ON A GENERALIZED CLM VORTICITY MODEL EQUATION

MARCUS WUNSCH

December 6, 2007

ABSTRACT. Reviewing several different model equations for the quasi-geostrophic equation, the Birkhoff-Rott equation, and the vorticity equation, we come up with a new 1D model equation interpolating, by means of a real parameter, between the former.

ACKNOWLEDGMENTS

This work would not have been possible without the invaluable support of my Japanese host professor at the Research Institute for Mathematical Sciences at Kyoto University, Hisashi Okamoto. I would also like to express my deepest gratitude to the Japanese Ministry of Education, Culture, Science, and Technology (Monbukagakusho) for financial support. Special thanks go to Takashi Sakajo, Professor at the Mathematical Department of Hokkaido University, for his inspiring numerical simulations of solutions to the DeGregorio Equation.

1. INTRODUCTION

These are preliminary notes to an article [[8]], written by H. Okamoto, T. Sakajo and the author of this contribution, to be published elsewhere.

We however choose a different approach here in that we emphasize the motivations that led us to our model equation.

It was not too short a path that took us to the generalized CLM vorticity model equation. We drew motivation from many (seemingly) quite different approaches, which, however, all become interrelated in our generalized model.

Let us mention the different model equations and their origins.

- Quasigeostrophic model equation
- Birkhoff-Rott model equation
- Vorticity model equation of Constantin-Lax-Majda
- Vorticity model equation of DeGregorio

Startled by the different asymptotic behavior of these similar (or not-so-similar) model equations, we inserted an artificial parameter by means of which we could interpolate between them all.

We give a short description of the model equations and their solutions in the following sections.

2. The Quasigeostrophic Equation in 2 Space Dimensions

We follow the exposition in [[1]].

The Quasigeostrophic Equation models the dynamics of a mixture of hot and cold air and the fronts between them. It reads as follows:

$$\begin{cases} \theta_t + (u \cdot \nabla)\, \theta = 0 \\ u = \nabla^{\perp}\, \psi, \theta = -(-\Delta)^{1/2}\, \psi \\ \theta(x,0) = \theta_0(x) \end{cases}$$

It follows that

$$u = -\nabla^{\perp}(-\Delta)^{-1/2}\, \theta = -R^{\perp}\, \theta,$$

where $R^{\perp}$ stands for the vector orthogonal to the Riesz transform in 2D:

$$R_j\theta(x,t) = (2\pi)^{-1}(PV) \int\limits_{\mathbb{R}^2} \frac{(x_j - y_j)\theta(y,t)}{|x-y|^3}\, dy,\ (j = 1, 2)$$

We therefore can rewrite the Quasigeostrophic Equation as

$$\begin{cases} \theta_t + \nabla \cdot [(R^{\perp}\theta)\theta] = 0 \\ \theta(x,0) = \theta_0(x) \end{cases}$$

since $\nabla \cdot R^{\perp}\theta = 0$.

2.1. **Derivation of the model equation.** In one space dimension, we perform the substitutions

$$\nabla \cdot \to \frac{\partial}{\partial x}$$
$$R^{\perp} \to H$$

to get

$$\theta_t + (\theta H \theta)_x = 0.$$

H stands for the Hilbert Transform

$$\begin{cases} H\omega(x) = \pi^{-1}(PV)\int\limits_{\mathbb{R}} \dfrac{\omega(y)}{x-y}\, dy \\ H\omega(x) = (2\pi)^{-1}(PV)\int\limits_{-\pi}^{\pi} \dfrac{\omega(y)}{\tan\frac{y}{2}}\, dy \end{cases}$$

More generally, one can study

$$\theta_t + \delta(\theta H\theta)_x + (1-\delta)\theta_x H\theta = 0, \qquad \delta \in (0,1] \tag{1}$$

2.2. **Nonexistence of solutions to the generalized Quasigeostrophic model equation.**

Theorem 2.1 ([1]). *Let $\theta_0 \in C^1[-\pi,\pi]$ be a non-constant periodic initial datum such that $\int_{-\pi}^{\pi}\theta_0\ dx = 0$. Then there is no $C^1[-\pi,\pi]\times[0,\infty)$ solution to (1).*

3. The Birkhoff-Rott Equations

The Birkhoff-Rott equations are integro-differential equations modeling the evolution of vortex sheets with surface tension.

$$\begin{cases} \overline{z}_t(\alpha,t) = (2\pi i)^{-1}(PV)\int\limits_{S} \dfrac{\gamma(\alpha')d\alpha'}{z(\alpha,t)-z(\alpha',t)} \\ \gamma_t = \sigma\kappa \end{cases}$$

where $z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$ represents the 2D vortex sheet parameterized by α. [cf. [3]]

3.1. **Derivation of the Birkhoff-Rott model equation.**

Here we follow the exposition in [[4]].

