

Random Point Fields for Para-Particles of order 3

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概要

Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction

The purpose of this note is to apply the method which we have developed in [T1a] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal system of finite fixed number of para-bosons and/or para-fermions in the bounded box in \mathbb{R}^d . Taking the thermodynamic limit, random point fields on \mathbb{R}^d are obtained. We will see that the point processes thus obtained are those of $\alpha = \pm 1/3$ given in [ShTa03].

We use the representation theory of the symmetric group. (cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on thermodynamic limit for para-fermion's case.

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2 Brief review on Representation of the symmetric group

We say that $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length n for the symmetric group \mathcal{S}_N if

$$\sum_{j=1}^n \lambda_j = N, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0.$$

We associate the Young frame $(\lambda_1, \lambda_2, \dots, \lambda_n)$ with the diagram of λ_1 -boxes in the first row, λ_2 -boxes in the second row, ..., and λ_n -boxes in the n -th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \dots, N$ to the N boxes of the frame.

Let M_p^N be the set of all the Young frames for \mathcal{S}_N which have lengths less than or equal to p . For each frame in M_p^N , let us choose one tableau from those on the frame. The choices are arbitrary but fixed. \mathcal{T}_p^N denotes the set of all the tableaux thus chosen. The row stabilizer of a tableau T is denote by $\mathcal{R}(T)$, i.e., the subgroup of \mathcal{S}_N consists of those elements that keep all rows of T invariant, and $\mathcal{C}(T)$ the column stabilizer whose elements preserve all columns of T .

Let us introduce the three elements

$$a(T) = \frac{1}{\#\mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \quad b(T) = \frac{1}{\#\mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \text{sgn}(\sigma)\sigma$$

and

$$e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \text{sgn}(\tau)\sigma\tau = c_T a(T)b(T)$$

of the group algebra $\mathbb{C}[\mathcal{S}_N]$ for each $T \in \mathcal{T}_p^N$, where d_T is the dimension of the irreducible representation of \mathcal{S}_N corresponding to T and $c_T = d_T \#\mathcal{R}(T)\#\mathcal{C}(T)/N!$. As is known,

$$a(T_1)\sigma b(T_2) = b(T_2)\sigma a(T_1) = 0 \quad (2.1)$$

hold for any $\sigma \in \mathcal{S}_N$ if $T_2 \dashv\vdash T_1$. The relations

$$a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1)e(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.2)$$

also hold for $T, T_1, T_2 \in \mathcal{T}_p^N$. For later use, let us introduce

$$d(T) = e(T)a(T) = c_T a(T)b(T)a(T) \quad (2.3)$$

for $T \in \mathcal{T}_p^N$. They satisfy

$$d(T)^2 = d(T), \quad d(T_1)d(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.4)$$

as is shown readily from (2.2) and (2.1). The inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{C}[\mathcal{S}_N]$ is defined by

$$\langle \sigma, \tau \rangle = \delta_{\sigma\tau} \quad \text{for } \sigma, \tau \in \mathcal{S}_N$$

and the sesqui-linearity.

The left representation L and the right representation R of \mathcal{S}_N on $\mathbb{C}[\mathcal{S}_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\sigma\tau = \sum_{\tau \in \mathcal{S}_N} g(\sigma^{-1}\tau)\tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau\sigma^{-1} = \sum_{\tau \in \mathcal{S}_N} g(\tau\sigma)\tau,$$

respectively. Here and hereafter we identify $g : \mathcal{S}_N \rightarrow \mathbb{C}$ and $\sum_{\tau \in \mathcal{S}_N} g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. They are extended to the representation of $\mathbb{C}[\mathcal{S}_N]$ on $\mathbb{C}[\mathcal{S}_N]$ as

$$L(f)g = fg = \sum_{\sigma, \tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} \left(\sum_{\tau} f(\sigma\tau^{-1})g(\tau) \right) \sigma$$

and

$$R(f)g = g\hat{f} = \sum_{\sigma, \tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} \left(\sum_{\tau} g(\sigma\tau)f(\tau) \right) \sigma,$$

where $\hat{f} = \sum_{\tau} \hat{f}(\tau)\tau = \sum_{\tau} f(\tau^{-1})\tau = \sum_{\tau} f(\tau)\tau^{-1}$.

