Stability and bifurcation analysis to dissipative cavity soliton of Lugiato-Lefever equation in one dimensional bounded interval

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Abstract. In this paper, a mathematically rigorous analysis for bifurcation structure of a spatially uniform stationary solution in nonlinear Schrödinger equations with a cubic nonlinearity and a dissipation, and with a detuning term is presented. Numerically, it has been reported that the "snake bifurcation" occurs, but this equation does not have the variational (Hamiltonian) structure. Therefore, both a variational technique capturing the ground state of conservative system and a dynamical system technique with reversible system of 1:1 resonance cannot be applied to the problem. We here ensure that the pitchfork type bifurcation happens only under the physically natural conditions by using the bifurcation theory with the symmetry, and moreover, make a much finer analysis at the codimension two bifurcation point to give a proof to the "fold bifurcation" around the singular point. In the consequence, the bending solution branch at least once has been captured in an adequate parameter area near the singularity, which means that a part of the global bifurcation structure is infinitesimally folding into the singularity with codimension two.

AMS classification scheme numbers: 34C23, 37L10, 70K20, 70K50

Submitted to: *Nonlinearity*

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1. Introduction

Dissipative cavity soliton is an optical localized spatial pattern caused by a kind of coherent structure in optical nonlinear medium (for example, a Kerr cavity medium). This is an optical soliton-like pattern, but which is different from the so-called "soliton" in conservative system, because this is because of proportion among input driving force and detuning and dissipation for light in nonlinear medium. This is also a kind of dissipative structure based on excitability of the nonlinear medium. Recently, a lot of reports have been done from both experimental and theoretical point of views of physics, for instance, in [2] and in [3]. Especially, we can see the brief history and the underlying nonlinear optics of cavity and feedback soliton in the review article written by Professors, T. Ackemann and W.J. Firth [1] and the references therein ([5, 11, 24] for example).

In this article, we are mainly concerned with clearing out a mathematical aspect of the phenomena and with giving mathematically rigorous proofs to basic theorems in one space dimensional bounded interval at a point where the homogeneous steady state loses its stability and makes a bifurcation, and moreover, with showing perspectives in the future from a viewpoint of mathematical physics with rigorous mathematical argument. One of interesting, but difficult points of this problem is lack of variational (Hamiltonian) structure, which is very useful and important technique ensuring existence of this kind of pulsating solutions. Because of the lack of this useful structure, we cannot apply the dynamical system technique of reversible system of 1:1 resonance, and also not apply the variational method as PDE technique by which, for instance, a ground state of conservative system of nonlinear Schrödinger equations with a cubic nonlinearity is captured.

We now introduce the model equation of initial-boundary value problem in one space dimension:

$$\frac{\partial E}{\partial t} = -(1+\mathrm{i}\theta)E + \mathrm{i}b^2\Delta E + E_{\mathrm{in}} + \mathrm{i}|E|^2E, \qquad x\in\Omega, t>0 \tag{1.1}$$

$$E(-\frac{1}{2},t) = E(\frac{1}{2},t), \quad \frac{\partial E}{\partial x}(-\frac{1}{2},t) = \frac{\partial E}{\partial x}(\frac{1}{2},t), \qquad t>0 \tag{1.2}$$

$$E(x,0) = E_0(x), \qquad x\in\Omega \tag{1.3}$$

where $\Omega = (-\frac{1}{2},\frac{1}{2}) \subset \mathbb{R}$, $\Delta = \partial^2/\partial x^2$ is the Laplacian and i is the imaginary unit. $\theta \geq 0$ is a detuning parameter and $b^2 \in \mathbb{R}$ is a diffraction constant, and both are constants. Suppose that the homogeneous driving field E_{in} is real and positive. Here, E denotes the slowly varying envelope of the electric field. (1.1) describes physically a unidirectional ring or Fabry-Perot cavity with plane mirrors containing a Kerr medium driven by a coherent plane-wave field (see Lugiato and Lefever, [20]).

Note that (1.1) has a homogeneous equilibrium point E_S given implicitly by

$$E_S = \frac{E_{\mathrm{in}}}{1+\mathrm{i}(\theta - I_S)}, \tag{1.4}$$

where $I_S = |E_S|^2$. Or, we can easily obtain

$$|E_{\text{in}}|^2 = I_S \left\{1 + (I_S - \theta)^2\right\}. \tag{1.5}$$

This cubic steady-state curve $I_S(|E_{\text{in}}|^2)$ is single-valued for $\theta < \sqrt{3}$ while it is multi-valued for $\theta > \sqrt{3}$ and leads to a hysteresis. Let us denote E_S as it when we take one of the homogeneous equilibrium states, even if there are two homogeneous states. We define an auxiliary complex field $A(x,t)$ by

$$E = E_S(1 + A), \tag{1.6}$$

and we consider the following equation near the homogeneous state E_S:

$$\frac{\partial A}{\partial t} = -(1 + \mathrm{i}\theta)A + \mathrm{i}b^2 \Delta A + \mathrm{i}I_s \left(2A + \bar{A} + A^2 + 2|A|^2 + |A|^2 A\right). \tag{1.7}$$

Obviously, $A = 0$ is a homogeneous equilibrium point of (1.7) and corresponds to E_S by the transformation. One of advantages is that (1.7) turns to be autonomous, but instead of this, the equation has $\bar{A}$ term so that we should be careful to analyze it a little.

The rest of this paper is composed of the following sections: In the section 2, we make an analysis of time evolution equation to determine the long time behavior of the solution roughly. There exists a finite dimensional global attractor of the dynamical system defined by (1.1). In the section 3, we make stability and bifurcation analysis about the homogeneous steady state of E_S. We get a theorem in which zero eigenvalue occurs at a certain critical value of I_S. Moreover, the dimension of zero eigenspace is two, but this has a kind of symmetry. Therefore the bifurcation analysis with a group symmetry can be applicable ([8] and [9]) for us to get the bifurcation theorem. The stability of the bifurcation solution will be determined by the theory. Moreover, we make a much finer analysis at the codimension two bifurcation point to give a proof to the "fold bifurcation" around the singular point. In the consequence, the bending solution branch at least once has been captured in an adequate parameter area near the singularity, which means that a part of the global bifurcation structure is infinitesimally folding into the singularity with codimension two, although this type analysis is only for "roll" solutions.

2. Existence of solutions and attractors

2.1. Formulation

We consider the Cauchy problem of the Lugiato-Lefever equation (1.1)-(1.3) on an interval $(-\frac{1}{2}, \frac{1}{2})$. It is a weakly dissipative equation, that is, the dissipation occurs only on the lowest-order terms. We mainly impose periodic boundary conditions. Remark that, however, the existence result is also valid for the homogeneous Dirichlet or Neumann boundary conditions on a finite interval.

By an appropriate rescaling, (1.1) can be rewritten as

$$\mathrm{i}u_t + u_{xx} + g(|u|^2)u + \mathrm{i}u = f, \quad x \in (0, L) \subset \mathbb{R}, \tag{2.1}$$

where $u(x,t) = E(x,t), f(x) = \mathrm{i}E_{\mathrm{in}}(x), L = 1/b$ and $g(\sigma) = \sigma - \theta(\sigma \geq 0)$. Now the boundary conditions are replaced by

$$u(0,t) = u(L,t), \qquad u_x(0,t) = u_x(L,t). \tag{2.2}$$

Define two functions related to g by

$$h(s) = sg(s), \qquad G(s) = \int_0^s g(\sigma)\,\mathrm{d}\sigma. \tag{2.3}$$

In addition, we set $G_+(s) = \max(G(s), 0)$ and $G_-(s) = \max(-G(s), 0)$. We obtain the following two conditions

- $$\lim_{s\to+\infty} \frac{G_+(s)}{s^3} = 0, \tag{2.4}$$
- there exists $\omega > 0$ such that
$$\limsup_{s\to+\infty} \frac{h(s) - \omega G(s)}{s^3} = 0. \tag{2.5}$$

This is precisely the case treated in [7].

Let us introduce some notations. Let $\mathcal{H} = L^2$ be the space of complex-valued L^2-functions on Ω equipped with the standard scalar product and norm. Let H^k be the subspace of $\mathcal{H}$ such that for $u \in \mathcal{H}$, u and $\frac{\partial^j u}{\partial x^j}$ belong to $\mathcal{H}$ for $j = 1, \ldots, k$, and $x \mapsto u(x)$ and $x \mapsto \frac{\partial^j u}{\partial x^j}(x)$ are L-periodic for $j = 1, \ldots k-1$. Let $\mathcal{A}$ be an unbounded linear operator on $\mathcal{H}$, $\mathcal{A}v = -v_{xx}$, with domain $D(\mathcal{A}) = H^2$. We denote by w_j and λ_j the eigenvectors and eigenvalues of $\mathcal{A}$ in $\mathcal{H}$

$$\begin{aligned} &\mathcal{A}w_j = \lambda_j w_j, \quad j \geq 1 \\ &0 \leq \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \to \infty \text{ as } j \to \infty. \end{aligned} \tag{2.6}$$

The powers $\mathcal{A}^s$ of $\mathcal{A}$, $s \in \mathbb{R}$, are well defined with domain $D(\mathcal{A}^s)$

$$D(\mathcal{A}^s) = \left\{ u \in \mathcal{H} \;;\; \sum_{j=1}^{\infty} \lambda_j^{2s} (u, w_j)^2 < \infty \right\}.$$

We have $\mathcal{V} = D(\mathcal{A}^{\frac{1}{2}}), \mathcal{V}' = D(\mathcal{A}^{-\frac{1}{2}})$(after identification of $\mathcal{H}$ and its dual $\mathcal{H}'$).

2.2. Existence results

According to Ghidaglia[7], existence and uniqueness of the solution to the Cauchy problem for (2.1) with initial condition $u(0) = u_0$:

Theorem 2.1. *For every $u_0 \in \mathcal{V}$ and f satisfying*

$$f \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{H}), \qquad f_t \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{V}'), \tag{2.7}$$

the Cauchy problem for (2.1) with initial condition $u(0) = u_0$ possesses a unique solution and for every $t \in \mathbb{R}$, the mapping $u_0 \to u(t)$ is continuous on $\mathcal{V}$. Moreover, if we have

$$f \in L^\infty(\mathbb{R}_+, \mathcal{H}), \qquad f_t \in L^\infty(\mathbb{R}_+, \mathcal{V}'),$$

then $u \in L^\infty(\mathbb{R}_+, \mathcal{V})$.

The proof is achieved by Faedo-Galerkin method. This theorem implies the existence of the group $\{S(t)\}_{t\geq 0}$, where

$$S(t) : u_0 \mapsto u(t), \tag{2.8}$$

is continuous from $\mathcal{V}$ into itself.

