A stochastic approach to the bipolaron model

József Lőrinczi
Wolfgang Pauli Institut, Fakultät für Mathematik, Universität Wien
Nordbergstr. 15, 1090 Wien, Austria
and
Zentrum Mathematik, Technische Universität München
Boltzmannstr. 3, 85747 Garching bei München, Germany
lorinczi@ma.tum.de

Norbert J. Mauser
Wolfgang Pauli Institut, Fakultät für Mathematik, Universität Wien
Nordbergstr. 15, 1090 Wien, Austria
Norbert.Mauser@univie.ac.at

February 9, 2006

Abstract

We present a path measure-based approach to the problem of ground state of the bipolaron model. We discuss the method and derive a functional measure for the bipolaron for all compact intervals of the line. We show that this measure is the mixture of Gaussian measures and a Gibbs measure with respect to two independent Brownian motions.

1 Introduction

Phonons are quantum particles carrying the vibration energy of an ionic crystal. Electrons interacting with phonons appear then as effectively embedded into an energy cloud ("dressed electrons"); in this state they are called polarons. A *bipolaron* system consists of two dressed electrons in which two forces compete. One is the Coulomb repulsion between the two electrons carrying the same negative charge. The other results from the coupling to the phonon cloud which tends to hold the two polarons together into a two-lobe system (acting like a flexible membrane).

Formally, the bipolaron Hamiltonian is

$$H_{\mathrm{bip}} = H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}} + H_{\mathrm{i}}. \tag{1.1}$$

Here

$$H_{\mathrm{p}} = (-1/2)\nabla_{x_1}^2 + (-1/2)\nabla_{x_2}^2 + \frac{g}{|x_1 - x_2|} \tag{1.2}$$

is the Hamiltonian of the two indistinguishable particles of unit mass (assumed spinless) at positions $x_1, x_2 \in \mathbb{R}^3$. The electrons interact via a Coulomb repulsion of strength $g > 0$. The operator

$$H_{\mathrm{f}} = \int_{\mathbb{R}^3} a^*(k)a(k)\mathrm{d}k \tag{1.3}$$

is the free field Hamiltonian featuring the usual Bose annihilation and creation operators a, a^* describing the phonons, and

$$H_{\mathrm{i}} \text{ “ = ” } \alpha \sum_{j=1}^{2} \int_{\mathbb{R}^3} \frac{1}{|k|} \left(e^{ik\cdot x_j} \otimes a(k) + e^{-ik\cdot x_j} \otimes a^*(k) \right) \mathrm{d}k \tag{1.4}$$

gives the interaction between the electrons and the phonon field. $\alpha < 0$ is the electron-phonon coupling parameter. We are going to make sense of these objects in the following section.

A basic question is for which values of parameters α and g does a ground state for the bipolaron exist. There is substantial literature on the subject in physics (for instance, [2, 6, 1, 15]), however, no mathematical proof of occurrence of a ground state seems to exist. The intuition is that though the Coulomb repulsion keeps the two electrons apart, a sufficiently strong coupling to the phonon field would balance this effect and a ground state of the system should exist. As in the case of Nelson's model studied in [3, 10, 11], having a path measure at hand should make possible to express the ground state of the system in terms of its density with respect to a product measure describing the uncoupled system. This is useful when the path measure can be constructed by independent probabilistic means, referring to no underlying ground state. In the present case, however, beside the competing interactions an extra difficulty is that the Hamiltonian is translation invariant, in contrast with the so far understood cases where an external potential was used to pin the ground state down.

In this note we explain how the path measure for any bounded subinterval of the real line can be constructed. In order to derive the relationship between the density of the two path measures and the bipolaron's ground state, this construction has to be extended over the full line. We refer to [7, 8] for more details and proofs.