The character of the irreducible representation of \mathcal{S}_N corresponding to the tableau $T \in \mathcal{T}_p^N$ is obtained by

$$\chi_T(\sigma) = \sum_{\tau \in \mathcal{S}_N} (\tau, L(\sigma)R(e(T))\tau) = \sum_{\tau \in \mathcal{S}_N} (\tau, \sigma\tau e(\widehat{T})).$$

We introduce a tentative notation

$$\chi_g(\sigma) \equiv \sum_{\tau \in \mathcal{S}_N} (\tau, L(\sigma)R(g)\tau) = \sum_{\tau, \gamma \in \mathcal{S}_N} (\tau, \sigma\tau\gamma^{-1})g(\gamma) = \sum_{\tau \in \mathcal{S}_N} g(\tau^{-1}\sigma\tau) \quad (2.5)$$

for $g = \sum_{\tau} g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. Then $\chi_T = \chi_{e(T)}$ holds.

Now let us consider representations of \mathcal{S}_N on Hilbert spaces. Let \mathcal{H}_L be a certain L^2 space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its N -fold Hilbert space tensor product. Let U be the representation of \mathcal{S}_N on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \quad \text{for } \varphi_1, \cdots, \varphi_N \in \mathcal{H}_L,$$

or equivalently by

$$(U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \quad \text{for } f \in \otimes^N \mathcal{H}_L.$$

Obviously, U is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend U for $\mathbb{C}[\mathcal{S}_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(\widehat{a(T)}) = U(a(T))$ and (2.2). So are $U(b(T))$'s, $U(d(T))$'s and $P_{pB} = \sum_{T \in \mathcal{T}_p^N} U(d(T))$. Note that $\text{Ran } U(d(T)) = \text{Ran } U(e(T))$ because of $d(T)e(T) = e(T)$, $e(T)d(T) = d(T)$.

3 Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider the quantum system of N para-bosons of order p in the box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$. We refer the literatures [MeG64, HaT69, StT70] for quantum mechanics of para-particles. (See also [OK69].) The arguments of these literatures indicate that the state space of our system is given by $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$, where $\mathcal{H}_L = L^2(\Lambda_L)$ with Lebesgue measure is the state space of one particle system in Λ_L . We need the heat operator $G_L = e^{\beta \Delta_L}$ in Λ_L , where Δ_L is the Laplacian in Λ_L with periodic boundary conditions.

It is obvious that there is a CONS of $\mathcal{H}_{L,N}^{pB}$ which consists of the vectors of the form $U(d(T))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Then, we define a point field of N free para-bosons of order p as in section 2 of [T1a] and its generating functional is given by

$$E_{L,N}^{pB}[e^{-\langle f, \xi \rangle}] = \frac{\text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N \tilde{G}_L) P_{pB}]}{\text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L) P_{pB}]},$$

where f is nonnegative continuous function on Λ_L and $\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}$.

Lemma 3.1

$$E_{L,N}^{pB}[e^{-\langle f, \xi \rangle}] = \frac{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N \tilde{G}_L) U(\sigma)]}{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L) U(\sigma)]} \quad (3.1)$$

$$= \frac{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{ \tilde{G}_L(x_i, x_j) \} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{ G_L(x_i, x_j) \} dx_1 \cdots dx_N} \quad (3.2)$$

Remark 1 : $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$ is determined by the choice of the tableaux T 's. The spaces corresponding to different choices of tableaux are different subspaces of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_T(\sigma)$ depends only on the frame on which the tableau T is defined.

Remark 2 : $\det_T A = \sum_{\sigma \in \mathcal{S}_N} \chi_T(\sigma) \prod_{i=1}^N A_{i\sigma(i)}$ in (3.2) is called immanant.

Proof : Since $\otimes^N G$ commutes with $U(\sigma)$ and $a(T)e(T) = e(T)$, we have

$$\begin{aligned} \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(d(T))) &= \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(e(T)) U(a(T))) \\ &= \text{Tr}_{\otimes^N \mathcal{H}_L} (U(a(T)) (\otimes^N G_L) U(e(T))) = \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(e(T))). \end{aligned} \quad (3.3)$$

On the other hand, we get from (2.5) that

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}_N} \chi_g(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(\sigma)) &= \sum_{\tau, \sigma \in \mathcal{S}_N} g(\tau^{-1} \sigma \tau) \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(\sigma)) \\
&= \sum_{\tau, \sigma} g(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(\tau \sigma \tau^{-1})) = \sum_{\tau, \sigma} g(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(\tau) U(\sigma) U(\tau^{-1})) \\
&= N! \sum_{\sigma} g(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(\sigma)) = N! \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G_L) U(g)), \tag{3.4}
\end{aligned}$$

where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^N G$. Putting $g = e(T)$ and using (3.3) and $P_{pB} = \sum_{T \in \mathcal{T}_p^N} U(d(T))$, the first equation is derived. The second one is obvious. \square

Now, let us consider the thermodynamic limit

$$L, N \rightarrow \infty, \quad N/L^d \rightarrow \rho > 0. \tag{3.5}$$

We need the heat operator $G = e^{\beta \Delta}$ on $L^2(\mathbb{R}^d)$. In the following, f is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in Λ_L for large enough L .