Several *a priori* estimates in L^2, H^1 and H^2 are derived for employing the Galerkin method. For example, multiplying (2.1) by $\bar{u}$ and integrating over Ω and taking the imaginary part, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{L^2}^2 + |u|_{L^2}^2 = \mathrm{Im}(f, u)_{L^2} \tag{2.9}$$

By using Schwarz and Young's inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_{L^2}^2 + |u|_{L^2}^2 \leq |f|_{L^2}^2, \tag{2.10}$$

from which we derive an *a priori* estimate of u in $L^\infty(\mathbb{R}_+, \mathcal{H})$:

$$|u(t)|_{L^2}^2 \leq |u_0|_{L^2}^2 \exp(-t) + |f|_{L^2}^2 \left(1 - \exp(-t)\right). \tag{2.11}$$

The absorbing set in L^2 is derived from these estimates. Let $\rho_0^2 = |f|_{L^2}^2$ and let ρ_0' be any number, $\rho_0' > \rho_0$. Then the ball $\mathcal{B}_0$ of L^2 centered at 0 of radius ρ_0' is an absorbing ball for the group $S(t)$. If $\mathcal{B}$ is included in the ball of L^2 centered at 0 of radius R, then $S(t)\mathcal{B} \subset \mathcal{B}_0$ for $t \geq t_0(\mathcal{B}, \mathcal{B}_0)$,

$$t_0 = \log \frac{R^2}{(\rho_0')^2 - \rho_0^2}. \tag{2.12}$$

Further estimates give absorbing sets in H^1 and H^2[7].

Proposition 2.1. *There exists a constant $\rho_1 > 0$ such that for every $R > 0$ and for every $u_0 \in \mathcal{V}$ with $\|u_0\|_{H^1}^2 \leq R^2$, there exists $t_1 > 0$ such that the solution of (2.1) satisfies $\|u(t)\|_{H^1}^2 \leq \rho_1^2$ for $t \geq t_1$. Therefore the ball $\mathcal{B}_1$ of $\mathcal{V}$ centered at 0 of radius ρ_1 is an absorbing ball for $S(t)$.*

Proposition 2.2. *There exists a constant $\rho_2 > 0$ such that for every $R > 0$ and for every $u_0 \in D(\mathcal{A})$ with $\|u_0\|_{H^2}^2 \leq R^2$, there exists $t_2 > 0$ such that the solution of (2.1) satisfies $\|u(t)\|_{H^2}^2 \leq \rho_2^2$ for $t \geq t_2$. Therefore the ball $\mathcal{B}_2$ of $D(\mathcal{A})$ centered at 0 of radius ρ_2 is an absorbing ball for $S(t)$.*

Existence of weak attractor in H^2 can be obtained by the argument in [7] (See also [25]). It has finite Hausdorff and fractal dimension as a subset of H^1. These results can be improved by the augment in [27]. The weak attractor is actually the strong attractor in H^2. The same result is also valid for the attractor in H^1.

Theorem 2.2. *If f is given in L^2, then the semigroup $\{S(t)\}_{t\geq 0}$ possesses a compact global attractor in H^1.*

3. Bifurcation analysis of homogeneous equilibrium point

(1.1) has a spatially homogeneous equilibrium point E_S given implicitly by (1.4):

$$E_S = \frac{E_{\text{in}}}{1 + \mathrm{i}(\theta - I_S)},$$

where $I_S = |E_S|^2$. Or, we can obtain the relation (1.5). We study the symmetry-breaking bifurcation of E_S.

3.1. Reformulation

We introduce a new parameter $\alpha \geq 0$ as a bifurcation parameter and let E_{in} be a function of α

$$E_{\text{in}}(\alpha) = \sqrt{\alpha \left\{1 + (\alpha - \theta)^2\right\}}.$$

We define E_α by

$$E_\alpha = \sqrt{\frac{\alpha}{1 + (\theta - \alpha)^2}} \left\{1 - \mathrm{i}(\theta - \alpha)\right\}. \tag{3.1}$$

E_α is a homogeneous equilibrium of (1.1) with $|E_\alpha|^2 = \alpha$.

Then we define an auxiliary complex field $A(x, t)$ by

$$E = E_\alpha(1 + A),$$

and, as we stated it in the introduction, (1.7) is derived near the homogeneous state E_α. We consider (1.7) on a finite interval $\Omega = (-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}$. The boundary conditions are given by

$$A(-\frac{1}{2}, t) = A(\frac{1}{2}, t), \qquad \frac{\partial A}{\partial x}(-\frac{1}{2}, t) = \frac{\partial A}{\partial x}(\frac{1}{2}, t). \tag{3.2}$$

Decomposing $A(x, t)$ into its real and imaginary parts by $A(x, t) = u_1(x, t) + iu_2(x, t)$, we have

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -b^2 \Delta u_2 - u_1 + (\theta - \alpha) u_2 - \alpha \left(2u_1 u_2 + u_2(u_1^2 + u_2^2)\right), \\ \frac{\partial u_2}{\partial t} &= b^2 \Delta u_1 + (3\alpha - \theta) u_1 - u_2 + \alpha \left(3u_1^2 + u_2^2 + u_1(u_1^2 + u_2^2)\right). \end{aligned} \tag{3.3}$$

We work on two Hilbert spaces, $\mathcal{X} = H^1(\Omega)^2, \mathcal{Y} = L^2(\Omega)^2$. $\mathcal{X}$ is dense in $\mathcal{Y}$. The space $\mathcal{Y}$ is equipped with the standard inner product

$$\langle u, v \rangle = \int_\Omega u(x)^T v(x) dx.$$

Let $F : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ be a nonlinear operator defined by

$$F(\alpha, u) \equiv \begin{pmatrix} -b^2 \Delta u_2 - u_1 + (\theta - \alpha) u_2 - \alpha \left(2u_1 u_2 + u_2(u_1^2 + u_2^2)\right) \\ b^2 \Delta u_1 + (3\alpha - \theta) u_1 - u_2 + \alpha \left(3u_1^2 + u_2^2 + u_1(u_1^2 + u_2^2)\right) \end{pmatrix}, \tag{3.4}$$

where $u = (u_1, u_2)^T$. The steady states of (3.3) are solutions to

$$F(\alpha, u) = 0. \tag{3.5}$$

Obviously, $A = 0$ corresponds to $u = u^o = (0, 0)^T$. We consider the bifurcation problem of the homogeneous equilibrium point $u = u^o$. The linearized equation of (3.3) near $u = u^o$ is given by

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -b^2\Delta u_2 - u_1 + (\theta - \alpha)u_2 \\ b^2\Delta u_1 + (3\alpha - \theta)u_1 - u_2 \end{pmatrix}. \tag{3.6}$$

We denote the linear operator in the right-hand-side by

$$\mathcal{L}u = B\Delta u + Cu, \tag{3.7}$$

where

$$B = \begin{pmatrix} 0 & -b^2 \\ b^2 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & \theta - \alpha \\ 3\alpha - \theta & -1 \end{pmatrix}.$$

3.2. Linearized eigenvalue problem

Now we consider the eigenvalue problem

$$\mathcal{L}\phi = \lambda\phi. \tag{3.8}$$

Lemma 3.1. *If $\alpha < 1$, all eigenvalues have negative real parts.*

Proof. Multiplying the both sides of (3.8) by $\phi = (\phi_1, \phi_2)^T$, integrating over Ω and taking real parts, we obtain

$$\begin{aligned}
\mathrm{Re}\lambda \int_\Omega |\phi|^2 dx &= \int_\Omega \mathcal{L}\phi \cdot \phi dx \\
&= b^2 \int_\Omega \left(\frac{\partial^2 \phi_1}{\partial x^2}\phi_2 - \phi_1 \frac{\partial^2 \phi_2}{\partial x^2}\right) dx - \int_\Omega (\phi_1^2 - 2\alpha\phi_1\phi_2 + \phi_2^2) dx \\
&= (\alpha - 1)\int_\Omega (\phi_1^2 + \phi_2^2) dx - \alpha \int_\Omega (\phi_1 - \phi_2)^2 dx \\
&\le (\alpha - 1)\int_\Omega |\phi|^2 dx.
\end{aligned}$$

Thus, if $\alpha < 1$, then $\mathrm{Re}\lambda < 0$. □

We consider the instability occurring at $\alpha = 1$. The viewpoint of symmetry helps our calculation[9, 10]. Let us recall some group theoretical terms: $\mathbf{O}(n)$ is the n-dimensional orthogonal group and $\mathbf{SO}(n)$ is the special orthogonal group. The group $\mathbf{O}(2)$ is generated by $\mathbf{SO}(2)$ together with the reflection.

Definition 3.1 (Γ-equivariance[9]). *Let Γ be a compact Lie group on a vector space V. The mapping $g : V \to V$ commutes with Γ or is Γ-equivariant if*

$$g(\gamma x) = \gamma g(x)$$

for all $\gamma \in \Gamma, x \in V$.

The nonlinear operator $F(\alpha,\cdot)$ is $\mathbf{O}(2)$-equivariant where $\mathbf{SO}(2)$ acts on $x\in\mathbb{R}$ by transition modulo the spatial period 1 and the reflection κ acts by $x\mapsto -x$. Then $\mathcal{L}$ also commutes with $\mathbf{O}(2)$. Commuting linear operators map isotypic components to isotypic components. By Fourier analysis we can write

$$u(x)=\sum_{n=-\infty}^{\infty} u_n \exp(2n\pi x\mathrm{i}),\quad u_n\in\mathbb{C}^2.$$

It follows that the subspaces

$$X_n=\left\{\exp(2n\pi x\mathrm{i})a+\text{c.c}\ :\ a\in\mathbb{C}^2\right\}, n=0,1,2,\dots$$

are the $\mathbf{O}(2)$-isotypic components of both $\mathcal{X}$ and $\mathcal{Y}$, where c.c. is complex conjugate. $\mathcal{L}$ maps each X_n into itself. Thus the eigenvalues of $\mathcal{L}$ are the union of all of the eigenvalues of $\mathcal{L}|_{X_n}$ for $n=0,1,\dots$. The problem can be reduced to

$$\mathcal{L}_n\psi_n=\lambda_n\psi_n,\quad n=0,1,2,\dots, \tag{3.9}$$

where $\psi_n\in\mathbb{R}^2$, and $\mathcal{L}_n$ is 2×2 matrix given by

$$\mathcal{L}_n=\begin{pmatrix} -1 & b^2k_n^2+\theta-\alpha \\ -b^2k_n^2+3\alpha-\theta & -1\end{pmatrix},\quad k_n=2n\pi.$$

Then two eigenfunctions associated with the eigenvalue λ_n can be given by $\phi_n=\psi_n\cos(k_nx)$ and $\phi_n=\psi_n\sin(k_nx)$. We can easily compute traces and determinants of $\mathcal{L}_n$:

$$\begin{aligned}\mathrm{tr}\mathcal{L}_n &= -2,\\ \det\mathcal{L}_n &= \left(b^2k_n^2-2\alpha+\theta\right)^2+1-\alpha^2.\end{aligned}$$

Since $\mathrm{tr}\mathcal{L}_n=-2$, a pair of purely imaginary eigenvalues cannot exist. Therefore we concentrate on the instability by zero eigenvalue, which exists if and only if $\det\mathcal{L}_n=0$ for some $n\in\mathbb{N}\cup\{0\}$. Such n is given by

$$n=n_\pm(\alpha)=\frac{1}{2\pi b}\sqrt{2\alpha-\theta\pm\sqrt{\alpha^2-1}}. \tag{3.10}$$

Here, we assume that $\theta\le 2$. Consider $n_\pm$ as real-valued functions of $\alpha\ge 1$. The following properties can be easily checked:

(i) n_+ is monotone increasing for $\alpha>1$ and $n_+\to\infty$ as $\alpha\to\infty$.