2 Bipolaron model in Fock space

In Fock space representation the particle space is the joint Hilbert space of the two electrons, $L^2(\mathbb{R}^3, \mathrm{d}x) \times L^2(\mathbb{R}^3, \mathrm{d}x)$. The underlying space for the Bose field is the symmetric

Fock space $\mathcal{F} := \oplus_{n=0}^{\infty} \mathcal{F}^{(n)}$ of functions $(f_0, f_1, ...)$ for which the direct sum norm

$$\|F\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{F}^{(n)}}^2$$

converges. The Fock sectors $\mathcal{F}^{(n)} := \otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^3, \mathrm{d}x)$, with $\otimes_{\mathrm{sym}}^0 L^2(\mathbb{R}^3, \mathrm{d}x) = \mathbb{C}$, are symmetric tensor products of $L^2(\mathbb{R}^3, \mathrm{d}x)$ spaces. $\mathcal{F}^{(n)}$ is spanned by linear combinations of functions of the form $f^{\otimes n} = f^{\otimes_{\mathrm{sym}} n}$, i.e., of $L^2(\mathbb{R}^{3n}, \mathrm{d}x)$ functions f that are symmetric in the sense that for each $k_1, \ldots, k_n \in \mathbb{R}^3$ and any permutation π of $\{1, \ldots, n\}$ it is true that $f(k_1, \ldots, k_n) = f(k_{\pi(1)}, \ldots, k_{\pi(n)})$.

On $\mathcal{F}$ linear operators $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^3, \mathrm{d}x)$, are defined, called Bose annihilation operator and creation operator, respectively. Since the linear hull of joint Fock sectors is dense in $\mathcal{F}$, it is sufficient to define these operators on $\mathcal{F}^{(n)}$. For $f^{\otimes n} \in \mathcal{F}^{(n)}$ and $g \in L^2(\mathbb{R}^3, \mathrm{d}x)$ write

$$\begin{aligned} a^*(g) f^{\otimes n} \equiv \left(\int a^*(k) g(k) \mathrm{d}k \right) f^{\otimes n} &= \sqrt{n+1} f^{\otimes n} \otimes_{\mathrm{sym}} g \in \mathcal{F}^{(n+1)}, \\ a(g) f^{\otimes n} \equiv \left(\int a(k) g(k) \mathrm{d}k \right) f^{\otimes n} &= \sqrt{n} \, \langle \bar{g}, f \rangle_{L^2(\mathbb{R}^d, \mathrm{d}x)} \, f^{\otimes (n-1)} \in \mathcal{F}^{(n-1)}, \quad \forall n > 0, \end{aligned}$$

while $(a(g))(\mathcal{F}^{(0)}) = 0$. Above we have

$$(f^{\otimes n} \otimes_{\mathrm{sym}} g)(k_1, \ldots, k_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \left(\prod_{j \neq i}^{n+1} f(k_j) \right) g(k_i).$$

Both operators have the domain $\{(f_0, f_1, \ldots) \in \mathcal{F} : \sum_{n=0}^{\infty} n \, \|f_n\|_{\mathcal{F}^{(n)}}^2 < \infty\}$. Furthermore, $\langle F, a(g) G \rangle_{\mathcal{F}} = \langle a^*(\bar{g}) F, G \rangle_{\mathcal{F}}$ with F, G in the above domain, and the canonical commutation relations

$$[a(f), a^*(g)] = \langle \bar{f}, g \rangle_{L^2(\mathbb{R}^3, \mathrm{d}x)}, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0$$

hold.

The free field operator $H_{\mathrm{f}} = \int_{\mathbb{R}^3} a^*(k) a(k) \mathrm{d}k$ is the differential second quantisation of the identity with

$$\left(\int_{\mathbb{R}^3} a^*(k) a(k) \, \mathrm{d}k \right) f^{\otimes n} = n f^{\otimes (n)}. \tag{2.1}$$

This operator is self-adjoint and coincides with the boson number operator.

We want now to give a self-adjoint description of the Hamiltonian (1.1). Since the right hand side of (1.4) is not well defined, we take the ultraviolet cutoff function $1_{\{|k|<\Lambda\}}$ with parameter $\Lambda > 0$ and write

$$H_{\mathrm{i}}(\Lambda) := \alpha \sum_{l=1}^{2} \int_{\mathbb{R}^3} \frac{1_{\{|k|<\Lambda\}}}{|k|} \left(e^{ik \cdot x_l} \otimes a(k) + e^{-ik \cdot x_l} \otimes a^*(k) \right) \mathrm{d}k. \tag{2.2}$$