We get the limiting random point field on \mathbb{R}^d for the low density region.

Theorem 3.2 *The finite random point field for para-bosons of order 3 defined above converge weakly to the random point field whose Laplace transform is given by*

$$\mathbb{E}_\rho^{3B} [e^{-\langle f, \xi \rangle}] = \text{Det} [1 + \sqrt{1 - e^{-f} r_* G (1 - r_* G)^{-1}} \sqrt{1 - e^{-f}}]^{-3}$$

in the thermodynamic limit, where $r_* \in (0, 1)$ is determined by

$$\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta |p|^2}}{1 - r_* e^{-\beta |p|^2}} = (r_* G (1 - r_* G)^{-1})(x, x),$$

if

$$\frac{\rho}{3} < \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta |p|^2}}{1 - e^{-\beta |p|^2}}.$$

Remark : The high density region $\rho \geq 3\rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [Tib] for the case of $p = 1$ and 2.

3.2 Para-fermions of order 3

For a Young tableau T , we denote by T' the tableau obtained by exchanging the rows and the columns of T , i.e., T' is the transpose of T . The transpose λ' of the frame λ can be defined similarly. Then, T' lives on λ' if T lives on λ . It is obvious that

$$\mathcal{R}(T') = \mathcal{C}(T), \quad \mathcal{C}(T') = \mathcal{R}(T).$$

The generating functional of the point process for N para-fermions of order p in the box Λ_L is given by

$$E_{L,N}^{pF} [e^{-\langle f, \xi \rangle}] = \frac{\sum_{T \in \mathcal{T}_p^N} \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N \tilde{G})U(d(T')))}{\sum_{T \in \mathcal{T}_p^N} \text{Tr}_{\otimes^N \mathcal{H}_L} ((\otimes^N G)U(d(T')))}$$

as in the case of para-bosons of order p . And the following expression also holds.

Lemma 3.3

$$E_{L,N}^{pF} [e^{-\langle f, \xi \rangle}] = \frac{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{T'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{T'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L)U(\sigma)]} \quad (3.6)$$

$$= \frac{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'} \{ \tilde{G}_L(x_i, x_j) \} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'} \{ G_L(x_i, x_j) \} dx_1 \cdots dx_N} \quad (3.7)$$

Theorem 3.4 *The finite random point fields for para-fermions of order 3 defined above converge weakly to the point field whose Laplace transform is given by*

$$E_{\rho}^{3F} [e^{-\langle f, \xi \rangle}] = \text{Det} [1 - \sqrt{1 - e^{-f}} r_* G (1 + r_* G)^{-1} \sqrt{1 - e^{-f}}]^3$$

in the thermodynamic limit (3.5), where $r_* \in (0, \infty)$ is determined by

$$\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta|p|^2}}{1 + r_* e^{-\beta|p|^2}} = (r_* G (1 + r_* G)^{-1})(x, x). \quad (3.8)$$

4 Proof of Theorem 3.4

In the rest of this paper, we use results in [TIIa] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [TIIa]. Let ψ_T be the character of the induced representation $\text{Ind}_{\mathcal{R}(T)}^{\mathcal{S}_N} [\mathbf{1}]$, where $\mathbf{1}$ is the one dimensional representation $\mathcal{R}(T) \ni \sigma \rightarrow 1$, i.e.,

$$\psi_T(\sigma) = \sum_{\tau \in \mathcal{S}_N} \langle \tau, L(\sigma) R(a(T)) \tau \rangle = \chi_{a(T)}(\sigma).$$

Since the characters χ_T and ψ_T depend only on the frame on which the tableau T lives, not on T itself, we use the notation χ_{λ} and ψ_{λ} ($\lambda \in M_p^N$) instead of χ_T and ψ_T , respectively.