(ii) (a) if $\theta\le\sqrt{3}$, then n_- is monotone decreasing for $1<\alpha<2/\sqrt{3}$ and monotone increasing for $\alpha>2/\sqrt{3}$.

(b) if $\sqrt{3}<\theta\le 2$, then n_- is monotone decreasing for $1<\alpha<\left(2\theta-\sqrt{\theta^2-3}\right)/3$ and monotone increasing for $\alpha>\left(2\theta+\sqrt{\theta^2-3}\right)/3$. n_- is not real-valued for $\left(2\theta-\sqrt{\theta^2-3}\right)/3<\alpha<\left(2\theta+\sqrt{\theta^2-3}\right)/3$.

Hence, there exists $\alpha\ge 1$ such that $n_+(\alpha)\in\mathbb{N}\cup\{0\}$ or $n_-(\alpha)\in\mathbb{N}\cup\{0\}$. Especially, we are interested in

$$\alpha_*=\min\left\{\alpha\ ;\ n_+(\alpha)\in\mathbb{N}\cup\{0\}\text{ or }n_-(\alpha)\in\mathbb{N}\cup\{0\}\right\}.$$

Later in subsection 3.3 and 3.4, we will focus on the case that zero eigenvalue occurs at $\alpha_* = \alpha^o \equiv 1$ to make a bifurcation analysis. On the other hand, in subsection 3.5 we will make a bifurcation analysis of it in the case of $\alpha_* = \alpha^o > 1$.

Theorem 3.1. *For any $b > 0$ and $0 \le \theta \le 2$, there exists $n \in \mathbb{N} \cup \{0\}$ and $\alpha_* \ge 1$ such that*

$$\alpha_* = \min \{\alpha \ ; \ n_+(\alpha) \in \mathbb{N} \cup \{0\} \ \text{or} \ n_-(\alpha) \in \mathbb{N} \cup \{0\}\},$$

where $n = n_+(\alpha_*)$ or $n = n_-(\alpha_*)$. Moreover, the following three properties hold:

- *if $\alpha < \alpha_*$, then u^o is exponentially stable.*
- *if $\alpha = \alpha_*$, then $\mathcal{L}$ has zero eigenvalue with the "n-mode" eigenfunctions.*
- *if $\alpha > \alpha_*$, then u^o is exponentially unstable at least for the direction of the "n-mode" eigenfunction.*

Furthermore,

(i) If $\theta = 2$, then $\alpha_* = 1$ and $n = 0$. Hence $\mathcal{L}$ has zero eigenvalue with the spatially homogeneous eigenfunction at $\alpha = 1$. Its geometric multiplicity is one.

(ii) If $0 \le \theta < 2$, then

(a) $\alpha_* = 1$ if and only if there exists $n \in \mathbb{N}$ such that $b = \sqrt{2-\theta}/2n\pi$.

(b) if $\theta \ge \sqrt{3}$ and $b > \left[(\theta - 2\sqrt{\theta^2-3})/6\pi^2\right]^{\frac{1}{2}}$, then $\mathcal{L}$ has zero eigenvalue with the spatially homogeneous eigenfunction at $\alpha = \left(2\theta - \sqrt{\theta^2-3}\right)/3$. Its geometric multiplicity is one.

(c) if $\alpha_* > 1$ and there exists a number $n \in \mathbb{N} \cup \{0\}$ such that

$$b = \sqrt{\frac{(2n^2-2n-1)\theta + 2\sqrt{\theta^2 - 3 + 4n(1+n)(n^2+n+\theta^2-3)}}{2\pi^2(2n^2-2n-1)(2n^2+6n+3)}},$$

then $\mathcal{L}$ has zero eigenvalue with the "n-mode" and "n+1-mode" eigenfunctions at $\alpha = \alpha_*$.

3.3. Lyapunov-Schmidt reduction with symmetry

We study the problem $F(\alpha, u) = 0$ for nonlinear operator $F : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ in a neighborhood of $\mathcal{O} = (\alpha^o, u^o)$. The Lyapunov-Schmidt reduction is a standard method in bifurcation theory[8]. The method reduces the problem to a finite-dimensional one. When the system has a certain symmetry, the reduced system can inherit the symmetry[8].

Suppose that $\theta < 2$ and $b = \sqrt{2-\theta}/2n\pi$ for some $n \in \mathbb{N}$. To study solutions to (3.5) and their stability in the neighborhood of $\mathcal{O}$, we apply the Lyapunov-Schmidt reduction.

We introduce some notations:

- Let $\mathcal{L}^o = (D_uF)(\alpha^o, u^o)$ be the linearized operator of F with respect to u.
- $\mathcal{N} = \ker(\mathcal{L}^o)$ is nullspace of $\mathcal{L}^o$, which is of course the zero eigenspace of $\mathcal{L}^o$.

- $\mathcal{R} = \mathrm{range}(\mathcal{L}^o)$ is the range of $\mathcal{L}^o$.
- Let $\mathcal{L}^{o*}$ be the adjoint operator of $\mathcal{L}^o$.
- $\mathcal{N}^* = \ker(\mathcal{L}^{o*})$ is nullspace of $\mathcal{L}^{o*}$.

Remark that if $\mathcal{R}$ is closed, then $\mathcal{R} = (\mathcal{N}^*)^\perp$. As $\mathcal{L}$ is elliptic, we get the following:

Lemma 3.2. *$\mathcal{L}^o$ is a Fredholm operator with index zero.*

Since $\mathcal{L}^o$ is Fredholm with index zero, $\mathcal{N}$ and $\mathcal{N}^*$ have same dimension $d = 2$. $\mathcal{X}$ and $\mathcal{Y}$ can be decomposed as

$$\begin{aligned} &\mathcal{X} = \mathcal{N} \oplus \mathcal{M}, \qquad \mathcal{N} \cap \mathcal{M} = \{0\} \\ &\mathcal{Y} = \mathcal{R} \oplus \mathcal{S}, \qquad \mathcal{R} \cap \mathcal{S} = \{0\}\,. \end{aligned}$$

Let $\mathcal{Q} : \mathcal{Y} \to \mathcal{R}$ be a projection onto $\mathcal{R}$ along $\mathcal{S}$.

We find the solution to (3.5) in the form of

$$\begin{cases} \alpha = \alpha^o + \nu, \\ u = u^o + v + w, \quad v \in \mathcal{N}, w \in \mathcal{M}, \end{cases}$$

in the neighborhood of $\mathcal{O}$. Decompose (3.5) into $\mathcal{QY}$ and $(I - \mathcal{Q})\mathcal{Y}$ components and consider the system

$$\begin{cases} \mathcal{Q}F(\alpha^o + \nu, u^o + v + w) = 0 \\ (I - \mathcal{Q})F(\alpha^o + \nu, u^o + v + w) = 0. \end{cases} \tag{3.11}$$

Thanks to the Implicit Function Theorem, in the neighborhood $\mathcal{I} \times \mathcal{U}$ of $(0, 0) \in \mathbb{R} \times \mathcal{N}$, there exists a unique mapping $w^o(\nu, v), w^o : \mathcal{I} \times \mathcal{U} \subset \mathbb{R} \times \mathcal{N} \to \mathcal{M}$, such that

$$w^o(0, 0) = 0, \qquad \mathcal{Q}F(\alpha^o + \nu, u^o + v + w^o(\nu, v)) = 0.$$

The problem (3.5) is reduced to finite dimensional problem

$$\Phi(\nu, v) = (I - \mathcal{Q})F(\alpha^o + \nu, u^o + v + w^o(\nu, v)) = 0. \tag{3.12}$$

Remark that the reduced equation (3.12) inherit $\mathbf{O}(2)$-symmetry, that is, Φ commutes with the action of $\mathbf{O}(2)$ (see Proposition 3.3. in [8], Chapter VII).

Choosing a proper basis for $\mathcal{N}$ and $\mathcal{S}$, we can consider $\Phi : \mathcal{I} \times \mathcal{U} \subset \mathbb{R} \times \mathcal{N} \to \mathcal{S}$ as $\Phi : \mathcal{I} \times \tilde{\mathcal{U}} \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for some $\tilde{\mathcal{U}} \subset \mathbb{R}$. To utilize the symmetry, we should choose the basis consistently. As we have already seen, $\mathcal{N}$ is spanned by

$$\phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos k_n x, \qquad \phi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin k_n x.$$

By the Fredholm alternative, we have $\mathcal{S} = \mathcal{R}^\perp = \mathcal{N}^* = \mathrm{span}\{\phi_1^*, \phi_2^*\}$, where $\phi_1^* = \phi_1, \phi_2^* = \phi_2$. Define the bifurcation map $g : (\nu, z) \mapsto g(\nu, z) \in \mathbb{R}^2$ by

$$g(\nu, z) = \begin{pmatrix} g_1(\nu, z) \\ g_2(\nu, z) \end{pmatrix} = \begin{pmatrix} \langle \phi_1^*, \Phi(\nu, z_1\phi_1 + z_2\phi_2) \rangle \\ \langle \phi_2^*, \Phi(\nu, z_1\phi_1 + z_2\phi_2) \rangle \end{pmatrix}, \tag{3.13}$$

where $z = (z_1, z_2)^T$ is contained in a small neighborhood of $z = (0, 0) \in \mathbb{R}^2$. Thus, by the Lyapunov-Schmidt reduction, we have

Lemma 3.3. *Solutions of (3.5) are locally in one-to-one correspondence with solutions of the finite system $g(\nu, z) = 0$, where g is defined by (3.13).*

Since $\mathbf{O}(2)$ acts linearly on $\mathcal{N}$, for each $\gamma \in \mathbf{O}(2)$ there is a $2{\times}2$ matrix $A(\gamma) = (a_{ij}(\gamma))_{i,j=1}^2$ such that

$$\gamma \cdot \phi_i = \sum_{j=1}^{2} a_{ji}(\gamma)\phi_j, \quad i = 1, 2. \tag{3.14}$$

Since we take $\phi_1^* = \phi_1, \phi_2^* = \phi_2$, we also have

$$\gamma \cdot \phi_i^* = \sum_{j=1}^{2} a_{ji}(\gamma)\phi_j^*, \quad i = 1, 2. \tag{3.15}$$

Then the bifurcation map given by (3.13) satisfies

$$g(\nu, A(\gamma)z) = A(\gamma)g(\nu, z), \tag{3.16}$$

where $A(\gamma)$ is the $2{\times}2$ matrix defined by (3.14) and (3.15).