In one space dimension, we perform the substitution $x_t(\alpha,t) = -H\theta$, where we identify $\gamma(\alpha,t)$ with θ. Letting $\sigma \to 0$, γ becomes constant along trajectories, so that for the 1D model we may conclude that

$$0 = \frac{\partial}{\partial t}\theta(x(\alpha,t),t) = \theta_x x_t + \theta_t \tag{2}$$

$$= \theta_t - \theta_x H\theta \tag{3}$$

3.2. **Nonexistence of the solution to the Birkhoff-Rott model equation.**

Theorem 3.1 ([3]). *Let $\theta(x,0) \in C_0^{1+\delta}(\mathbb{R})$ be a positive and compactly supported initial datum for the BR-Model Equation. Then there is no global in time, locally bounded (in space) solution.*

Sketch of the Proof. For the sake of completeness, we provide a short sketch of the proof given in [[3]] The Mellin Transform

$$\begin{aligned} M\theta(\lambda) &= \int_0^\infty x^{i\lambda}\theta(x)\,\frac{dx}{x} \\ \Rightarrow \int_0^\infty \overline{f(x)}g(x)\,\frac{dx}{x} &= (2\pi)^{-1}\int_{-\infty}^{\infty} \overline{Mf(\lambda)}Mg(\lambda)\,d\lambda \end{aligned}$$

yields

$$\begin{aligned} -\int_0^\infty \frac{\theta_x(x)H\theta(x)}{x^\alpha}\,dx &= (2\pi)^{-1}\int_{-\infty}^{\infty}\overline{F(\lambda)}m_s(\lambda)F(\lambda)\,d\lambda \qquad \text{for} \\ F(\lambda) &= M\left(\frac{\theta}{\cdot^{\alpha/2}}\right)(\lambda) \end{aligned}$$

Using the identity

$$\int_0^\infty t^{-\beta}\frac{dt}{t-1} = \pi\ \cot(\pi\ \beta),$$

one can show that for sufficiently smooth even functions θ

$$Re[m_s(\lambda)] = \frac{\lambda \sinh(\pi\lambda) + \frac{\alpha}{2}\sin\left(\frac{1}{2}\pi\alpha\right)}{\cosh(\pi\lambda) + \cos\left(\frac{1}{2}\pi\alpha\right)}.$$

For $\alpha \in [-2,2],\ \alpha \neq 0,\ Re[m_s(\lambda)] > C_\alpha$ is strictly positive, so that

$$-\int_0^\infty \frac{\theta_x(x) H\theta(x)}{x^\alpha}\,dx \geq C_\alpha \int_{\mathbb{R}} \frac{|\theta(x)|^2}{|x|^{\alpha+1}}\,dx$$

This works similarly for odd θ and, by decomposition into a sum of an even and an odd part, for general sufficiently smooth functions.

After a change of coordinates

$$x' = x - x_M(t), \qquad t' = t,$$

one obtains that for $g = \bar{\theta}_{max} - \bar{\theta}, \quad \bar{\theta}(x',t') = \theta(x' + x_M(t), t)$

$$\frac{d}{dt'}\left(\int_{-L}^{L} \frac{g(x',t')}{|x'|^\alpha}\,dx'\right) \geq C_{L,\alpha}\left(\int_{-L}^{L} \frac{g(x',t')}{|x'|^\alpha}\,dx'\right)^2.$$

This is sufficient for the blow-up of $||\theta_x(.,t)||_\infty$.

4. The Vorticity Equation in Three Space Dimensions

Here we follow the presentations in [[5]] and in [[7]].