This gives rise to the well-defined cutoff-Hamiltonian $H_{\text{bip}}(\Lambda) = H_{\text{p}} \otimes 1 + 1 \otimes H_{\text{f}} + H_{\text{i}}(\Lambda)$. Next we take the operator

$$G_\Lambda = -\frac{1}{\sqrt{2}} \sum_{l=1}^{2} \int_{\mathbb{R}^3} \beta_\Lambda(k) \left(e^{ik \cdot x_l} \otimes a(k) + e^{-ik \cdot x_l} \otimes a^*(k) \right) \mathrm{d}k, \tag{2.3}$$

with

$$\beta_\Lambda(k) = \frac{1_{\{|k|<\Lambda\}}}{|k|(1+|k|^2/2)}, \tag{2.4}$$

and consider Gross transform obtained as a conjugation map with respect to the exponential of G_Λ [12, 14]. This transform is a unitary map turning H_{bip} into a Hamiltonian with minimal coupling. A direct calculation shows that the field operators resp. the particle momenta are transformed in the following way:

$$e^{G_\Lambda} a(k) e^{-G_\Lambda} = a(k) + \beta_\Lambda(k) \left(e^{-ik \cdot x_1} + e^{-ik \cdot x_2} \right)$$

and the hermitian conjugate of this expression for a^*, respectively

$$e^{G_\Lambda} p_l e^{-G_\Lambda} = p_l + A_\Lambda(x_l) + A_\Lambda^*(x_l),$$

where we used $l = 1, 2$ for distinguishing the particles, and have that

$$A_\Lambda(x_l) = -\frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} k \beta_\Lambda(k) e^{ik \cdot x_l} a(k) \mathrm{d}k. \tag{2.5}$$

Thus we obtain

$$\begin{aligned} H_{\text{G}}(\Lambda) := e^{G_\Lambda} H_{\text{bip}}(\Lambda) e^{-G_\Lambda} \quad &= \quad \sum_{l=1}^{2} \{ \frac{1}{2}(p_l^2 - p_l \cdot A(x_l) - A^*(x_l) \cdot p_l \\ &\qquad +2A(x_l)^2 + 2A^*(x_l)^2 + A(x_l) \cdot A^*(x_l) \} \\ &\qquad + \frac{g}{|x_1 - x_2|} + H_{\text{f}} \\ &\qquad + V_{\text{eff}}(x_1, x_2; \Lambda) + E_\Lambda. \end{aligned} \tag{2.6}$$

The extra terms are given by

$$V_{\text{eff}}(x_1, x_2; \Lambda) := -\alpha^2 \int_{\mathbb{R}^3} \frac{\beta_\Lambda(k)}{|k|} \cos k \cdot (x_1 - x_2) \, \mathrm{d}k \tag{2.7}$$

and

$$E_\Lambda := \int_{\mathbb{R}^3} \frac{\beta_\Lambda(k)}{|k|} \mathrm{d}k. \tag{2.8}$$

V_{eff} is an effective interaction between the particles, and E_Λ is an additive constant term.

Notice that $\beta_\Lambda \in L^2(\mathbb{R}^3, \mathrm{d}x)$, for all positive Λ, including infinity. Similarly, since $k\beta_\Lambda \in L^2(\mathbb{R}^3, \mathrm{d}x)$, $\forall\, 0 \leq \Lambda \leq \infty$, A_Λ and A_Λ^* are well defined symmetric operators also in the limit $\Lambda \to \infty$. Also, $\exists\, E_\infty = \lim_{\Lambda \to \infty} E_\Lambda$, moreover we have that

$$\lim_{\Lambda \to \infty} V_{\text{eff}}(x_1, x_2; \Lambda) = \frac{1}{|x_1 - x_2|} \left(C_1 + C_2 \, e^{-|x_1 - x_2|} \right), \quad C_1 < 0, \; C_2 > 0. \tag{2.9}$$

Consider the form

$$Q_\Lambda(\phi, \psi) := (\phi, H_{\rm i}(\Lambda)\psi), \quad \phi, \psi \in \mathrm{Dom}\, H_{\rm i}(\Lambda). \tag{2.10}$$