Let δ be the frame $(p-1, \dots, 2, 1, 0) \in M_p^N$. Generalize ψ_μ to those $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{Z}^p$ which satisfies $\sum_{j=1}^p \mu_j = N$ by

$$\psi_\mu = 0 \quad \text{for } \mu \in \mathbb{Z}^p - \mathbb{Z}_+^p$$

and

$$\psi_\mu = \psi_{\pi\mu} \quad \text{for } \mu \in \mathbb{Z}_+^p \quad \text{and} \quad \pi \in \mathcal{S}_p \quad \text{such that} \quad \pi\mu \in M_p^N,$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Then the determinantal form [JK81] can be written as

$$\chi_\lambda = \sum_{\pi \in \mathcal{S}_p} \text{sgn} \pi \psi_{\lambda + \delta - \pi\delta}. \quad (4.1)$$

Let us recall the relations

$$\chi_{T'}(\sigma) = \text{sgn} \sigma \chi_T(\sigma), \quad \varphi_{T'}(\sigma) = \text{sgn} \sigma \psi_T(\sigma),$$

where

$$\varphi_{T'}(\sigma) = \sum_{\tau} \langle \tau, L(\sigma) R(b(T')) \tau \rangle = \chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\text{Ind}_{\mathcal{C}(T')}^{\mathcal{S}_N}[\text{sgn}]$, where sgn is the representation $\mathcal{C}(T') = \mathcal{R}(T) \ni \sigma \mapsto \text{sgn} \sigma$. Then we have a variant of (4.1)

$$\chi_{\lambda'} = \sum_{\pi \in \mathcal{S}_p} \text{sgn} \pi \varphi_{\lambda' + \delta' - (\pi\delta)'}. \quad (4.2)$$

Now let us consider the denominator of (3.6). Let $T \in \mathcal{T}_p^N$ lives on $\mu = (\mu_1, \dots, \mu_p) \in M_p^N$. Thanks to (3.4) for $g = b(T')$, we have

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_N} \varphi_{T'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(\sigma)) &= N! \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(b(T'))) \\ &= N! \prod_{j=1}^p \text{Tr}_{\otimes^{\mu_j} \mathcal{H}_L}((\otimes^{\mu_j} G)A_{\mu_j}), \end{aligned}$$

where $A_n = \sum_{\tau \in \mathcal{S}_n} \text{sgn}(\tau) U(\tau) / n!$ is the anti-symmetrization operator on $\otimes^n \mathcal{H}_L$. In the last step, we have used

$$b(T') = \prod_{j=1}^p \sum_{\sigma \in \mathcal{R}_j} \frac{\text{sgn} \sigma}{\#\mathcal{R}_j} \sigma,$$

where \mathcal{R}_j is the symmetric group of μ_j numbers which lie on the j -th row of the tableau T . Now (4.2) yields

$$\sum_{\sigma \in \mathcal{S}_N} \chi_{\lambda'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] = \sum_{\pi \in \mathcal{S}_p} \text{sgn} \pi \sum_{\sigma \in \mathcal{S}_N} \varphi_{\lambda' + \delta' - (\pi\delta)' }(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G_L)U(\sigma))$$

$$= N! \sum_{\pi \in \mathcal{S}_p} \text{sgn } \pi \prod_{j=1}^p \text{Tr}_{\otimes^{\lambda_j - j + \pi(j)} \mathcal{H}_L} \left((\otimes^{\lambda_j - j + \pi(j)} G_L) A_{\lambda_j - j + \pi(j)} \right).$$

Here we understand that $\text{Tr}_{\otimes^n \mathcal{H}_L} ((\otimes^n G) A_n) = 1$ if $n = 0$ and $= 0$ if $n < 0$ in the last expression. Let us recall the defining formula of Fredholm determinant

$$\text{Det}(1 + J) = \sum_{n=0}^{\infty} \text{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n J) A_n]$$

for a trace class operator J . We use it in the form

$$\text{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n G_L) A_n] = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}} \text{Det}(1 + z G_L), \quad (4.3)$$

where $r > 0$ can be set arbitrary. Note that the right hand side equals to 1 for $n = 1$ and to 0 for $n < 0$. Then we have the following expression of the denominator of (3.6)

$$\begin{aligned} & \sum_{\lambda \in \mathcal{M}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{\lambda'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L) U(\sigma)] \\ &= N! \sum_{\lambda \in \mathcal{M}_p^N} \sum_{\pi \in \mathcal{S}_p} \text{sgn } \pi \oint \cdots \oint_{S_r(0)^p} \prod_{j=1}^p \frac{\text{Det}(1 + z_j G_L) dz_j}{2\pi i z_j^{\lambda_j - j + \pi(j) + 1}}. \\ &= N! \sum_{\lambda \in \mathcal{M}_p^N} \oint \cdots \oint_{S_r(0)^p} \frac{[\prod_{1 \leq i < j \leq p} (z_i - z_j)] [\prod_{j=1}^p \text{Det}(1 + z_j G_L)] dz_1 \cdots dz_p}{\prod_{j=1}^p 2\pi i z_j^{\lambda_j + p - j + 1}}. \end{aligned} \quad (4.4)$$

The similar formula for the numerator also holds.