Lemma 3.4. *The $2{\times}2$ matrix $A(\gamma)$ defined by (3.14) and (3.15) is determined as follows:*

(i) For $\xi \in \mathbf{SO}(2)$, the matrix $A(\xi)$ is given by

$$A(\xi) = \begin{pmatrix} \cos k_n\xi & -\sin k_n\xi \\ \sin k_n\xi & \cos k_n\xi \end{pmatrix}.$$

(ii) For the reflection κ, the matrix $A(\kappa)$ is given by

$$A(\kappa) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore $A(\gamma)$ defines the action on $\mathbb{R}^2$ of $\mathbf{O}(2)$.

Proof. (i) For a function $u(x)$ and $\xi \in \mathbf{SO}(2)$, $\xi \cdot u(x) = u(x - \xi)$. By the sum and difference formulas, we obtain

$$\xi \cdot \phi_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(k_n(x - \xi)\right) = (\cos k_n\xi)\phi_1 + (\sin k_n\xi)\phi_2,$$

$$\xi \cdot \phi_2(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\left(k_n(x - \xi)\right) = -(\sin k_n\xi)\phi_1 + (\cos k_n\xi)\phi_2.$$

(ii) For a function $u(x)$, $\kappa \cdot u(x) = u(-x)$. By the negative angle formula, we obtain

$$\kappa \cdot \phi_1(x) = \phi_1(x), \quad \kappa \cdot \phi_2(x) = -\phi_2(x).$$

□

As mentioned above, the bifurcation map g satisfies (3.16). It implies that g is $\mathbf{O}(2)$-equivariant.

Lemma 3.5. *For the bifurcation map g defined by (3.13), there is a smooth function $p(\nu, \xi)$ such that*

$$g(\nu, z) = zp(\nu, |z|^2), \tag{3.17}$$

where $|z|$ is the standard norm in $\mathbb{R}^2$,that is, $|z|^2 = z_1^2 + z_2^2$.

Proof. Since g is also $\mathbf{SO}(2)$-equivariant, there exists smooth functions $p(\nu, \xi), q(\nu, \xi)$ such that

$$g(\nu, z) = p(\nu, |z|^2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + q(\nu, |z|^2) \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}, \tag{3.18}$$

(see [8], chapter VIII). Remark that $\mathbf{O}(2)$ is generated by $\mathbf{SO}(2)$ and reflection κ. We can easily get $A(\kappa)g(\nu, A(\kappa)z) = A(\kappa)^2 g(\nu, z) = g(\nu, z)$, where $A(\kappa)$ is the 2×2 matrix defined in the previous lemma. Substituting (3.18), we obtain

$$A(\kappa)g(\nu, A(\kappa)z) = p(\nu, |z|^2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + q(\nu, |z|^2) \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}.$$

This formula can equal $g(\nu, z)$ only if $q(\nu, |z|^2) = 0$. Hence (3.17) holds. □

Then we study bifurcation solutions in the neighborhood of $\mathcal{O}$. We need the Fréchet derivatives of F at $\mathcal{O}$. Let us denote the derivatives at $\mathcal{O}$ by $(D_u F)^o, (D_\alpha F)^o, \ldots$.

Lemma 3.6. *For the nonlinear operator $F(\alpha, u)$ defined in (3.5), the Fréchet derivatives at $\mathcal{O} = (\alpha^o, u^o)$ are given as follows: for $u = (u_1, u_2)^T, v = (v_1, v_2)^T, w = (w_1, w_2)^T \in \mathcal{X}$,*

$$\begin{aligned}
&(D_\alpha F)^o = 0, \quad (D_\alpha^2 F)^o = 0, \\
&(D_u F)^o u = \mathcal{L}^o u, \quad ((D_{\alpha u} F)^o)u = \begin{pmatrix} -u_2 \\ 3u_1 \end{pmatrix}, \\
&(D_u^2 F)^o(u, v) = \begin{pmatrix} -2(u_2 v_1 + u_1 v_2) \\ 2(3u_1 v_1 + u_2 v_2) \end{pmatrix}, \\
&(D_u^3 F)^o(u, v, w) = \begin{pmatrix} -2(u_2 v_1 + u_1 v_2)w_1 - 2(u_1 v_1 + 3u_2 v_2)w_2 \\ 2(3u_1 v_1 + u_2 v_2)w_1 + 2(u_2 v_1 + u_1 v_2)w_2 \end{pmatrix}.
\end{aligned}$$

Now we calculate the Taylor expansion of $g(\nu, z)$ around $(\nu, z) = (0, 0)$. First, the Taylor expansion of F is given by

$$\begin{aligned}
F(\alpha^o + \nu, u^o + v) = F(\alpha^o, u^o) &+ \nu(D_\alpha F)^o + (D_u F)^o v \\
&+ \frac{\nu^2}{2}(D_\alpha^2 F)^o + \nu((D_{\alpha u} F)^o)v + \frac{1}{2}(D_u^2 F)^o(v, v) + O(3),
\end{aligned} \tag{3.19}$$

where $O(3)$ is the higher order terms of $|\nu|, |v|$.

Then $w^o(\nu, z)$ is determined as follows:

Lemma 3.7. *In a neighborhood of $(\nu, z) = (0, 0)$, the Taylor expansion of $w^o(\nu, z) \in \mathcal{QX}$ is given by*

$$\begin{aligned} w^o(\nu, z) = &-(\mathcal{QL}^o)^{-1}\mathcal{Q}\{\nu z_1(D_{\alpha u}F)^o\phi_1 + \nu z_2(D_{\alpha u}F)^o\phi_2 \\ &+ \frac{z_1^2}{2}(D_u^2F)^o(\phi_1, \phi_1) + z_1z_2(D_u^2F)^o(\phi_1, \phi_2) + \frac{z_2^2}{2}(D_u^2F)^o(\phi_2, \phi_2)\} + O(3). \end{aligned} \tag{3.20}$$

Proof. Substituting $v = z_1\phi_1 + z_2\phi_2 + w$ into (3.19) and taking the projection $\mathcal{Q}$. Substituting $w = w^o(\nu, z) = \sum w_{ijk}\nu^i z_1^j z_2^k, (w_{ijk} \in \mathcal{QX})$ into the resulting equation and equating each term, we obtain (3.20). Remark that $\mathcal{QL}^o : \mathcal{QX} \to \mathcal{QY}$ is invertible according to the Fredholm property. □

We compute the Taylor expansion of g defined by

$$\begin{pmatrix} g_1(\nu, z) \\ g_2(\nu, z) \end{pmatrix} = \begin{pmatrix} \langle \phi_1^*, F(\alpha^o + \nu, u^o + z_1\phi_1 + z_2\phi_2 + w^o(\nu, z_1\phi_1 + z_2\phi_2))\rangle \\ \langle \phi_2^*, F(\alpha^o + \nu, u^o + z_1\phi_1 + z_2\phi_2 + w^o(\nu, z_1\phi_1 + z_2\phi_2))\rangle \end{pmatrix}, \tag{3.21}$$

around $(\nu, z) = (0, 0)$. Since the function g has the form (3.17), we only have to consider the case $z_2 = 0$. Taylor coefficients are

$$g_{j,k} = \frac{\partial^{j+k}g}{\partial\nu^j\partial z_1^k}(0, 0), \quad j, k \geq 1.$$

Remark that $g_{0,0}$ and $g_{0,1}$ are 0 because we have

$$\begin{aligned} g_{0,0} &= g(0, 0) = \langle \phi_1^*, F(\alpha^o, u^o)\rangle = 0, \\ g_{0,1} &= \frac{\partial g}{\partial z_1}(0, 0) = \langle \mathcal{L}^{o*}\phi_1^*, \phi_1 + D_zw^o(0, 0)\rangle = 0. \end{aligned}$$

Similarly, we obtain

Lemma 3.8. *The coefficients $g_{j,k}$ are given as follows:*

$$\begin{aligned} &g_{0,0} = 0, \quad g_{0,1} = 0, \quad g_{1,0} = 0, \quad g_{2,0} = 0, \\ &g_{1,1} = \langle \phi_1^*, (D_{\alpha u}F)^o\phi_1\rangle, \quad g_{0,2} = \langle \phi_1^*, (D_u^2F)^o(\phi_1, \phi_1)\rangle, \\ &g_{0,3} = \langle \phi_1^*, (D_u^3F)^o(\phi_1, \phi_1, \phi_1) - 3(D_u^2F)^o(\phi_1, (\mathcal{QL}^o)^{-1}\mathcal{Q}(D_u^2F)^o(\phi_1, \phi_1))\rangle. \end{aligned}$$

Proof. Substituting $v = z_1\phi_1 + w^o(\nu, z_1\phi_1)$ into (3.19) and taking into account Lemma 3.6, we get

$$\begin{aligned} F(\alpha^o + \nu, u^o + v) = \mathcal{L}^o w^o + \nu z_1(D_{u\alpha})^o\phi_1 + \frac{z_1^2}{2}(D_u^2F)^o(\phi_1, \phi_1) \\ + \nu(D_{u\alpha})^o w^o + z_1(D_u^2F)^o(\phi_1, w^o) + \frac{z_1^3}{3!}(D_u^3F)^o(\phi_1, \phi_1, \phi_1) + \cdots. \end{aligned}$$

Substitute (3.20) and take a product with ϕ_1^*. Remark that $\langle \phi_1^*, \mathcal{L}^o w^o\rangle = \langle \mathcal{L}^{o*}\phi_1^*, w^o\rangle = 0$. Then we obtain

$$\begin{aligned} g(\nu, z_1, 0) = &\nu z_1\langle \phi_1^*, (D_{\alpha u}F)^o\phi_1\rangle + \frac{z_1^2}{2}\langle \phi_1^*, (D_u^2F)^o(\phi_1, \phi_1\rangle \\ &+ \frac{\nu^3}{3!}\langle \phi_1^*, (D_u^3F)^o(\phi_1, \phi_1, \phi_1) - 3(D_u^2F)^o(\phi_1, (\mathcal{QL}^o)^{-1}\mathcal{Q}(D_u^2F)^o(\phi_1, \phi_1))\rangle + \cdots. \end{aligned}$$

□

Now we can compute $g_{1,1}, g_{0,2}$ and $g_{0,3}$ explicitly.

Lemma 3.9. *$g_{1,1}, g_{0,2}$ and $g_{0,3}$ are given by*

$$g_{1,1} = 1, \quad g_{0,2} = 0, \quad g_{0,3} = \frac{2(30\theta - 41)}{3(2-\theta)^2}.$$

Therefore the bifurcation map is represented as

$$g(\nu, z) = z\left(\nu + \frac{30\theta - 41}{9(2-\theta)^2}|z|^2 + \cdots\right), \tag{3.22}$$

in a neighborhood of $(\nu, z) = (0, 0)$.

