[I2] Ichinose, T., *On path integral for the radial Dirac equation*, to appear in the Proceedings of the 8-th International Conference "Path Integrals. From Quantum Information to Cosmology", Prague, June 6–10, 2005.