Now let us concentrate on the case of $p = 3$.

To make the thermodynamic limit procedure explicit, let us take a sequence $\{L_N\}_{N \in \mathbb{N}}$ which satisfies $N/L_N^d \rightarrow \rho$ as $N \rightarrow \infty$. In the followings, $r = r_k \in [0, \infty)$ denotes the unique solution of

$$\text{Tr } r G_{L_N} (1 + r G_{L_N})^{-1} = k \quad (4.5)$$

for $0 \leq k \leq N$. We suppress the N dependence of r_k . The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of r . See Lemma I.3.2, for details. We put

$$v_k = \text{Tr} [r_k G_{L_N} (1 + r_k G_{L_N})^{-2}]. \quad (4.6)$$

Now let us introduce

$$\mathcal{D}_{k,l,m} = \oint \oint \oint_{S_r(0)^3} \frac{[\prod_{j=1}^3 \text{Det}(1 + z_j G_{L_N})] (z_1 - z_2)(z_2 - z_3)}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3,$$

for $k, l, m \in \mathbb{Z}$. Note that $\mathcal{D}_{k,l,m} = 0$ if at least one of k, l, m is negative. Summing over λ_1 and λ_3 in (4.4) for $p = 3$, we get

$$\sum_{\lambda \in M_3^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{\lambda'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_{L_N}} [(\otimes^N G_{L_N}) U(\sigma)] = N! \left(\sum_{l=1}^{[N/3]+1} \mathcal{D}_{N+3-2l, l, l-1} + \sum_{l=[N/3]+2}^{[N/2]+1} \mathcal{D}_{l, l, N+2-2l} \right).$$

Since $r > 0$ of the contour $S_r(0)$ is arbitrary, we may change the complex integral variables $z_j = r_j \eta_j$ with $\eta_j \in S_1(0)$ for $j = 1, 2, 3$. Thanks to the property of Fredholm determinant, we have

$$\text{Det}[1 + z_j G_{L_N}] = \text{Det}[1 + r_j G_{L_N}] \text{Det}[1 + (\eta_j - 1) r_j G_{L_N} (1 + r_j G_{L_N})^{-1}]$$

Now, we can put

$$\mathcal{F}_{k,l,m} = \frac{r_0^{3k_0} v_0^{5/2}}{\text{Det}[1 + r_0 G_{L_N}]^3} \mathcal{D}_{k,l,m} = R_{k,l,m} v_0^{5/2} I_{k,l,m},$$

where

$$R_{k_1, k_2, k_3} = \prod_{j=1}^3 \frac{r_0^{k_0} \text{Det}[1 + r_j G_{L_N}]}{r_j^{k_j} \text{Det}[1 + r_0 G_{L_N}]}$$

and

$$I_{k_1, k_2, k_3} = \oint \oint \oint_{S_1(0)^3} \left(\prod_{j=1}^3 \text{Det}[1 + (\eta_j - 1) r_j G_{L_N} (1 + r_j G_{L_N})^{-1}] \right) \\ \times (r_1 \eta_1 - r_2 \eta_2)(r_2 \eta_2 - r_3 \eta_3) \frac{d\eta_1 d\eta_2 d\eta_3}{(2\pi i)^3 \eta_1^{k_1+1} \eta_2^{k_2+1} \eta_3^{k_3+1}}.$$

Here $k_0 = (N + 2)/3$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ satisfy $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 3k_0$. We use the abbreviation r_ν and v_ν for r_{k_ν} and v_{k_ν} ($\nu = 0, 1, 2, 3$), respectively. Here, let us recall that $r_0 \rightarrow r_*$ in the thermodynamic limit because of $k_0/L^d \rightarrow \rho/3$, (3.8) and Lemma I.3.5.