Proof. First, we compute $g_{1,1}$. As $(D_{\alpha u}F)^o\phi_1$ is given by

$$(D_{\alpha u}F)^o\phi_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cos(2n\pi x),$$

we get

$$g_{1,1} = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\cos^2(2n\pi x)dx = 1.$$

Next, we compute $g_{0,2}$. $(D_u^2F)^o(\phi_1, \phi_1)$ is given by

$$(D_u^2F)^o(\phi_1, \phi_1) = \begin{pmatrix} -4 \\ 8 \end{pmatrix} \cos^2(2n\pi x).$$

We get

$$\begin{aligned} g_{0,2} &= \langle \phi_1^*, (D_u^2F)^o(\phi_1, \phi_1)\rangle \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 4\cos^3(2n\pi x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos(6n\pi x) + 3\cos(2n\pi x))\, dx = 0. \end{aligned}$$

Finally, we consider $g_{0,3}$. We have

$$\begin{aligned} \mathcal{Q}(D_u^2F)^o(\phi_1, \phi_1)) &= (I - \mathcal{P})(D_u^2F)^o(\phi_1, \phi_1)) \\ &= (D_u^2F)^o(\phi_1, \phi_1)) - \langle \phi_1^*, (D_u^2F)^o(\phi_1, \phi_1))\rangle\phi_1 \\ &= \begin{pmatrix} -4 \\ 8 \end{pmatrix} \cos^2(2n\pi x) \\ &= \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \cos^2(4n\pi x). \end{aligned}$$

Then we solve $\mathcal{L}^o\psi = \mathcal{Q}(D_u^2F)^o(\phi_1,\phi_1))$. Although $\mathcal{L}^o$ is not invertible, this system has a solution because the right-hand-side is orthogonal to $\mathcal{N}^*$. The solution can be obtained in the form $\psi = \psi_0 + \psi_{2n}\cos(4n\pi x)$, where ψ_0 and ψ_{2n} are solutions to

$$\begin{pmatrix} -1 & \theta-1 \\ 3-\theta & -1 \end{pmatrix}\psi_0 = \begin{pmatrix} -2 \\ 4 \end{pmatrix},$$
$$\begin{pmatrix} -1 & b^2k_{2n}^2+\theta-1 \\ -b^2k_{2n}^2+3-\theta & -1 \end{pmatrix}\psi_{2n} = \begin{pmatrix} -2 \\ 4 \end{pmatrix},$$

where $k_{2n} = 4n\pi$ and $\psi_0, \psi_{2n} \in \mathbb{R}^2$. ψ_0 and ψ_{2n} are found to be

$$\psi_0 = \frac{2}{(2-\theta)^2}\begin{pmatrix} 3-2\theta \\ 1-\theta \end{pmatrix}, \quad \psi_{2n} = \frac{2}{9(2-\theta)^2}\begin{pmatrix} 6\theta-13 \\ 3\theta-7 \end{pmatrix}.$$

Since $\langle\phi_1^*,\psi\rangle = 0$, $\mathcal{Q}\psi$ is given by $\mathcal{Q}\psi = \psi_0 + \psi_{2n}\cos(4n\pi x)$. Now we obtain

$$(D_u^2F)(\phi_1,\psi) = \frac{2}{9(2-\theta)^2}\left\{\begin{pmatrix} 45\theta-52 \\ -105\theta+134 \end{pmatrix}\cos(2n\pi x) + \begin{pmatrix} 20-9\theta \\ 21\theta-46 \end{pmatrix}\cos(6n\pi x)\right\}.$$

Thus we get

$$g_{0,3} = \langle\phi_1^*, -3(D_u^2F)(\phi_1,\psi)\rangle = \frac{2(30\theta-41)}{3(2-\theta)^2}.$$

□

Theorem 3.2. *The set of solutions to the bifurcation equation (3.21) near* $(\nu,z) = (0,0)$ *is given by*

$$\left\{(\nu,z)\ ;\ \nu = -\frac{30\theta-41}{9(2-\theta)^2}|z|^2 + o(|z|^2)\right\} \cup \{(\nu,z)\ ;\ z=0\}.$$

Proof. The bifurcation map (3.13) can be written as (3.17). Hence $g(\nu,z) = 0$ is equivalent to $z_1 = z_2 = 0$ or $p(\nu,|z|^2) = 0$. The branch of nontrivial solutions corresponds to solutions to the latter condition.

Now $\Sigma = \mathbb{Z}_2(\kappa) = \{1,\kappa\}$ is a subgroup of $\mathbf{O}(2)$ and its fixed-point subspace $\mathrm{Fix}(\Sigma)$,

$$\mathrm{Fix}(\Sigma) = \left\{z \in \mathbb{R}^2\ ;\ \sigma z = z, \quad \forall\sigma\in\Sigma\right\},$$

is one-dimensional. Recall that $\mathbf{O}(2)$ acts on $\mathbb{R}^2$ absolutely irreducibly. Since g is $\mathbf{O}(2)$-equivariant, there exists a real-valued function $c : \nu \mapsto c(\nu)$ such that $(D_zg)(\nu,0) = c(\nu)I$. As we have $c(0) = 0$ and $c'(0) = 1 \neq 0$. Thus we can apply the Equivariant Branching Lemma[9, 10]. It follows that there exists a unique branch of nontrivial solutions to $g(\nu,z) = 0$ in $\mathbb{R}\times\mathrm{Fix}(\Sigma)$. The solution set to $p = 0$ consists of the group orbit through points on this branch. Taking the previous lemmas into account, we get the statement. □

Thus the occurrence of zero eigenvalue of $u^o = 0$ at $\alpha = 1$ leads to a pitchfork of revolution bifurcation. The cycles of equilibria exist for $\alpha > 1$ if $\theta < \frac{41}{30}$, otherwise for $\alpha < 1$. As discussed in [10] Chapter 7.2, the bifurcation with (homogeneous) Neumann boundary conditions on an interval $(0, \frac{1}{2})$ can be treated as the restriction of $\mathbf{O}(2)$-equivariant maps to $\mathrm{Fix}(\mathbb{Z}_2(\kappa))$. It follows that with NBC on $(0, \frac{1}{2})$ the equilibrium point u^o undergoes a pitchfork bifurcation at $\alpha = 1$.

Let us study the change of stability along each branch of solution. The symmetry forces the nontrivial branch to have zero eigenvalue. By the isotypic decomposition for isotropy subgroup Σ, we can restrict the stability problem on $\mathrm{Fix}(\Sigma)$.

Let $s \in \mathbb{R}$ be a parameter which parametrizes a branch of solutions. In our cases, $s = \nu$ or z_1. Consider a family of eigenvalue problem

$$\mathcal{L}(s)\phi(s) = \zeta(s)\phi(s), \quad \phi \in \mathcal{X}, s \in \mathbb{R}.$$

Eigenvalues on z_1-(ν-)branch is denoted by $\zeta^{z_1}(z)\,(\zeta^{\nu}(\nu))$. By a straightforward calculation we get

Lemma 3.10. *The following three hold:*

- $$\frac{d\zeta^{\nu}}{d\nu}(0) = g_{1,1} > 0 \tag{3.23}$$
- $$\frac{d\zeta^{z_1}}{dz_1}(0) = g_{0,2} = 0 \tag{3.24}$$
- $$\frac{d^2\zeta^{z_1}}{dz_1^2}(0) = \frac{2}{3}g_{0,3} = -2\frac{d\zeta^{\nu}}{d\nu}(0)\frac{d^2\nu^{z_1}}{dz_1^2}(0) \tag{3.25}$$

(3.23) says that trivial equilibrium point $u = 0$(ν-branch) loses its stability at $\alpha = 1$. On the other, z_1-branch is tangent to z_1-axis with order 2. If $g_{0,3} < 0$, then ζ^{z_1} is negative in a small neighborhood of the bifurcation point. If $g_{0,3} > 0$, then ζ^{z_1} is positive. Therefore,

- if $\theta < \frac{41}{30}$, then $u = 0$ undergoes supercritical pitchfork bifurcation and stable branch arises for $\alpha > 1$.
- if $\theta > \frac{41}{30}$, then $u = 0$ undergoes subcritical pitchfork bifurcation and unstable branch arises for $\alpha < 1$.

Theorem 3.3. *Assume $\theta < 2$ and there exists $n \in \mathbb{N}$ such that $b = \sqrt{2-\theta}/2n\pi$. Consider (3.3) with PBC on an interval $(-\frac{1}{2}, \frac{1}{2})$. Then the homogeneous equilibrium point $u^o = 0$ undergoes a pitchfork of revolution bifurcation at $\alpha = 1$. u^o is linearly stable for $\alpha < 1$ and unstable for $\alpha > 1$.*

(i) If $\theta < \frac{41}{30}$, then the bifurcation is supercritical. A unique branch of nontrivial solutions with isotropy subgroup $\mathbb{Z}_2$ arises for $\alpha > 1$, which consists of neutral stable solutions.

(ii) If $\theta > \frac{41}{30}$, then the bifurcation is subcritical. A unique branch of nontrivial solutions with isotropy subgroup $\mathbb{Z}_2$ arises for $\alpha < 1$, which consists of unstable solutions.

3.4. Codim 2 Bifurcation at $(\alpha, \theta) = (1, \frac{41}{30})$

As shown in the previous subsection, the bifurcation map $g(\nu, z)$ have a degeneracy of nonlinear term at $(\alpha, \theta) = (1, \frac{41}{30})$. That is, cubic terms of $g(0, z)$ vanish for $\theta = \frac{41}{30}$. We focus on the bifurcation near this codim 2 point $(\alpha, \theta) = (1, \frac{41}{30})$. Here we apply the center manifold reduction near $\alpha = 1$ [18].

We redefine bifurcation parameters. Let $\nu_1 = \alpha - 1$, $\nu_2 = \theta - \frac{41}{30}$ and $\nu = (\nu_1, \nu_2)$.

Consider a suspended system

$$\begin{cases} \dfrac{\mathrm{d}\nu}{\mathrm{d}t} = 0, \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = F(\nu, u). \end{cases} \tag{3.26}$$

We have the decomposition $\mathcal{X} = \mathcal{N} \oplus \mathcal{M}$. Let $\mathcal{P}$ be a projection defined by $\mathcal{P}u = \sum \langle \phi_j^*, u \rangle \phi_j$. Let $\mathcal{Q} = I - \mathcal{P}$. Remark that the real parts of spectrum of $\mathcal{QL}^o$ are negative. Decompose F by projections $\mathcal{P}$ and $\mathcal{Q}$:

$$\begin{cases} \dfrac{\mathrm{d}\nu}{\mathrm{d}t} = 0, \\ \dfrac{\mathrm{d}z_i}{\mathrm{d}t} = \langle \phi_i^*, F(\nu, \sum z_j(t)\phi_j + v(t)) \rangle \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{Q}F(\nu, \sum z_j(t)\phi_j + v(t)). \end{cases} \tag{3.27}$$

According to the center manifold theorem, there exists four dimensional center manifold $v = V(\nu, z) \in \mathcal{QX}$. The reduced dynamics on a parameter-dependent center manifold is given by

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \langle \phi_i^*, F(\nu, \sum z_j \phi_j + V(\nu, z)) \rangle, \qquad i = 1, 2. \tag{3.28}$$

Due to the $\mathbf{O}(2)$-symmetry, this reduced vector field is also $\mathbf{O}(2)$-invariant [23]. Hence there exists a real-valued function p such that

$$\frac{dz}{dt} = zp(\nu, |z|^2). \tag{3.29}$$

Thus we can restrict the problem to the subspace of even functions to compute the Taylor expansion of the reduced vector field.