From the Euler Equation

$$v_t + (v\cdot\nabla)v = -\nabla p + f$$

we derive the vorticity equation

$$\frac{D}{Dt}\omega := \omega_t + (v\cdot\nabla)\omega = (\omega\nabla)v \tag{4}$$

$$\begin{cases} \omega = \nabla\times v \\ \nabla\cdot v = 0 \end{cases} \tag{5}$$

The solution is given by the well-known Biot-Savart Formula

$$v(x,t) = (4\pi)^{-1} \int \nabla \frac{1}{|x-x'|} \times \omega(x',t)dx'$$

We decompose ∇v into its symmetric part $S(v)$ and its antisymmetric part $R(v)$ and observe that

$$R(v)\ \omega = 0,$$

so that

$$\frac{D}{Dt}\omega = S(v)\omega =: D(\omega)\omega.$$

Substituting

$$\begin{aligned} D(\omega) &\rightarrow H\omega \\ \frac{D}{Dt} &\rightarrow \frac{\partial}{\partial t}, \end{aligned}$$

and defining $v(t,x) = \int_{-\infty}^{x} \omega(t,y)\, dy$, we get

$$\omega_t = \omega H \omega \tag{6}$$

[[2]]

4.1. **Nonexistence for the CLM Model Equation.** Due to its remarkably simple algebraic structure, one can state the following about the CLM Model Equation.

Theorem 4.1 ([7])**.** *Suppose that $\omega_0(x)$ is a smooth function decaying sufficiently rapidly as $|x| \to \infty$. Then the solution to the model vorticity equation is explicitly given by*

$$\omega(t,x) = \frac{4\omega_0(x)}{[2 - t\ H\omega_0(x)]^2 + t^2\omega_0^2(x)}$$

As a consequence, we can conclude the subsequent

Corollary 4.2 ([7]). *The smooth solution to the model equation blows up in finite time if and only if the set* $Z := \{x | \omega_0(x) = 0 \ \wedge \ H\omega_0(x) > 0\}$ *is non-empty. The blow-up time is given explicitly by* $T = 2/M$, $M := \sup_{\omega_0(x)=0} (H\omega_0)_+(x)$.

4.2. **CLM model equation with viscosity.** To complete the CLM Model Equation, Schochet [[10]] added an artificial viscosity term

$$\omega_t = \omega H\omega + \nu\omega_{xx}$$

However, there are serious drawbacks, most importantly

- The energy of the solution is unbounded.
- The explosion time T_ν can be earlier than in the inviscid case.

4.3. **The Wegert-Murthy model equation.** In order to tackle with the shortcomings, several approaches have been proposed. In [[11]], the authors proposed the addition of a dissipative term:

$$\begin{aligned} \omega_t &= \omega H\omega - \varepsilon H\omega_x \qquad \text{in} \qquad \mathbb{R} \times \mathbb{R}_+ \\ \omega(x,0) &= \omega_0(x) \end{aligned}$$

and proved the

Theorem 4.3. *Let* ω_0 *be a non-constant Hölder-continuous periodic function such that* $\int_0^{2\pi} \omega_0(x)\, dx = 0$. *Then*

- *The blow-up time* $T_\varepsilon(\omega_0)$ *is a monotonously increasing function of* ε.

- *For each initial datum ω_0 there exists a positive ε_* such that $T_\varepsilon(\omega_0) = +\infty$ if $\varepsilon > \varepsilon_*$*

This result has been generalized by [[9]] for more general dissipative terms of the form $(-\Delta)^{\alpha/2}\omega$, $\alpha \in [0, 2]$.

5. The DeGregorio Vorticity model equation

The author of [[5]] chose a different strategy for a 1D model of the vorticity equation:

- Leave the material derivative $\frac{D}{Dt}$.
- Set $v_x = H\omega$, so that v is NOT the antiderivative of ω.

The equation therefore reads

$$\frac{D}{Dt}\omega = \frac{\partial}{\partial t}\omega + v\omega_x = \omega v_x = \omega H\omega \tag{7}$$

6. A Generalilzed CLM Vorticity Model Equation

In order to capture simultaneously the features of the model equations for the Quasigeostrophic Equation, the Birkhoff-Rott Equation, and for the Vorticity Equation, we proposed the following generalized CLM vorticity model equation [[8]] with an interpolating parameter $a \in \mathbb{R}$:

$$\omega_t = v_x\omega - av\omega_x \begin{cases} a = 1 \text{...DeGregorio's Equation} \\ a = 0 \text{...Vorticity Equation} \\ a = -1 \text{...Quasigeostrophic Equation,} \\ \text{Birkhoff-Rott Equation} \end{cases}$$

Using a theorem by [[6]], in

$$H^s(S^1)/\mathbb{R} = \left\{ f \;\middle|\; f = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \text{ where } \sum_{n=1}^{\infty}(a_n^2 + b_n^2)n^{2s} < \infty \right\},$$

one can prove

Theorem 6.1. *For all $\omega_0 \in H^1(S^1)/\mathbb{R}$, there exists a T_a depending only on the parameter a and $\|\omega_{0,x}\|$ such that there is a unique $\omega \in C^0([0,T_a]; H^1(S^1)/\mathbb{R}) \cap C^1([0,T_a]; L^2(S^1)/\mathbb{R})$ with $\omega(0) = \omega_0$.*

Analogously to the Beale-Kato-Majda Condition for the 3D Vorticity equation [[7]], we find that the subsequent theorem holds true.

