# Angular Momentum and its Expectation Value

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In this paper, for a physical system of a family of hydrogen atoms, we study the natural statistical phenomena of the angular momenta of electrons in hydrogen atoms.

As a theoretical model of a system of hydrogen atoms, we consider the family of hydrogen atoms , each electron of which is moving in the Coulom potential $V(r) = -\dfrac{e^2}{r}$, $(r = ||\boldsymbol{r}||)$ with its center at the the nucleus of the hydrogen atom. In this case, each electron $\rho$ has the total energy $\mathcal{E}(\rho) = \dfrac{1}{2m}\boldsymbol{p}(\rho) - \dfrac{e^2}{r}$, where $m$ and $e$ denote the mass and the electric charge of the electron respectively. We denote this system of hydrogen atoms by $\Omega = \Omega(\mathcal{B}, P)$. Then $\Omega$ is assumed to be a probability space, whose elementary event is an electron $\rho$ in a hydrogen atom.

Let $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = {}^t(L_x, L_y, L_z)$ be the angular momentum of an electron $\rho$. Here we consider $\boldsymbol{L}$ to be a vector valued natural random variable defined on $\Omega$.

Here a hydrogen atom is a combined system of a hydrogen nucleus and its inner electron. We call this state of hydrogen atom as the bound state. Thus, here, we do not study the scattering state of a hydrogen atom because we consider just a system of hydrogen atoms.

Then the system of hydrogen atoms $\Omega$ in the bound state has the following structure in the stationary state.

Namely, we have the direct sum decomposition in the folowing :

$$\Omega = \sum_{n=1}^{\infty} \Omega_n, \ = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \Omega_{nl} = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \Omega_{nlm},$$

$$\Omega_n = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \Omega_{nlm}, \ \Omega_{nl} = \sum_{m=-l}^{l} \Omega_{nlm},$$

$$P(\Omega_n) = p_n, \ (n = 1, 2, \cdots), \ \sum_{n=1}^{\infty} p_n = 1,$$

$$p_{nl} = P(\Omega_{nl}) = \sum_{m=-l}^{l} |c_{nlm}|^2, \ p_{nlm} = P(\Omega_{nlm}) = |c_{nlm}|^2.$$

Then we consider the complete orthonormal system $\{\psi_{nlm}\}$ of the eigenfunctions of the Schrödinger equation in the bound state and denote the Fourier type coefficients of the eigenfunction expansion of an $L^2$-density as $\{c_{nlm}\}$.

The energy expectation of the proper subsystem $\Omega_{nlm}$ is equal to

$$E_{\Omega_{nlm}}(\mathcal{E}(\rho)) = \int \psi_{nlm}(\boldsymbol{r})^* H \psi_{nlm}(\boldsymbol{r}) d\boldsymbol{r} = \mathcal{E}_n, \ (|m| \leq l, 0 \leq l \leq n-1; n \geq 1).$$

Then, the energy expectation $\overline{E}$ of the total system of hydrogen atoms is equal to

$$\overline{E} = E(\mathcal{E}(\rho)) = -\frac{me^4}{2\hbar^2} \sum_{n=1}^{\infty} \frac{1}{n^2} p_n.$$

Therefore the system of hydrogen atoms in the bound state is realized as the mixed system of proper subsystems. The subsystem $\Omega_n$ with the energy expectation $\mathcal{E}_n$ is the mixed system of $n^2$ proper subsystems $\Omega_{nlm}$, $(|m| \leq l, \ 0 \leq l \leq n-1)$. The ratio of mixing of those subsystems $\Omega_n$ is equal to the ratio of the sequence $\{p_n\}_{n=1}^{\infty}$.

Then, because the electron in a hydrogen atom is moving by the action of Coulomb force, $L^2$-densities $\psi_{nlm}(\boldsymbol{r}, t)$ and $\psi(\boldsymbol{r}, t)$ are varying with time variation. Then the Fourier type coefficients $\{c_{nlm}\}$ are also varying with time variation. Accompanying with this, the values of $\{p_n\}$ are also varying with time variation.

Therefore, each hydrogen atom which composes the proper subsystem with mean energy $\mathcal{E}_n$ varies its belonging to some proper subsystem according to time variation.

This is the meaning of energy expectation values of the system of hydrogen atoms.

Next we study the meaning of the expectation values of angular momenta of inner electrons.

The expectation value of angular momentum $\boldsymbol{L}^2$ of the proper subsystem $\Omega_{nlm}$ is equal to

$$E_{\Omega_{nlm}}[\boldsymbol{L}^2] = \int \psi_{nlm}(\boldsymbol{r})^* \hat{\boldsymbol{L}}^2 \psi_{nlm}(\boldsymbol{r}) d\boldsymbol{r} = l(l+1)\hbar^2.$$

The expectation value of angular momentum $\boldsymbol{L}^2$ of the total system $\Omega$ is also equal to

$$E[\boldsymbol{L}^2] = \hbar^2 \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} l(l+1)|c_{nlm}|^2.$$

The expectation value of the $z$-component $L_z$ of angular momentum of the proper subsystem $\Omega_{nlm}$ is equal to

$$E_{\Omega_{nlm}}[L_z] = m\hbar.$$

The expectation value of the $z$-component $L_z$ of angular momentum of the total system $\Omega$ is also equal to

$$E[L_z] = \hbar \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} m|c_{nlm}|^2.$$

This is the meaning of the expectation values of angular momentum of the system of hydrogen atoms.

In the stationary state, the total system $\Omega$ of hydrogen atoms is the mixed system of proper subsystems $\{\Omega_{nlm}; |m| \leq l, \ 0 \leq l \leq n-1, \ n \geq 1\}$. The ratio of this mixing is equal to the ratio of $\{p_{nlm}; |m| \leq l, \ 0 \leq l \leq n-1, \ n \geq 1\}$.

Then, the proper subsystem $\Omega_{nlm}$ is the subsystem of hydrogen atoms with the expectation value of energy $\mathcal{E}_n$, the expectation value of angular momentum $l(l+1)\hbar^2$, and the expectation value of the $z$-component of angular momentum $m\hbar$.

The proper subsystem of hydrogen atoms $\Omega_n$ with the expectation value of energy $\mathcal{E}_n$ is the mixed system of proper subsystems $\{\Omega_{nlm}; |m| \leq l, \ 0 \leq l \leq n-1\}$. The ratio of this mixing is equal to the ratio of $\{p_{nlm}; |m| \leq l, \ 0 \leq l \leq n-1\}$.

The proper subsystem of hydrogen atoms $\Omega_{nl}$ with the expectation value of energy $\mathcal{E}_n$ and the expectation value of angular momentum $l(l+1)\hbar^2$ is the mixed system of proper subsystems $\{\Omega_{nlm}; |m| \leq l$. The ratio of this mixing is equal to the ratio of $\{p_{nlm}; |m| \leq l\}$.

Therefore the belonging of each hydrogen atom to some proper subsystem is varying according to time variation.

Such a phase of varitation of the state of hydrogen atoms is known by solving the Schrödinger equation.

For the precise consideration, we refer to Ito [1], Chapter 4.

# References

[1]   Y. Ito, Natural Statistical Physics, preprint, 2009, (in Japanese).

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