We need some Taylor coefficients of the reduced vector field $\frac{dz}{dt} = f(\nu, z) = \sum f_{ij} \nu_1^i z^j$. Remark that the coefficients of $\nu_1^i z^{2m}$-terms $(i, m \in \mathbb{N})$ vanish due to the $\mathbb{Z}_2$-symmetry. The center manifold satisfies the homological equation

$$\mathcal{Q}F(\nu, z\phi_1 + V(\nu, z)) = D_z V(\nu, z) \langle \phi_1^*, F(\nu, z\phi_1 + V(\nu, z)) \rangle. \tag{3.30}$$

Expand $V(\nu, z)$ as $V(\nu, z) = \sum V_{ij} \nu_1^i z^j$. Here we need $V_{02}, V_{03}, V_{04}, f_{11}, f_{03}$ and f_{05}.

Substituting this into (3.30) and equating each term, we obtain a set of equations

$$\mathcal{QL}^o V_{02} + \frac{1}{2}\mathcal{Q}(D_u^2 F)^o(\phi_1, \phi_1) = 0,$$

$$\mathcal{QL}^o V_{03} + \mathcal{Q}(D_u^2 F)^o(\phi_1, V_{02}) + \frac{1}{6}\mathcal{Q}(D_u^3 F)^o(\phi_1, \phi_1, \phi_1) = 2f_{02}V_{02},$$

$$\begin{aligned}\mathcal{QL}^o V_{04} = &- \left[(D_u^2 F)^o(\phi_1, V_{03}) + \frac{1}{2}(D_u^2 F)^o(V_{02}, V_{02}) + \frac{1}{2}(D_u^3 F)^o(\phi_1, \phi_1, V_{02})\right] \\ &+ 2f_{03}V_{02} + 3f_{02}V_{03}.\end{aligned}$$

On the other, the Taylor coefficients of $f(\nu, z)$ is given by

$$\begin{aligned} f_{11} =&\langle \phi_1^*, (D_{\alpha u}F)^o \phi_1 \rangle, \\ f_{03} =&\langle \phi_1^*, (D_u^2 F)^o(\phi_1, V_{02}) + \frac{1}{6}(D_u^3 F)^o(\phi_1, \phi_1, \phi_1) \rangle, \\ f_{05} =&\langle \phi_1^*, (D_u^2 F)^o(\phi_1, V_{04}) + (D_u^2 F)^o(V_{02}, V_{03}) + \frac{1}{2}(D_u^3 F)^o(\phi_1, \phi_1, V_{03}) \\ &+ \frac{1}{2}(D_u^3 F)^o(\phi_1, V_{02}, V_{02}) \rangle. \end{aligned}$$

It is possible to compute these coefficients successively. As in the previous subsections, we have

$$f_{11} = 1, \quad f_{03} = \frac{30\theta - 41}{9(2-\theta)^2}.$$

Furthermore, we can compute V_{02}, V_{03} and V_{04} as

$$V_{02} = \frac{1}{(2-\theta)^2}\begin{pmatrix} 2\theta - 3 \\ \theta - 1 \end{pmatrix} + \frac{\cos(4n\pi x)}{9(2-\theta)^2}\begin{pmatrix} 13 - 6\theta \\ 7 - 3\theta \end{pmatrix}$$

$$V_{03} = \frac{\cos(2n\pi x)}{12(2-\theta)^2}\begin{pmatrix} 26 - 14\theta - 9\theta^2 \\ -(26 - 14\theta - 9\theta^2) \end{pmatrix} + \frac{\cos(6n\pi x)}{288(2-\theta)^4}\begin{pmatrix} 525 - 574\theta + 192\theta^2 - 18\theta^3 \\ 317 - 374\theta + 144\theta^2 - 18\theta^3 \end{pmatrix}$$

$$\begin{aligned} V_{04} = &\frac{1}{162(2-\theta)^6}\begin{pmatrix} -6390 + 90004\theta - 13470^2 - 2961\theta^3 + 1512\theta^4 - 243\theta^5 \\ +6866 - 115700\theta - 50490^2 - 234\theta^3 + 297\theta^4 \end{pmatrix} \\ &+ \frac{\cos(4n\pi x)}{11664(2-\theta)^6}\begin{pmatrix} -56383 + 1317720 - 1557360^2 + 1082700^3 - 398520^4 + 58320^5 \\ -179965 + 3829590 - 2944980^2 + 975780^3 - 118260^4 \end{pmatrix} \\ &+ \frac{\cos(8n\pi x)}{291600(2-\theta)^6}\begin{pmatrix} 683425 - 11676120 + 7398720^2 - 2055780^3 + 210600^4 \\ 400075 - 7096770 + 4715820^2 - 1391580^3 + 153900^4 \end{pmatrix} \end{aligned}$$

Finally it can be found out that f_{05} is given by

$$f_{05} = \frac{-262440^6 + 2332880^5 + 5326560^4 - 16578000^3 + 7971480^2 + 19751640 - 1767245}{23328(2-\theta)^6}.$$

Lemma 3.11. *Expansion of $f(\nu, z)$ as a Taylor series with respect to z at $z = 0$ yields*

$$\frac{dz}{dt} = f_1(\nu)z + f_3(\nu)|z|^2 z + f_5(\nu)|z|^4 z + O(|z|^7), \tag{3.31}$$

where coefficients satisfy

$$f_1(0) = 0, \quad \frac{\partial f_1}{\partial \nu_1}(0) = 1, \quad f_3(0) = 0, \quad f_5(0) < 0.$$

Proof. Because of $\mathbf{O}(2)$-symmetry, $|z|^6$-term vanishes.

$f_1(0) = 0$ since the expansion has no $\nu_1^0 z$-term. We have already seen $\frac{\partial f_1}{\partial \nu_1}(0) = f_{11} = 1$ and $f_3(0)$ vanishes. Finally we have $f_5(0) = f_{05}|_{\theta=\frac{41}{30}} = -\frac{3067411529}{376367048} < 0.$ □

Let us introduce new parameters $\mu = (\mu_1, \mu_2)$ by

$$\mu_1 = f_1(\nu), \quad \mu_2 = f_3(\nu).$$

This transform is regular at $\nu = 0$. Indeed, we get

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial \nu_1} & \frac{\partial f_1}{\partial \nu_2} \\ \frac{\partial f_3}{\partial \nu_1} & \frac{\partial f_3}{\partial \nu_2} \end{pmatrix} \Bigg|_{\nu=0} \neq 0.$$

Hence we can write ν in terms of μ near the origin and obtain the equation

$$\frac{dz}{dt} = \mu_1 z + \mu_2 |z|^2 z + F_5(\mu)|z|^4 z + O(|z|^7),$$

where $F_5(\mu) = f_5(\nu(\mu))$.

Then, rescaling

$$y = \sqrt[4]{|F_5(\mu)|}z, \quad y \in \mathbb{R}^2$$

and defining the parameters

$$\beta_1 = \mu_1, \quad \beta_2 = \sqrt{|F_5(\mu)|}\mu_2,$$

yields the normal form

$$\frac{dy}{dt} = \beta_1 y + \beta_2 |y|^2 y - |y|^4 y + O(|y|^7). \tag{3.32}$$

Write the system in polar coordinates (ρ, φ), where $y_1 = \rho\cos\varphi, y_2 = \rho\sin\varphi$:

$$\begin{aligned} \frac{d\rho}{dt} &= \rho(\beta_1 + \beta_2\rho^2 - \rho^4 + O(\rho^6)), \\ \frac{d\varphi}{dt} &= 0. \end{aligned} \tag{3.33}$$

There two equations are independent. We have to study non-negative solutions to the first equation. Since it is one-dimensional, the problem is to study the number and stability of equilibria.

First, the system always has the trivial equilibrium $\rho = 0$. It is obvious that $\{(\beta_1, \beta_2) : \beta_1 = 0\}$ is the bifurcation curve (line) of the trivial equilibrium. $\rho = 0$ is stable if $\beta_1 < 0$, while it is unstable if $\beta_1 > 0$.

Next, we consider nontrivial equilibria of (3.33). Nontrivial equilibria should satisfy

$$h(\rho^2, \beta) \equiv \beta_1 + \beta_2\rho^2 - \rho^4 + O(\rho^6) = 0.$$

Therefore the problem is to find non-negative solutions to

$$h(\xi, \beta) = \beta_1 + \beta_2\xi - \xi^2 + O(\xi^3) = 0, \tag{3.34}$$

in a small neighborhood of $(\xi, \beta) = (0, 0) \in \mathbb{R} \times \mathbb{R}^2$. We apply scaling procedures as shown in [4].

Lemma 3.12. *There is a neighborhood U of $(\xi, \beta) = (0,0)$ and a constant $c > 0$ such that any solution of (3.34) in U must satisfy*

$$|\xi| \le c\left(|\beta_1|^{\frac{1}{2}} + |\beta_2|\right).$$

Proof. We prove it by use of a contradiction. If the consequence is not be satisfied, then there exist sequences of solutions and parameters: $\{(\xi_n, \beta_{1n}, \beta_{2n})\}_{n=1}^{\infty}$ corresponding to (3.34) such that, if $n \to \infty$, then

$$(\xi_n, \beta_{1n}, \beta_{2n}) \to (0,0,0), \text{ and } \left(\frac{|\beta_{1n}|}{|\xi_n|^2}, \frac{|\beta_{2n}|}{|\xi_n|}\right) \to (0,0).$$

But, this is a contradiction with the equation (3.34). □

Consider the solutions of (3.34) along the β_1-axis. Suppose that $\beta_2 = 0, \beta_1 \neq 0$ and consider

$$\beta_1 - \xi^2 + O(\xi^3) = 0. \tag{3.35}$$

Rescaling (3.35) by $\xi = |\beta_1|^{1/2}\zeta$, we get

$$\begin{cases} 1 - \zeta^2 + O(|\beta_1|^{1/2}) = 0 & \text{for } \beta_1 > 0 \text{ small,} \\ -1 - \zeta^2 + O(|\beta_1|^{1/2}) = 0 & \text{for } \beta_1 < 0 \text{ small.} \end{cases} \tag{3.36}$$

By Lemma 3.12, to find small solution of (3.35) is equivalent to finding all solutions of (3.36) in $\mathbb{R}$. If $\beta_1 = 0$, then the first equation of (3.36) has two real solutions $\zeta = \pm 1$, while the second one has no real solutions. Hence, by the Implicit Function Theorem, there exists two distinct solutions $\xi_j(\beta_1) = |\beta_1|^{1/2}\zeta_j(\beta_1), \zeta_j(0) = (-1)^j, j = 0, 1$ of (3.35) for $\beta_1 > 0$, and (3.36) has no real solutions for small $\beta_1 < 0$. Furthermore, $\beta_1 = 0$ is a fold bifurcation point of (3.35). Remark that only $\xi_0(\beta_1), \beta_1 \ge 0$ gives the non-negative solutions.