Then whenever $|\alpha|$ is small enough, we find $C_1 < 1$ and $C_2 \in \mathbb{R}$ such that for all $0 \leq \Lambda \leq \infty$

$$|Q_\Lambda(\phi, \phi)| \ \leq \ C_1(\Lambda)(\phi, H_0\phi) + C_2(\Lambda)(\phi, \phi), \tag{2.11}$$

where we denoted by H_0 the sum of terms in the first three lines of (2.6). The above considerations and Nelson's Theorem B then imply

Proposition 2.1 *For sufficiently small $|\alpha|$ the $\Lambda \to \infty$ limit in strong resolvent sense of the operator $H_{\rm G}(\Lambda)$ exists and is self-adjoint.*

Denote this limit by $H_{\rm G}$. By taking then inverse Gross transform, we identify the self-adjoint operator $\lim_{\Lambda\to\infty} e^{-G_\Lambda} H_{\rm G}\, e^{G_\Lambda}$ as the UV cutoff-free bipolaron Hamiltonian. The argument can be generalized for a system of arbitrary finite $N \geq 2$ polarons coupled to the given phonon field; then the formulas above extend in a straightforward way.

3 Bipolaron model in function space

As it is well known, the Feynman-Kac formula allows to associate with the exponential of $-(1/2)\nabla^2$ a Brownian motion on the space of continuous functions. In the bipolaron's case there are two indistinguishable particles, thus $H_{\rm p}$ generates two independent Brownian motions X_t and Y_t on $C(\mathbb{R}, \mathbb{R}^3)$ each. We denote by $w(X,Y)$ the product Wiener measure $\mathcal{W}(X) \otimes \mathcal{W}(Y)$ on the joint path space, and by w_T its restriction to $C([-T,T], \mathbb{R}^3 \times \mathbb{R}^3)$.

For the field, by using the Wiener-Itô isomorphism $\mathcal{F}$ can be mapped into an L^2 space of distributions weighted by a Gaussian measure. This is constructed in the following way. Take the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions over Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^3)$. Consider a random process $\mathbb{R} \ni t \mapsto \phi_t \in \mathcal{S}'(\mathbb{R}^3)$, and for $f \in \mathcal{S}(\mathbb{R}^4)$ write

$$\phi(f) = \int_{\mathbb{R}^4} f(t,x)\phi_t(x)\mathrm{d}t\mathrm{d}x. \tag{3.1}$$

We define the process such that $\{\phi(f), f \in \mathcal{S}(\mathbb{R}^4)\}$ is an Ornstein-Uhlenbeck process with measure γ. This measure is Gaussian with zero mean and covariance

$$\begin{aligned} \mathbb{E}_\gamma[\phi(f_1)\phi(f_2)] &= \big((-\partial_t^2 - \nabla^2)^{-1} f_1, f_2\big)_{L^2(\mathbb{R}^4, \mathrm{d}t\mathrm{d}x)} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{\overline{\widehat{f}_1(\kappa,k)}\widehat{f}_2(\kappa,k)}{1+k^2}\mathrm{d}k d\kappa. \end{aligned} \tag{3.2}$$

γ is a probability measure on $\mathcal{S}'(\mathbb{R}^4)$ endowed with its associated Borel σ-field. Since (3.2) can be extended to test functions of the form $f_t^h(x', t') = h(x')\delta(t'-t)$ with $h \in \mathcal{S}(\mathbb{R}^3)$, the process can be conveniently chosen to be taking values in a suitable Hilbert space. To do this, we define the random process $\phi_t(h) = \phi(f_t^h)$, with $h \in \mathcal{S}(\mathbb{R}^3)$, $t \in \mathbb{R}$. From (3.2) it then follows that

$$\mathbb{E}_\gamma[\phi_s(h_1)\phi_t(h_2)] = \frac{1}{2}e^{-|s-t|}\int_{\mathbb{R}^3} \overline{\widehat{h}_1(k)}\widehat{h}_2(k)\mathrm{d}k. \tag{3.3}$$

Clearly, $\phi_t(h)$ is a stationary Gaussian process. Its stationary measure χ defined on $\mathcal{S}'(\mathbb{R}^3)$ is itself Gaussian with mean 0 and covariance given by (3.3) at $s = t$. Moreover, the process $\phi_t(h)$ is time-reversible and Markovian.