Define a sequence $\{f_N\}_{N \in \mathbb{N}}$ of nonnegative functions on \mathbb{R} by

$$f_N(x) = \begin{cases} \mathcal{F}_{l, l, N+2-2l} & \text{for } \sqrt{N+2}x \in [l-1 - (N+2)/3, l - (N+2)/3] \\ & \text{and } l = [N/3] + 2, \dots, [N/2] + 1 \\ \mathcal{F}_{N+3-2l, l, l-1} & \text{for } \sqrt{N+2}x \in [l-1 - (N+2)/3, l - (N+2)/3] \\ & \text{and } l = 1, 2, \dots, [N/3] + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the denominator of (3.6) becomes

$$N! \sqrt{N+2} \frac{\text{Det}[1 + r_0 G_{L_N}]^3}{r_0^{3k_0} v_0^{5/2}} \int_{-\infty}^{\infty} f_N(x) dx$$

Let us introduce $\tilde{\mathcal{D}}_{k,l,m}$, $\tilde{\mathcal{F}}_{k,l,m}$ and \tilde{f}_N using \tilde{G}_{L_N} instead of G_{L_N} in $\mathcal{D}_{k,l,m}$, $\mathcal{F}_{k,l,m}$ and f_N and so on, we have the expression

$$\mathbb{E}_{L,N}^{3F} [e^{-\langle f, \xi \rangle}] = \frac{\text{Det}[1 + \tilde{r}_0 \tilde{G}_{L_N}]^3 r_0^{3k_0} v_0^{5/2} \int_{-\infty}^{\infty} \tilde{f}_N(x) dx}{\text{Det}[1 + r_0 G_{L_N}]^3 \tilde{r}_0^{3k_0} \tilde{v}_0^{5/2} \int_{-\infty}^{\infty} f_N(x) dx}.$$

From Lemma I.3.6, we have

$$\frac{\tilde{v}_0}{v_0} \rightarrow 1 \quad (4.7)$$

in the thermodynamic limit. Similarly, we get

$$\frac{r_0^{k_0} \text{Det}[1 + \tilde{r}_0 \tilde{G}_{L_N}]}{\tilde{r}_0^{k_0} \text{Det}[1 + r_0 G_{L_N}]} \rightarrow \text{Det}[1 - \sqrt{1 - e^{-f}} r_* G (1 + r_* G)^{-1} \sqrt{1 - e^{-f}}]$$

from the proof of Theorem I.3.1 (see Eq. (a-c), where we should read N as k_0 , z_N as r_0 and $\alpha = -1$). Thus Theorem 3.4 is proved, if we get the following lemma

Lemma 4.1 *Under the thermodynamic limit,*

$$\int_{-\infty}^{\infty} \tilde{f}_N(x) dx, \int_{-\infty}^{\infty} f_N(x) dx \rightarrow \int_{-\infty}^{\infty} e^{-2\rho x^2/\kappa} \frac{dx}{(2\pi)^{3/2}}$$

holds, where

$$\kappa = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta|p|^2}}{(1 + r_* e^{-\beta|p|^2})^2}.$$

Proof: Let k, r, v satisfy the relations

$$k = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-1}], \quad v = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-2}] \quad (4.8)$$

and $k \in [0, \infty)$.

1° There exist positive constants c_1 and c_2 which depend only on the density ρ such that

$$r_j \leq c_1, \quad r_j - r_l \leq c_1 \frac{k_j - k_l}{k_l}, \quad c_2 k_j \leq v_j \leq k_j,$$

hold for $k_j, k_l > 0$ satisfying $k_j > k_l$.

We have $v \leq k$ and $r \leq r_N$ for $k \leq N$. Recall r_N converge to the constant r^* which determined by

$$\int \frac{dp}{(2\pi)^d} \frac{r^* e^{-\beta|p|^2}}{1 + r^* e^{-\beta|p|^2}} = \rho.$$

Then $\{r_N\}$ is bounded from above. Hence we have $r \leq r_N \leq c_1$ and $v \geq k/(1 + r_N) \geq k/(1 + c_1)$ since $0 \leq G_{L_N} \leq 1$. Thanks to $dk/dr = v/r \geq k/c_1$, we get $c_1 \int_{k_l}^{k_j} dk/k \geq \int_{r_k}^{r_j} dr$, which yields the second inequality. \diamond

2° There exist positive constants c'_0, c'_1 and c'_2 which depend only on ρ such that

$$A_{k,n} = \oint_{S_1(0)} \text{Det}[1 + (\eta - 1)rG_{L_N}(1 + rG_{L_N})^{-1}] \frac{(\eta - 1)^n d\eta}{2\pi i \eta^{k+1}} \quad (n = 0, 1, 2, k = 0, 1, \dots, N)$$

satisfy

$$A_{k,0} = (1 + o(1))/\sqrt{2\pi v}, \quad A_{k,2} = (-1 + o(1))/\sqrt{2\pi v^3} \quad \text{for large } k \leq N$$

and

$$\begin{aligned} |A_{k,0}| &\leq c'_0/\sqrt{1+k}, & |A_{k,1}| &\leq c'_1/\sqrt{1+k}^3, \\ |A_{k,2}| &\leq c'_2/\sqrt{1+k}^3 & \text{for all } k &= 0, 1, \dots, N. \end{aligned}$$