Next, consider the solutions along the β_2-axis. Suppose that $\beta_1 = 0, \beta_2 \neq 0$ and consider

$$\beta_2\xi - \xi^2 + O(\xi^3) = 0. \tag{3.37}$$

Rescaling (3.37) by $\xi = \beta_2\eta$, we get

$$\eta - \eta^2 + O(\beta_2) = 0. \tag{3.38}$$

By Lemma 3.12, to find small solution of (3.37) is equivalent to find all solutions of (3.38) in $\mathbb{R}$. If $\beta_2 = 0$, then $\eta = 0$ and $\eta = 1$ are solutions of (3.38). It should be noted that $\mathbf{Z}_2$-symmetry forces $\eta = 0$ to be a solution for any β_2 small. As in the previous case, the Implicit Function Theorem implies that there exist two distinct solutions $\eta_0(\beta_2), \eta_0(0) = 1$ and $\eta_1(\beta_2) = 0$ of (3.38). Then $\tilde{\xi}_j(\beta_2) = \beta_2\eta_j(\beta_2), j = 0, 1$ gives two distinct solutions of (3.37). Remark that there exist positive solutions $\tilde{\xi}_0(\beta_2)$ only if $\beta_2 > 0$.

We now find the bifurcation curve of (3.34). Rescale (3.34) by $\xi = |\beta_1|^{1/2}\zeta$ and $\beta_2 = \gamma_2|\beta_1|^{1/2}$:

$$\begin{cases} 1 + \gamma_2\zeta - \zeta^2 + O(|\beta_1|^{1/2}) = 0 & \text{for } \beta_1 > 0 \text{ small} \\ -1 + \gamma_2\zeta - \zeta^2 + O(|\beta_1|^{1/2}) = 0 & \text{for } \beta_1 < 0 \text{ small.} \end{cases} \tag{3.39}$$

Since (3.39) and the equation resulting from the differentiation in ζ of (3.39) have no solutions for β_1, γ_2 small, there exist no multiple solutions of (3.39) for β_1, γ_2 small. Therefore, no bifurcation occurs near β_1-axis. Thus, it is necessary that $|\gamma_2|$ should be somewhat large. To avoid the non-compactness of the range of γ_2, let us reparametrize by $\xi = \beta_2\eta$ and $\beta_1 = \gamma_1\beta_2^2$, $\gamma_1 = \gamma_2^{-2}$ in (3.34):

$$\gamma_1 + \eta - \eta^2 + O(\beta_2) = 0,$$

where the moduli of β_2 and γ_1 are small, and $\eta \in \mathbb{R}$. The bifurcation curve is given by (3.40) and the following derivative in η of (3.41):

$$\gamma_1 + \eta - \eta^2 + O(\beta_2) = 0, \tag{3.40}$$

$$1 - 2\eta + O(\beta_2) = 0. \tag{3.41}$$

If $\beta_2 = 0$, then (3.40), (3.41) has the unique solution $\eta = \frac{1}{2}, \gamma_1 = -\frac{1}{4}$. The Implicit Function Theorem implies the existence of solutions $\eta^*(\beta_2), \gamma_1^*(\beta_2)$ with $\eta^*(0) = \frac{1}{2}, \gamma_1^*(0) = -\frac{1}{4}$ near $\beta_2 = 0$. Thus the bifurcation curve is given by $\beta_1 = \gamma_1^*(\beta_2)\beta_2^2$ while the solutions along this curve are given by $\xi^*(\beta_2) = \eta^*(\beta_2)\beta_2$. Remark that $\beta_2 > 0$ is necessary for positive solutions.

Finally we consider the solutions away from the bifurcation curve. Suppose $\gamma_1 \neq \gamma_1^*(\beta_2)$ and consider (3.40). If $\beta_2 = 0$, then the solutions of (3.40) is given by $\eta = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\gamma_1}\right)$. If $1 + 4\gamma_1 > 0$, then there exists two distinct solutions in $\mathbb{R}$: two positive solutions for $-\frac{1}{4} < \gamma_1 < 0$, while only one positive solution for $\gamma_1 \geq 0$. If $1 + 4\gamma_1 < 0$, then no solutions in $\mathbb{R}$ exists. Again, the Implicit Function Theorem implies the existence of solutions of (3.40) near $\beta_2 = 0$. Remark that if β_2 and η have the same sign, then $\xi = \beta_2\eta$ gives a positive solution of (3.34).

The above discussion yields the number of equilibria in each parameter region in the neighborhood of $\beta = 0$. It is also possible to determine the stability of these equilibria. Thus we obtain the following lemma.

Lemma 3.13. *There is a small neighborhood of $(\rho, \beta) = (0, 0)$ in which the following properties hold: There exists two bifurcation curves of (3.33)*

$$P = \{(\beta_1, \beta_2) \ : \ \beta_1 = 0\},$$
$$S = \left\{(\beta_1, \beta_2) \ : \ \beta_1 = \gamma_1^*(\beta_2)\beta_2^2, \beta_2 \geq 0\right\},$$

and which divide the neighborhood of $\beta = 0$ into three regions

$$D_1 = \{(\beta_1, \beta_2) \ : \ \beta_1 < 0, \beta_2 < 0\} \cup \left\{(\beta_1, \beta_2) \ : \ \beta_1 < \gamma_1^*(\beta_2)\beta_2^2, \beta_2 > 0\right\},$$
$$D_2 = \{(\beta_1, \beta_2) \ : \ \beta_1 > 0\},$$
$$D_3 = \left\{(\beta_1, \beta_2) \ : \ \gamma_1^*(\beta_2)\beta_2^2\beta_1 < 0, \beta_2 > 0\right\}.$$

(3.33) has the trivial equilibrium for any β. It is stable for $\beta_1 < 0$ and unstable for $\beta_1 > 0$. D_1 contains no equilibria other than the trivial one. On the other, there exists a stable nontrivial equilibrium point in D_2. In D_3, there exists a pair of nontrivial equilibria, one is stable and the other is unstable. This pair of equilibria undergoes fold bifurcation at the parameter on S.

Remark 3.1. *It can be shown that the truncated system*

$$\frac{dy}{dt} = \beta_1 y + \beta_2 |y|^2 y - |y|^4 y.$$

is a topological normal form for (3.32) (refer to [18]), which means that this is locally topologically equivalent to (3.32) in a neighborhood of the origin. Moreover, due to our discussions in this section, the neighborhood obtains the bending branch of solution, if β_1 and β_2 are small enough.

On the other hand, it seems that this property can be proved indirectly via this topological normal form, because of uniformness of the radius of the neighborhood for β_1 and β_2. But, here we have immediately constructed the bending branch of solution for β_1 and β_2 small enough. This is because it is clearer than the indirect way.

Thus the bifurcation of the original system near the codimension 2 point $(\alpha, \theta) = (1, \frac{41}{30})$ is summarized as follows:

Theorem 3.4. *Assume $\theta < 2$ and there exists $n \in \mathbb{N}$ such that $b = \sqrt{2-\theta}/2n\pi$. Consider (3.3) with PBC on an interval $(-\frac{1}{2}, \frac{1}{2})$. Then there exists a small neighborhood U_θ of $\theta = \frac{41}{30}$ which satisfies the following properties: fix $\theta \in U_\theta$ and consider of α as a bifurcation parameter, then*

(i) if $\theta < \frac{41}{30}$, then u^o undergoes supercritical pitchfork of revolution bifurcation at $\alpha = 1$. A family of nontrivial equilibria with $\mathbb{Z}_2$-symmetry arises for $\alpha > 1$. It consists of neutrally stable equilibria.

(ii) if $\theta > \frac{41}{30}$, then u^o undergoes subcritical pitchfork of revolution bifurcation at $\alpha = 1$. A family of nontrivial equilibria with $\mathbb{Z}_2$-symmetry arises for $\alpha < 1$. It consists of unstable equilibria. Furthermore, these equilibria undergo fold bifurcation at some $\alpha_f < 1$. That is, this unstable branches coexist with a family of neutrally stable nontrivial equilibria for $\alpha > \alpha_f$, and collide and disappear at $\alpha = \alpha_f$.

3.5. Codim 1 Bifurcation at $\alpha > 1$

In the previous subsections, we have studied the bifurcation of E_S at $\alpha = 1$. Here we consider the case $\alpha_* > 1$ (see Theorem 3.1). We assume that for given b and $\theta < 2$ the linearized operator $\mathcal{L}$ has zero eigenvalue with "n-mode" eigenfunction at $\alpha = \alpha_* > 1$, where $n \neq 0$. In this subsection, we set $\mathcal{O} = (\alpha_*, u^o)$ and use the same notations. The bifurcation point α_* is given by

$$\alpha_* = \frac{1}{3}\left[2\left\{(2n\pi b)^2 + \theta\right\} - \sqrt{\left\{(2n\pi b)^2 + \theta\right\}^2 - 3}\right]. \tag{3.42}$$

Remark that if $n_-(\alpha_*) = n$, then $1 \le \alpha \le 2/\sqrt{3}$ is necessary.

We can perform the same type of Lyapunov-Schmidt reduction as in the previous subsections. The discussion can be similarly applied. Only the coefficient of the bifurcation map differs. As $\mathbf{O}(2)$-symmetry is also valid, we only have to consider on the subspace of even functions.

First we choose the basis of the kernel of $\mathcal{L}^o$. Set $l = b^2k_n^2 + \theta - \alpha_*$. Since $\det \mathcal{L}_n = 0$, we get $l^{-1} = -b^2k_n^2 + 3\alpha_* - \theta$. Then $\mathcal{N} = \operatorname{span}\{\phi\}$ and $\mathcal{N}^* = \operatorname{span}\{\phi^*\}$ are given by

$$\phi = \begin{pmatrix} l \\ 1 \end{pmatrix} \cos k_n x, \quad \phi^* = \begin{pmatrix} l^{-1} \\ 1 \end{pmatrix} \cos k_n x. \tag{3.43}$$

Note that $\langle \phi^*, \phi \rangle = 1$.

Let $\nu = \alpha - \alpha_*$. The Taylor expansion of the bifurcation map $g_1(\nu, z)$ is given by $g_1(\nu, z) = g_{11}\nu z + \cdots$. For any $m \in \mathbb{N}$, z^{2m}-terms vanish due to the $\mathbb{Z}_2$-symmetry.

We calculate the coefficient g_{11}:

$$g_{11} = \langle \phi^*, (D_{\alpha u}F)^o \phi \rangle = \frac{3l - l^{-1}}{2}.$$

Since $l = b^2k_n^2 + \theta - \alpha_*$ and $(2n_\pm \pi b)^2 = 2\alpha_* - \theta \pm \sqrt{\alpha_*^2 - 1}$, we get

$$g_{11} = \begin{cases} \alpha_* + 2\sqrt{\alpha_*^2 - 1} & \text{if } n_+(\alpha_*) \in \mathbb{N}, \\ \alpha_* - 2\sqrt{\alpha_*^2 - 1} & \text{if } n_-(\alpha_*) \in \mathbb{N}. \end{cases}$$

It is easy to check that $\alpha_* + 2\sqrt{\alpha_*^2 - 1}$ is positive for $1 \le \alpha_*$, and $\alpha_* - 2\sqrt{\alpha_*^2 - 1}$ is positive for $1 \le \alpha_* < \frac{2}{\sqrt{3}}$. Therefore, in generic, g_{11} is positive. $g_{11} > 0$ and $g_{02} = 0$ imply the occurrence of the pitchfork bifurcation at $\alpha = \alpha_*$. Furthermore, for periodic boundary conditions case, the Equivariant Branching Lemma can be applied.