Next we construct a Hilbert space such that ϕ_t takes its values from it and $t \mapsto \phi_t$ is norm-continuous with probability 1. Let D be a positive self-adjoint operator in $L^2(\mathbb{R}^d, \mathrm{d}x)$ given by a jointly continuous symmetric kernel $(D\widehat{\phi})(k) = \int_{\mathbb{R}^d} D(k,k')\widehat{\phi}(k')\mathrm{d}k'$, with $\ker D = \{0\}$. Define

$$\|\phi\|^2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} D(k_1,k_2)\overline{\widehat{\phi}(k_1)}\widehat{\phi}(k_2)\mathrm{d}k_1\mathrm{d}k_2$$

with $\overline{\widehat{\phi}(k)} = \widehat{\phi}(-k)$. We denote by $\mathcal{B}_D$ the completion of $\mathcal{S}(\mathbb{R}^3)$ with respect to this norm. Clearly,

$$\mathbb{E}_\gamma\left[\|\phi_t\|^2\right] = \mathbb{E}_\chi\left[\|\phi_0\|^2\right] = \frac{1}{4}\int_{\mathbb{R}^3}\frac{D(k,k)}{|k|}\mathrm{d}k.$$

Hence, once the right hand side in the above equality is finite, the measure χ is concentrated on the space $\mathcal{B}_D$ and the random process ϕ_t takes its values from this set. Moreover, since

$$\mathbb{E}_\gamma(\|\phi_t-\phi_s\|^4_{\mathcal{B}_D}) = \frac{1}{4}\left(\int_{\mathbb{R}^3}\frac{D(k,k)}{|k|}(1-e^{-|t-s|})\mathrm{d}k\right)^2 + \\ \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{D(k_1,k_2)^2}{|k_1||k_2|}(1-e^{-|s-t|})^2\mathrm{d}k_1\mathrm{d}k_2,$$

we obtain

$$\mathbb{E}_\gamma\left[\|\phi_t-\phi_s\|^4\right] \;\leq\; C\;|t-s|^2. \tag{3.4}$$

By an application of Kolmogorov's criterion we thus conclude that γ-almost all paths of the process $t \mapsto \phi_t$ are norm continuous in $\mathcal{B}_D$. We denote the restriction to $\mathcal{B}_D$ of the Gaussian path measure again by γ. A convenient choice is $D(k_1,k_2) = |k_1|_1\bar{D}(k_1,k_2)|k_2|_1$, where $\bar{D}$ is the integral kernel of the operator $(-\nabla_k^2 + |k|^2)^{-4}$, and

$$|k|_1 = \begin{cases} |k|, & \text{if } |k| < 1, \\ 1, & \text{otherwise.}\end{cases}$$

The Wiener-Itô isomorphism is the transform

$$\mathcal{J} := \mathcal{F} \to L^2(\mathcal{B}_D, \mathrm{d}\chi), \quad f_1 \otimes_{\mathrm{sym}} \ldots \otimes_{\mathrm{sym}} f_n \mapsto :\prod_{i=1}^n \phi((\sqrt{2}f_i)^\vee):\;. \tag{3.5}$$

Here Wick polynomials with respect to χ appear defined recursively as

$$
\begin{aligned}
:\phi(h)^0: &= 1, \\
:\phi(h): &= \phi(h), \\
:\phi(h_1)\dots\phi(h_n): &= \frac{1}{\sqrt{n}} :\phi(h_1)\dots\phi(h_{n-1}): \phi(h_n) - \\
&\quad - \frac{1}{\sqrt{n(n-1)}} \sum_{i=1}^{n-1} \left(\int \phi(h_i)\phi(h_n)\, \mathrm{d}\chi(\phi) \right) :\prod_{j\neq i}^{n-1} \phi(h_j): \ .
\end{aligned}
$$

The free field operator transforms under the Wiener-Itô isomorphism as $\mathcal{J}\, H_{\mathrm{f}}\, \mathcal{J}^{-1} := \tilde{H}_{\mathrm{f}}$. A simple calculation shows that

$$
\tilde{H}_{\mathrm{f}} \ :\phi(h_1)\dots\phi(h_n): = \sum_{i=1}^{n} \ :\phi((\widehat{h}_i)^{\vee}) \prod_{j\neq i}^{n} \phi(h_j): = n \ :\phi(h_1)\dots\phi(h_n): \ . \tag{3.6}
$$

Also, it can be shown that $\tilde{H}_{\mathrm{f}}$ is the generator of the Ornstein-Uhlenbeck process ϕ_t [12].