As in the proof of Proposition I.A.2, let us put

$$h_k(x) = \chi_{[-\pi\sqrt{v}, \pi\sqrt{v}]}(x) e^{-ikx/\sqrt{v}} \text{Det}[1 + (e^{ix/\sqrt{v}} - 1)rG_{L_N}(1 + rG_{L_N})^{-1}].$$

Then, we have

$$|h_k(x)| \leq e^{-2x^2/\pi^2} \in L^1(\mathbb{R}) \quad (4.9)$$

and

$$h_k(x) = \chi_{[-\pi\sqrt{v}, \pi\sqrt{v}]}(x) e^{-x^2/2} e^\delta \rightarrow e^{-x^2/2} \quad \text{as } N \geq k \rightarrow \infty \quad (4.10)$$

where $|\delta| \leq 4|x^3|/9\sqrt{3v}$.

Setting $\eta = \exp(ix/\sqrt{v})$, we have

$$A_{k,n} = \int_{-\infty}^{\infty} \frac{(e^{ix/\sqrt{v}} - 1)^n h_k(x)}{2\pi\sqrt{v}} dx.$$

Then, $|A_{k,0}| \leq c'/\sqrt{v} \leq c''/\sqrt{k}$ for $k = 1, 2, \dots, N$. On the other hand, Cauchy's integral formula yields $A_{0,0} = 1$, readily. So we get the bound $|A_{k,0}| \leq c'_0/\sqrt{1+k}$.

Now the asymptotic behavior of $A_{k,0}$ can be derived by the use of dominated convergence theorem and (4.10).

For $n = 1$, we have

$$A_{k,1} = \frac{i}{2\pi v} \int_{-\infty}^{\infty} x h_k(x) dx + R,$$

where

$$|R| \leq \int \frac{x^2}{4\pi\sqrt{v^3}} h_k(x) dx = O(1/\sqrt{v^3}).$$

The integrand of first term can be written as

$$x h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) x e^{-x^2/2} + \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) x (e^\delta - 1) e^{-x^2/2}$$

$$+\chi_{[-\pi\sqrt{v},-\pi\sqrt{v}/3]\cup[\pi\sqrt{v}/3,\pi\sqrt{v}]}(x)\pi\sqrt{v}h_k(x).$$

The integral of the first term of the right hand side is 0. While the second term is bounded by $|x\delta|h(x)$, since $|e^\delta - 1| \leq |\delta|e^{\delta v_0}$. For the third term, we use (4.9). Then we get the bound $|\int xh_k(x) dx| \leq c'''/\sqrt{v}$ for $k \geq 1$. Together with $A_{0,1} = 0$, we get the bound for $A_{k,1}$. Similarly, the formulae for $A_{k,2}$ are obtained. \diamond

3° Let $(k_1, k_2, k_3) \in \mathbb{Z}_+$ satisfies

$$k_1 \geq k_2 \geq k_3, \quad k_1 + k_2 + k_3 = 3k_0 = N + 2$$

and

$$k_1 = k_2 \quad \text{or} \quad k_2 = k_3 + 1.$$

The estimate

$$|v_0^{5/2} I_{k_1, k_2, k_3}| \leq c \left(\frac{k_0}{1 + k_3} \right)^{5/2} \leq c' e^{(k_0 - k_3)^2 / 4k_0}$$

holds for all such (k_1, k_2, k_3) and

$$v_0^{5/2} I_{k_1, k_2, k_3} = \frac{v_0^{5/2} (1 + o(1))}{(2\pi)^{3/2} v_1^{1/2} v_2^{3/2} v_3^{1/2}}$$

holds for large enough N and (k_1, k_2, k_3) , where c, c' are positive constants depending only on ρ .

