Scaling relations for percolation in the 2D high temperature Ising Model

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We consider the percolation problem for Ising model on the two-dimensional square lattice $\mathbf{Z}^2$. For $T > T_c$ and $h \in \mathbf{R}$, there exists a unique Gibbs measure $\mu_{T,h}$. The (+)-cluster containing the origin is denoted by $\mathbf{C}_0^+$. For each $T > 0$, the critical external field is defined by

$$h_c(T) := \inf\{h : \mu_{T,h}(\#\mathbf{C}_0^+ = \infty) > 0\}.$$

It is known that $h_c(T) > 0$ whenever $T > T_c$. Hereafter we fix a $T > T_c$, and abbreviate $\mu_{T,h}$ to μ_h and $h_c(T)$ to h_c, respectively. The expectation under μ_h is denoted by E_h.

The following power laws are widely believed to hold:

- Percolation probability $\theta(h) := \mu_h(\#\mathbf{C}_0^+ = \infty) \approx (h - h_c)^\beta$ as $h \searrow h_c$.
- Mean cluster size:
$$\chi(h) := E_h[\#\mathbf{C}_0^+ : \#\mathbf{C}_0^+ < \infty] \approx |h - h_c|^{-\gamma} \quad \text{as } h \to h_c.$$
- Correlation length:
$$\xi(h) := \left[\frac{1}{\chi(h)} \sum_{v \in \mathbf{Z}^2} |v|^2 \mu_h\big(\mathbf{O} \overset{+}{\leftrightarrow} v,\, \#\mathbf{C}_0^+ < \infty\big)\right]^{1/2} \approx |h - h_c|^{-\nu} \quad \text{as } h \to h_c.$$
  * For $S(n) = [-n, n]^2$, we define
$$L(h, \varepsilon_0) := \begin{cases} \min\big\{n : \mu_h\big(\text{LS of } S(n) \overset{+}{\leftrightarrow} \text{RS of } S(n)\big) \geq 1 - \varepsilon_0\big\} & (h > h_c), \\ \min\big\{n : \mu_h\big(\text{LS of } S(n) \overset{+}{\leftrightarrow} \text{RS of } S(n)\big) \leq \varepsilon_0\big\} & (h < h_c). \end{cases}$$
  Then $\xi(h) \asymp L(h, \varepsilon_0)$.
- One-arm probability: $\pi_{h_c}(n) := \mu_{h_c}\big(\mathbf{O} \overset{+}{\leftrightarrow} \partial S(n)\big) \approx n^{-1/\delta_{\mathrm{r}}}$.
- Connectivity function: $\tau_{h_c}(n) := \mu_{h_c}\{\mathbf{O} \overset{+}{\leftrightarrow} (n, 0)\} \approx n^{-\eta}$.

We derive some scaling relations, provided the exponents exist; for 2D Bernoulli percolation, these relations are proved by Kesten.