For transforming the interaction Hamiltonian we take the conjugation map with respect to $1 \otimes \mathcal{J}$. Then

$$
\tilde{H}_{\mathrm{i}}(x, y, \phi; \Lambda)\, (v \otimes \ :\phi(h)^n:) = \alpha \left(\int_{\mathbb{R}^3} \phi(q) (\pi_x^{\Lambda}(q) + \pi_y^{\Lambda}(q)) \mathrm{d}q \right) (v \otimes \ :\phi(h)^n:), \tag{3.7}
$$

is obtained, with $v \in \mathrm{Dom}\, \tilde{H}_{\mathrm{p}}$ and

$$
\pi_x^{\Lambda}(q) = \int_{\mathbb{R}^3} \frac{1 - \cos(\Lambda |q - x|)}{|q - x|^2} \mathrm{d}q. \tag{3.8}
$$

Extending (3.7) by linearity, we find that $\tilde{H}_{\mathrm{i}}(\Lambda)$ is the multiplication operator

$$
\tilde{H}_{\mathrm{i}}(x, y, \phi; \Lambda) : \quad (x, y, \phi) \ \mapsto \ \alpha(\phi(\pi_x^{\Lambda}) + \phi(\pi_y^{\Lambda})). \tag{3.9}
$$

The full Hamiltonian on function space thus reads

$$
\tilde{H} = \tilde{H}_{\mathrm{p}} \otimes 1 + 1 \otimes \tilde{H}_{\mathrm{f}} + \tilde{H}_{\mathrm{i}}, \tag{3.10}
$$

(with $\tilde{H}_{\mathrm{p}} = H_{\mathrm{p}}$) similarly to (1.1). The operator $\tilde{H}_{\mathrm{p}} \otimes 1 + 1 \otimes \tilde{H}_{\mathrm{f}}$ on $L^2(\mathbb{R}^6 \times \mathcal{B}_D, w \otimes \gamma)$ is the generator of a stationary Markov process.

4 Path measure for the bipolaron

The stochastic processes (X_t, Y_t) and ϕ_t describe the free particles and free field, respectively, on the joint particle-field path space $\Omega := C(\mathbb{R}, \mathbb{R}^6 \times \mathcal{B}_D)$. Denote by $\mu_T^0 = w_T \otimes \gamma_T$ the so obtained path measure for the uncoupled system. Here we meant by γ_T the restriction of γ to $C([-T, T], B_D)$. Also, denote $\mathrm{d}\omega = \mathrm{d}x\mathrm{d}y \otimes \mathrm{d}\chi$.

Theorem 4.1 *For all $T > 0$ and $F, G \in L^2(\Omega, \mathrm{d}\mu_T^0)$ we have*

$$\langle F, e^{-2T\tilde{H}} G\rangle = \int \bar{F}(X_{-T}, Y_{-T}, \phi_{-T}) e^{-\int_{-T}^{T} \tilde{H}_{\mathrm{i}}(X_s, Y_s, \phi_s)\mathrm{d}s} G(X_T, Y_T, \phi_T)\mathrm{d}\mu_T^0(X, Y, \phi). \tag{4.1}$$

Idea of proof: Separate the interaction free part of the Hamiltonian into $H_0 = \tilde{H}_{\mathrm{p}} \otimes 1 + 1 \otimes \tilde{H}_{\mathrm{f}}$. Then by Trotter's formula

$$\left\langle F, \left(e^{-(2T/n)H_0} e^{-(2T/n)\tilde{H}_{\mathrm{i}}}\right)^n G\right\rangle = \tag{4.2}$$
$$\int \bar{F}(X_{-T}, Y_{-T}, \phi_{-T})\ e^{-(2T/n)\sum_{j=1}^{n} \tilde{H}_{\mathrm{i}}(X_{s_j}, Y_{s_j}, \phi_{s_j})}\ G(X_T, Y_T, \phi_T)\mathrm{d}\mu_T^0(X, Y, \phi).$$

