Spectral analysis of a scalar quantum field model on a Lorentzian manifold

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We are concerned with spectral properties of a scalar quantum field model defined on a curved space-time. E. Nelson investigated the so-called Nelson model in 1964 and he removed UV cutoff of the model. We extend the Nelson model to a model H on a static Lorentzian manifold. This model includes variable boson mass decaying to zero and the dispersion relation $h^{1/2}$ is realized as a pseudodifferential operator. It is given by

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(h^{1/2}) + H_I,$$

where

$$K = -\sum_{i,j} \partial_i A^{ij}(X)\partial_j + V(X),$$

$$h = -\sum_{i,j} \frac{1}{c(x)}\partial_i a^{ij}(x)\partial_j \frac{1}{c(x)} + m^2(x).$$

$d\Gamma$ denotes the second quantization and H_I a scalar field operator with some UV cutoff ρ, which is given by the sum of a creation operator and an annihilation operator:

$$H_I = \frac{1}{\sqrt{2}}\left(a^\dagger(h^{-1/4}\rho(\cdot - X)) + a(h^{-1/4}\rho(\cdot - X))\right).$$

It is assumed that $V(X)$ is a confining potential, i.e., $V(X) \to \infty$ as $|X| \to \infty$, and massless, i.e., $\inf \sigma(h) = 0$. The standard Nelson model is defined with $h = -\Delta_x$ and $K = -\Delta_X + V(X)$.

We show that the model has a ground state when $m(x) \geq a\langle x\rangle^{-1}$ with some $a > 0$. On the other hand when $m(x) \leq a\langle x\rangle^{-1-\epsilon}$ for any $\epsilon > 0$, the absence of ground state is shown. Finally we remove UV cutoff and define the self-adjoint operator associated with the model without UV cutoff. The existence of ground state is show by a compactness argument, the absence of ground state by Feynman-Kac formula and Kipnis-Varadhan theorem, and the removal of UV cutoff by pseudodifferential computations.