In fact, expanding

$$(r_1\eta_1 - r_2\eta_2)(r_2\eta_2 - r_3\eta_3) = (r_1(\eta_1 - 1) - r_2(\eta_2 - 1) + r_1 - r_2)(r_2(\eta_2 - 1) - r_3(\eta_3 - 1) + r_2 - r_3),$$

in the integrand of I_{k_1, k_2, k_3} , we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows. \diamond

4°

$$R_{k_1, k_2, k_3} = e^{-\sum_{j=1}^3 (k_0 - k_j)^2 / 2v'_j}$$

holds where $v'_j = \text{Tr}[r'_j G_{L_N} (1 + r'_j G_{L_N})^{-2}]$ for a certain middle point r'_j between r_0 and r_j . Especially, we have the bound

$$R_{k_1, k_2, k_3} \leq e^{-(k_0 - k_3)^2 / 2k_0}.$$

Recall that G_{L_N} is a non-negative trace class self-adjoint operator. If we put

$$\psi(t) = \log \text{Det}[1 + e^t G_{L_N}] = \text{Tr}[\log(1 + e^t G_{L_N})],$$

we have

$$\psi'(t) = \text{Tr}[e^t G_{L_N} (1 + e^t G_{L_N})^{-1}], \quad \psi''(t) = \text{Tr}[e^t G_{L_N} (1 + e^t G_{L_N})^{-2}].$$

In the equality

$$\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s) ds + t_0(\psi'(t_0) - \psi'(t)),$$

apply

$$\int_t^{t_0} (s - t_0)\psi''(s) ds = \int_t^{t_0} ds \int_{t_0}^s du \psi''(s) \frac{\psi''(u)}{\psi''(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)},$$

where u_c is a middle point of t and t_0 .

Then we get

$$\begin{aligned} \frac{e^{t_0\psi'(t_0)}}{e^{t\psi'(t)}} \frac{\text{Det}[1 + e^t G_{LN}]}{\text{Det}[1 + e^{t_0} G_{LN}]} &= e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} \\ &= e^{t_0(\psi'(t_0) - \psi'(t)) - (\psi'(t) - \psi'(t_0))^2 / 2\psi''(u_c)}. \end{aligned}$$

Set $e^t = r_j$ and $e^{t_0} = r_0$, then $\psi'(t) = k_j$, $\psi'(t_0) = k_0$, $\psi''(t) = v_j$ and $\psi''(t_0) = v_0$. Taking the product of those equalities for $j = 1, 2$ and 3 , we get the desired expression, since $3k_0 = k_1 + k_2 + k_3$. \diamond

5° Recall that the functions $\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L)$ ($k \in \mathbb{Z}^d$) constitute a C.O.N.S. of $L^2(\Lambda_L)$, where $G_L \varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2} \varphi_k^{(L)}$ holds for all $k \in \mathbb{Z}^d$. Thus, we obtain

$$\frac{v_0}{L^d} = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \left(\frac{2\pi}{L} \right)^d \frac{r_0 e^{-\beta|2\pi k/L|^2}}{1 + r_0 e^{-\beta|2\pi k/L|^2}} \rightarrow \kappa,$$

in the thermodynamic limit, since $k_0/L^d \rightarrow \rho/3$ and $r_0 \rightarrow r_*$ hold.

From 3° and 4°, we have a bound

$$|F_{k_1, k_2, k_3}| \leq c' e^{-(k_0 - k_3)^2 / 4k_0} \quad (4.11)$$

and

$$F_{k_1, k_2, k_3} = \frac{v_0^{5/2} (1 + o(1))}{(2\pi)^{3/2} v_1^{1/2} v_2^{3/2} v_3^{1/2}} e^{-\sum_j (k_0 - k_j) / 2v'_j} \quad (4.12)$$

for large N, K_1, k_2, k_3 , where v'_j is a mean value which we have written $\psi''(u_c)$ in 4°. For $l = 1, 2, \dots, [N/3] + 1$, $\sqrt{N+2}x \in [l-1 - (N+2)/3, l - (N+2)/3]$ implies $|l-1 - (N+2)/3| \geq \sqrt{N+2}|x|$, hence we get the bound

$$f_N(x) = F_{N+3-2l, l, l-1} \leq c' e^{-(N+2)x^2 / 4k_0} \leq c' e^{-3x^2 / 4}.$$

We also get $f_N(x) \leq c' \exp(-3x^2/4)$ for the other cases, similarly.

For fixed $x \in \mathbb{R}$, we choose $l \in \mathbb{Z}$ such that $\sqrt{N+2}x \in [l-1-(N+2)/3, l-(N+2)/3)$, then we have $v_j/v_0 \rightarrow 1$ ($j = 1, 2, 3$) and

$$\sum_{j=1}^3 \frac{(k_0 - k_j)^2}{v_j'} = \frac{4N}{v_0} x^2 + o(1).$$

Hence, we get $f_N(x) \rightarrow (2\pi)^{-3/2} \exp(-2\rho x^2/\kappa)$ in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for f_N . Because of (4.7), the one for \tilde{f}_N can be proved similarly. \square

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