Further we use a pointwise approximation to $\tilde{H}_{\mathrm{i}}$ and monotone convergence. A version for Schrödinger operators of this argument appeared in [13]. $\square$

The right hand side (4.1) allows to identify a joint particle-field path measure. Since the argument above can be repeated for normalized functions $F/\|F\|_2$, this measure can be normalized and written as a probability measure on Ω:

$$\mathrm{d}\mu_T = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T} \tilde{H}_{\mathrm{i}}(X_t, Y_t, \phi_t)\mathrm{d}t\right) \mathrm{d}\mu_T^0. \tag{4.3}$$

Here

$$Z_T = \int \exp\left(-\int_{-T}^{T} \tilde{H}_{\mathrm{i}}(X_t, Y_t, \phi_t)\mathrm{d}t\right) \mathrm{d}\mu_T^0 \tag{4.4}$$

is the normalizing partition function.

Our next result describes the structure of the path measure.

Theorem 4.2 *For any $T > 0$ and $F \in L^2(C([-T, T], \mathbb{R}^6 \times \mathcal{B}_D, \mathrm{d}\mu_T))$ we have*

$$\int F(X, Y, \phi)\mathrm{d}\mu_T(X, Y, \phi) = \int \left(\int F(X, Y, \phi)\mathrm{d}\bar{\gamma}_T^{X,Y}(\phi)\right) \mathrm{d}\nu_T(X, Y). \tag{4.5}$$

Here $\bar{\gamma}_T^{X,Y}$ is a Gaussian probability measure on $C([-T, T], \mathcal{B}_D)$ to be given below, and ν_T is a probability measure on $C([-T, T], \mathbb{R}^3 \times \mathbb{R}^3)$ defined by

$$\mathrm{d}\nu_T(X, Y) = \frac{1}{Z_T} e^{-\mathcal{H}_T(X,Y)}\mathrm{d}w_T(X, Y), \tag{4.6}$$

with $Z_T = \int e^{-\mathcal{H}_T(X,Y)}\mathrm{d}w_T(X, Y)$,

$$\mathcal{H}_T(X, Y) = \mathcal{E}_T(X, Y) + g \int_{-T}^{T} \frac{\mathrm{d}t}{|X_t - Y_t|}, \tag{4.7}$$

where

$$\mathcal{E}_T(X, Y) = \alpha^2 \int_{-T}^{T}\int_{-T}^{T} \mathcal{E}^W(X_t, Y_s, t - s)\mathrm{d}s\mathrm{d}t, \tag{4.8}$$

$$\mathcal{E}^W(X_t, Y_s, u) = W(X_s - X_t, u) + 2W(X_s - Y_t, u) + W(Y_s - Y_t, u), \tag{4.9}$$

and

$$W(x,t) = -\frac{1}{4|x|} e^{-|t|}. \tag{4.10}$$

Furthermore, ν_T satisfies the DLR conditions.

Idea of proof: Fix $\bar{X}, \bar{Y} \in C(\mathbb{R}, \mathbb{R}^3)$ and denote by $\mu_T^{\bar{X},\bar{Y}}$ the measure μ_T conditional on $\{(X, Y, \phi) = (\bar{X}, \bar{Y}, \phi)\}$. With this conditioning $\mu_T^{\bar{X},\bar{Y}}$ is a Gaussian measure on $C(\mathbb{R}, \mathcal{B}_D)$ with mean

$$\int \phi_t(f) \mathrm{d}\mu_T^{\bar{X},\bar{Y}}(\phi) = -\int_{-T}^{T} e^{-|t-s|} \mathrm{d}s \int \frac{\overline{\widehat{f}(k)} e^{ik\cdot(\bar{X}_s - \bar{Y}_s)}}{2|k|} \mathrm{d}k$$

and covariance equal to that of γ. This Gaussian measure is $\bar{\gamma}_T^{X,Y}$.