Theorem 3.5. *Let $b > 0$ and $0 \le \theta < 2$. Assume the following three:*

(i) $\theta < \sqrt{3}$ *or* $b < \left(\theta - 2\sqrt{\theta^2 - 3} + \sqrt{(2\theta - \sqrt{\theta^2 - 3})^2 - 9}\right)^{\frac{1}{2}} / 2\sqrt{3}\pi.$

(ii) *(a)* $b \ne \sqrt{\sqrt{3} - \theta}/2\pi$ *or* $\theta > 7\sqrt{3}/9$.
(b) $b \ne \sqrt{\sqrt{3} - \theta}/4\pi$ *or* $\theta > 7\sqrt{3}/15$.
(c) $b \ne \sqrt{\sqrt{3} - \theta}/6\pi$ *or* $\theta > \sqrt{3}/7$.

(iii) *there does not exist* $n \in \mathbb{N} \cup \{0\}$ *such that*

$$b = \sqrt{\frac{(2n^2 - 2n - 1)\theta + 2\sqrt{\theta^2 - 3 + 4n(1+n)(n^2 + n + \theta^2 - 3)}}{2\pi^2(2n^2 - 2n - 1)(2n^2 + 6n + 3)}}.$$

Then there exists $\alpha_* \ge 1$ at which the homogeneous equilibrium of (3.3) with PBC on $(-\frac{1}{2}, \frac{1}{2})$ generically undergoes pitchfork of revolution bifurcation.

Remark 3.2. *The assumption 1 implies that zero eigenvalue with spatially homogeneous eigenfunction does not occur at the bifurcation point. The assumption 2 implies that $\alpha_* \ne \frac{2}{\sqrt{3}}$ or $n_-(\frac{2}{\sqrt{3}}) \notin \mathbb{N} \cup \{0\}$. The assumption 3 implies that mode interaction does not occur.*

3.6. Spatially homogeneous equilibria

Here we give some remarks on spatially homogeneous equilibria.

If $\theta < \sqrt{3}$, then the input field E_{in} and equilibrium E_S have one-to-one correspondence. For such case, we need not make a special mention.

We assume $\theta > \sqrt{3}$. Given E_{in} the cubic equation $E_{\mathrm{in}}^2 = I_S\left[1 + (\theta - I_S)^2\right]$ has one, two or three positive roots. We can construct the homogeneous equilibrium from E_{in} and root I_S by

$$E_S = \frac{E_{\mathrm{in}}}{1 + \mathrm{i}(\theta - I_S)}.$$

Now we parametrize E_{in} by $E_{\mathrm{in}}(\alpha) = \sqrt{\alpha\left[1 + (\theta - \alpha)^2\right]}$. The system always has an equilibrium E_α with $|E_\alpha|^2 = \alpha$. Moreover, for E_{in} given by $E_{\mathrm{in}} = E_{\mathrm{in}}(\alpha)$, we can solve the cubic equation $E_{\mathrm{in}}(\alpha)^2 = I_S\left[1 + (\theta - I_S)^2\right]$ for I_S:

$$I_S = \alpha, \quad \frac{1}{2}\left(2\theta - \alpha \pm \sqrt{-3\alpha^2 + 4\theta\alpha - 4}\right).$$

We define $I_\pm(\alpha) = \left(2\theta - \alpha \pm \sqrt{-3\alpha^2 + 4\theta\alpha - 4}\right)/2$. The functions $I_+(\alpha)$ and $I_-(\alpha)$ are real and positive for $2\left(\theta - \sqrt{\theta^2 - 3}\right)/3 \le \alpha \le 2\left(\theta + \sqrt{\theta^2 - 3}\right)/3$. In this interval, the system has equilibria with $|E|^2 = I_\pm(\alpha)$. $I_+(\alpha)$ and $I_-(\alpha)$ are roots of

$$\left(I_s + \frac{\alpha - \theta}{2}\right)^2 + \frac{3}{4}\left(\alpha - \frac{2}{3}\theta\right)^2 - \frac{\theta^2 - 3}{3} = 0, \tag{3.44}$$

which defines an ellipse on (α, I_S)-plane as an implicit function. The ellipse and the line $\{(\alpha, I_S) \ : \ I_S = \alpha\}$ intersect at $\alpha = \left(2\theta \pm \sqrt{\theta^2 - 3}\right)/3$.

It is possible to express spatially homogeneous equilibria of (3.3) in terms of α. If $2\left(\theta - \sqrt{\theta^2 - 3}\right)/3 \le \alpha \le 2\left(\theta + \sqrt{\theta^2 - 3}\right)/3$, then (3.3) possesses additional two equilibria $u^+ = (u_1^+, u_2^+)$ and $u^- = (u_1^-, u_2^-)$ given by

$$\begin{cases} u_1^\pm = \dfrac{I_\pm(\alpha)(\theta - I_\pm(\alpha))(I_\pm(\alpha) - \alpha)}{\alpha\left\{1 + (\theta - \alpha)^2\right\}}, \\ u_2^\pm = \dfrac{I_\pm(\alpha)(I_\pm(\alpha) - \alpha)}{\alpha\left\{1 + (\theta - \alpha)^2\right\}}. \end{cases}$$

Remark that our bifurcation analysis has been focused on the first bifurcation. A more careful treatment is required if a modulational instability occurs after "0-mode" instability.

4. Discussions and concluding remarks

In this paper we have mainly performed a mathematical rigorous study about a spatially pulsative structure of Lugiato-Lefever equation, that is, nonlinear Schrödinger equation with dissipation and detuning, and with cubic nonlinearity. One of the characteristic properties about this problem is lack of variational (Hamiltonian) structure. Because of the lack, we must state that it is difficult to construct the pulsating solution mathematically rigorously, especially in the entire infinite interval.

We have first indicated existence theorem of time global solution and the global attractor with finite dimensions in appropriate space of functions. This is a simple application of Prof. Ghidaglia's theorem in [7], but due to this theorem, we concentrate to construct a "soliton-like" stationary solution as an important object composing the global attractor, and stability and bifurcation analysis in one dimensional bounded interval.

Next, we have made a stability and bifurcation analysis to a homogeneous steady state. Especially, in a bounded interval with periodic boundary condition or with homogeneous Neumann boundary condition, the linearized eigenvalue problem near the homogeneous steady state obtains 0 eigenvalue with nontrivial spatial mode eigenfunctions and loses its stability. It is found out that $\alpha_* = 1$, which is the minimum value of dispersion curve, becomes the bifurcation point only when the diffraction constant b^2 takes appropriate discrete values. Moreover, at these discrete values, we have proved rigorously that a pulsating solution is bifurcating from the homogeneous steady state and stable-unstable pair of pulsating solutions coexists in some parameter region via bifurcation theory with group symmetry (See [8] and [9]) to get Theorem 3.3. This means that such a kind of pulsating structure has a preferable distance between a pulse and the next pulse, because the appropriate discrete values of b^2 can be regarded as appropriate size of the interval by use of a simple rescaling. This is meaning that the pulse solutions must be packed suitably in the space, and it is one of interesting properties about this problem.

On the other hand, if "soliton" means only one pulse solution in the entire space generally, the pulsating solution under consideration here should be called "roll" solution with periodic structure, which is slightly different from "soliton" solution. We should study the problem on the entire line of $\mathbb{R}$, if we would like to make a research about a "soliton" solution as a solitary wave. Very few are known about the dissipative cavity soliton in this rigorous meaning. In numerical simulations, usually FFT algorithm has been utilized and naturally it requires the periodic boundary condition (See [21]). In mathematical point of view, there is a very strong tool by which we prove the existence of "soliton" solution and its stability analysis in one space dimension. That is a method of homoclinic bifurcation analysis in reversible 1:1 resonance vector fields (for instance, see [6, 15, 16, 26], and so forth). Generically speaking, the "soliton" solution can exist near the "roll" solution via the theory, but this method cannot be simply applicable to the problem, because this equation dose not have an important conservation law (for example, like the first integral). Also here, the lack of variational structure affects analysis to make the problem difficult, but interesting mathematically (See also [17]).

Finally, let us discuss about "snake bifurcation". As Professors Ackemann and Firth have made a numerical simulation about it in [1], this has "snake bifurcation" structure, which means a series of lots of saddle-node bifurcations corresponding to increasing or decreasing pulses (See also [12]). In fact, if the parameters are taken as ones just outside of parameters region of this "snake bifurcation" region, then increasing or decreasing pulses can be comprehensible as transition process of trajectory passing through "traces" of stationary states of standing pulses in the phase space. This type of dynamics had been already pointed out in [22] by the time equal to or more than ten years ago in the context of self-replicating pattern of pulses in the Gray-Scott model. Moreover, Swift-Hohenberg equations have also this interesting bifurcation structure, which has been studied very well recently, for instance, refer to [14], [13], and [19]. But Swift-Hohenberg equation has Hamiltonian structure, it is surely proved that 1:1 resonant Hopf bifurcation actually happens rigorously, which is a quite different point from Lugiato-Lefever equations.

We have made numerical simulations about such a kind of interesting transition process to ensure it for ourselves. From this viewpoint, understanding this structure globally and mathematically is very interesting and important. But, it is difficult to construct the global bifurcation structure mathematically rigorously. Instead of that, as the first step, we have made a kind of singularity analysis about it near the bifurcation point with codimension two to get Theorem 3.4. This rigorously means that we made a center manifold reduction in which the dynamical system can be make a reduction to the topological normal form possessing important informations about the bifurcation branches near the bifurcation points with two codimensions. We have analyzed it to get the form whose stationary solution means "the first bending solution" in some parameter regions. Generally speaking, singularity with higher codimensions has often very important information condensed infinitesimally about the global structure of bifurcation. In this problem, we have also applied the idea to get an interesting and crucial information about the structure of bifurcation, although it is for the "roll solutions".

Acknowledgments

The first author is adopted as a Research Fellow of the Japan Society for the Promotion of Science and supported by Grant-in-Aid for JSPS Fellows, No. 20 · 3310. Moreover, whole of the work is supported by Grant-in-Aid for JSPS Fellows, No. 19204012 (The research representative is the third author). The second author must state that a part of this work has been done during his staying at IMA (Institute for mathematics and its application) in Minnesota University from November in 2008 to January in 2009. The second author discussed a little about the difficulty coming from the lack of Hamiltonian structure in Lugiato-Lefever equation with Professor Arnd Scheel at that time. His special thanks are going to go to the professor.

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