Theorem 6.2. *Suppose that $\omega(0) \in H^1(S^1)/\mathbb{R}$, that the solution exists in $[0,T)$, and that*

$$\int_0^T \|H\omega(t)\|_\infty \, dt < \infty.$$

Then the solution exists in $0 \leq t \leq T + \delta$ with some$\delta > 0$.

Proof. Differentiating in x yields

$$\omega_{xt} = \omega H\omega_x + (1-a)\omega_x H\omega - a\omega_{xx}v,$$

so that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\omega(.,t)||_{H^1}^2 \\
&\quad= \int \omega\omega_x H\omega_x \, dx + (1-a)\int \omega_x^2 H\omega \, dx - a\int \omega_x\omega_{xx} v \, dx \\
&\quad= \int H\omega H(\omega_x H\omega_x) \, dx + (1-a)\int \omega_x^2 H\omega \, dx + \frac{a}{2}\int \omega_x^2 H\omega \, dx \\
&\quad= \frac{1}{2}\int H\omega(H\omega_x^2 - \omega_x^2) \, dx + \left(1-\frac{a}{2}\right)\int \omega_x^2 H\omega \, dx \\
&\quad\leq \frac{2-a}{2} \, ||H\omega||_\infty \, ||\omega||_{H^1}
\end{aligned}$$

The following identities have been used

$$\begin{aligned}
\int H\omega H v \, dx &= \int \omega v \, dx \\
\int H\left(\omega H\omega\right) \, dx &= \frac{1}{2}\left[\int (H\omega)^2 - \omega^2 \, dx\right]
\end{aligned}$$

Grönwall's Lemma therefore yields

$$||\omega(.,t)||_{H^1}^2 \leq ||\omega(.,0)||_{H^1}^2 \ \exp\left((2-a)\int_0^t ||H\omega(.,s)||_\infty \, ds\right).$$

7. Concluding Remarks

As mentioned in the introduction, these are but preliminary notes to [[8]]. In this article, the interested reader will find many stimulating numerical simulations of solutions to the DeGregorio vorticity model equation, further analytical results, and conjectures on the global existence or blow-up, respectively, for the generalized CLM vorticity model equation, depending on the value of the interpolating real parameter a.

References

[1] D. Chae, A. Córdoba, D. Córdoba, M. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, Adv. of Math. 194 (2005), 203–223

[2] P. Constantin, F. Lax, A. Majda, A simple one-dimensional model for the three-dimensional vorticity equation. Comm. Pure Appl. Math. 38 (1985), no. 6, 715–724

[3] A. Córdoba, D. Córdoba, M. A. Fontelos, Integral inequalities for the Hilbert transform applied to a nonlocal transport equation. J. Math. Pures Appl. (9) 86 (2006), no. 6, 529–540.

[4] A. Córdoba, D. Córdoba, M. A. Fontelos, Formation of Singularities for a Transport Equation with Nonlocal Velocity. Ann. of Math. (2) 162 (2005), no. 3, 1377–1389

[5] S. DeGregorio, On a one-dimensional model for the three-dimensional vorticity equation. J. Statist. Phys. 59 (1990), no. 5-6, 1251–1263

[6] T. Kato, C. Y. Lai, Nonlinear evolution equations and the Euler flow, J. Func. Anal., 56 (1984), 15-28

[7] A. Majda, A. Bertozzi, Vorticity and incompressible flow. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002

[8] H. Okamoto, T. Sakajo, M. Wunsch, On a Generalization of the Constantin-Lax-Majda Equation, preprint.

[9] T. Sakajo, On global solutions for the Constantin-Lax-Majda equation with a generalized viscosity term. Nonlinearity 16 (2003), no. 4, 1319–1328

[10] S. Schochet, Explicit solutions of the viscous model vorticity equation. Comm. Pure Appl. Math. 39 (1986), no. 4, 531–537

[11] E. Wegert, A. S. Vasudeva Murthy, Blow-up in a modified Constantin-Lax-Majda model for the vorticity equation. Z. Anal. Anwendungen 18 (1999), no. 2, 183–191

Marcus Wunsch — Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502

E-mail address: marcus.wunsch@univie.ac.at