Now, since $\phi \mapsto \tilde{H}_{\mathrm{I}}(x, y, \phi; \Lambda)$ is linear, see (3.9), and γ is a Gaussian measure, the integration with respect to ϕ can be explicitly done. By making the Gaussian integral we are led to (4.8) and (4.10). We prove exponential integrability of (4.7) by factorizing the exponent using Hölder's inequality, noticing that $\zeta_t := X_t - Y_t$ is another Brownian motion, and applying to each factor Itô's formula to get

$$(d-1) \int_{-T}^{T} \frac{dt}{|\zeta_t|} = |\zeta_T| + \int_{-T}^{T} \frac{\zeta_t}{|\zeta_t|} d\zeta_t.$$

The stochastic integral can be shown to have bounded quadratic variation. Thus the probability measure (4.6) is obtained and is the marginal over the particles of the full path measure μ_T. Finally, we show by direct inspection that for all $T > 0$ the family $\{\nu_S : 0 < S \leq T\}$ satisfies the DLR conditions, i.e. ν_T is a Gibbs measure on path space with respect to interaction (4.7). $\square$

To conclude, we note that the $T \to \infty$ weak local limit of the Gaussian measures $\bar{\gamma}_T^{X,Y}$ is easy to obtain. The same problem for μ_T is, however, far less trivial (see [8]). Convergence of these two measures implies then convergence of the path measure μ_T as well.

Acknowledgments: This work was supported by the Austrian Ministry of Science (BM:BWK) via its grant for the Wolfgang Pauli Institute and by the Austrian Science Foundation (FWF) via the START Project Y-137-TEC. J.L. thanks M. Gubinelli, M. Hirokawa, F. Hiroshima, T. Miyao and H. Spohn for many interesting discussions on the bipolaron and related models as well as on path measures. Also, he thanks H. Sugita for a travel grant to Japan (Grant-in-Aid for Scientific Research (A) 17204009 of JSPS of the Japanese Government), and K.R. Itô for splendid hospitality at RIMS, Kyoto.

References

[1] J. Adamowski: Formation of Fröhlich bipolarons, *Phys. Rev.* **B39**, 3649-52, 1989

[2] F. Bassani, M. Geddo, G. Iadonisi and D. Ninno: Variational calculus of bipolaron binding energies, *Phys. Rev.* **B43**, 5296-5306, 1991

[3] V. Betz, F. Hiroshima, J. Lőrinczi, Minlos R.A. and H. Spohn: Ground state properties of the Nelson Hamiltonian — A Gibbs measure-based approach, *Rev. Math. Phys.* **14**, 173-198, 2002

[4] F. Hiroshima and H. Spohn: Enhanced binding through coupling to a quantum field, *Ann. H. Poincaré* **2**, 1159-1187, 2001

[5] M. Hirokawa, F. Hiroshima and H. Spohn: Binding energy for hydrogen-like atoms in the Nelson model without cutoffs, *Adv. Math.*, **191**, 339, 2005

[6] G. Iadonisi, F. Bassani and G. Strinati: Polaronic effects on exciton states with different angular momenta, *Phys. Status Solidi*, **B153**, 611, 1989

[7] J. Lőrinczi and N.J. Mauser: Functional integral for the bipolaron, submitted for publication, 2006

[8] J. Lőrinczi and N.J. Mauser: Ground state of the bipolaron, work in progress, 2006

[9] J. Lőrinczi and R.A. Minlos: Gibbs measures for Brownian paths under the effect of an external and a small pair potential, *J. Stat. Phys.* **105**, 605-647, 2001

[10] J. Lőrinczi, R.A. Minlos and H. Spohn: The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3**, 1-28, 2002

[11] J. Lőrinczi, R.A. Minlos and H. Spohn: Infrared regular representation of the three dimensional massless Nelson model, *Lett. Math. Phys.* **59**, 189-198, 2002

[12] E. Nelson: Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5**, 1190–1197, 1964

[13] B. Simon: *Functional Integration and Quantum Physics*, Second Edition, AMS Chelsea Publ., 2005

[14] H. Spohn: *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, 2004

[15] D. Verbist, M.A. Smondryev, F.M. Peeters and J.T. Devreese: Strong-coupling analysis of large bipolarons in two and three dimensions, *Phys. Rev.* **B45**, 5262-69, 1992