Wigner matrices with random potential

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1 Introduction

Consider large matrices whose entries are random variables. Famous examples of such matrices are Wigner matrices: a Wigner matrix is an $N \times N$ real or complex matrix $W = (w_{ij})$ whose entries are independent random variables with mean zero and variance $1/N$, subject to the symmetry constraint $w_{ij} = \overline{w}_{ji}$. The empirical density of eigenvalues converges to the Wigner semicircle law in the large N limit. Under some additional moment assumptions on the entries this convergence also holds on very small scales: denoting by $G_W(z) = (W - z)^{-1}$, $z \in \mathbb{C}^+$, the resolvent or Green function of W, the convergence of the empirical eigenvalue distribution on scale η at an energy $E \in \mathbb{R}$ is equivalent to the convergence of the averaged Green function $m_W(z) = N^{-1}\mathrm{Tr}\, G_W(z)$, $z = E + \mathrm{i}\eta$. The convergence of $m_W(z)$ at the optimal scale N^{-1}, up to logarithmic corrections, the so-called *local semicircle law*, was established for Wigner matrices in a series of papers [11, 12, 13], where it was also shown that the eigenvectors of Wigner matrices are completely delocalized. The proof is based on a self-consistent equation for $m_W(z)$ and the continuity of the Green function $G(z)$ in the spectral parameter z. Precise estimates on the averaged Green function $m_W(z)$ and on the eigenvalue locations are essential ingredients for proving bulk universality [14, 15] and edge universality [16] for Wigner matrices. (See also [29, 30].)

Poisson statistics for systems represents the other extreme. It corresponds to diagonal matrices with i.i.d. random entries. While the eigenvalues of the Wigner matrix are strongly correlated, the diagonal randomness makes eigenvalues independent, hence uncorrelated. Physically, the diagonal matrix may represent an on-site random potential on a lattice system. Compared to the mean-field nature of the Wigner matrix, which is in the weak disorder- or the delocalization regime, the diagonal randomness also provides a good example in the strong disorder- or the localization regime. It is conjectured that, after quantization, classically integrable systems correspond to Poisson statistics whereas classically chaotic systems correspond to random matrix statistics. In terms of quantum chaos, the diagonal matrix describes the 'regular' part, while the Wigner matrix is a good model for the 'chaotic' part.

It is thus natural to consider the interpolation of the two, i.e., the $N \times N$ random matrix

$$H = \lambda V + W\,, \qquad \lambda \in \mathbb{R}\,, \tag{1.1}$$

where V is a real diagonal random matrix, or a 'random potential', and W a standard Hermitian or symmetric Wigner matrix independent of V. Here, W is properly normalized so that the typical eigenvalues of V and W are of the same order. The parameter λ determines the relative strength of each part in this model.

For $\lambda \sim 1$ the eigenvalue density is not solely determined by V or W in the limit $N \to \infty$, but can be described by a functional equation for the Stieltjes transforms of the limiting eigenvalue distributions of V and W; see [24]. In general, this eigenvalue distribution, referred to as the *deformed semicircle law*, is different from the semicircle distribution. The equal strength of V and W makes it non-trivial to find the nature of the interpolation H. For example, the eigenvectors are completely delocalized for W whereas they are localized for V, hence the eigenvector localization/delocalization problem requires deep investigation of the model.

When W belongs to the Gaussian Unitary Ensemble (GUE), H is called the *deformed GUE*, and it can describe Dyson Brownian motion [8] on the real line; see, e.g., [19]. There have been many important works with various scales of λ: Related to symmetry-breaking, transition statistics for eigenvalues in the bulk, especially the nearest neighbor spacing, were studied in [17] for $\lambda \sim N^{1/2}$. In this situation, the diagonal part λV controls the average density, while the GUE part induces fluctuation of eigenvalues. For $\lambda \lesssim 1$, it was shown in [26] that universality of eigenvalue correlation functions holds in the bulk of the spectrum. Concerning the edge behaviour, it was shown in [20] that the transition from the Tracy-Widom to the standard Gaussian distribution occurs on the scale $\lambda \sim N^{-1/6}$. For $\lambda \ll N^{-1/6}$, the Tracy-Widom distribution for the edge eigenvalues was established in [27].

There exists, for some choices of V, yet another transition for the limiting behaviour of the largest eigenvalues μ_1 of H as λ changes: For simplicity, we assume that the distribution of the entries of V is centered and is given by the density

$$\mu(v) := Z^{-1}(1+v)^{\mathfrak{a}}(1-v)^{\mathfrak{b}} d(v) \mathbb{1}_{[-1,1]}(v)\,, \tag{1.2}$$

where $-1 \le \mathfrak{a}, \mathfrak{b} < \infty$, d is a strictly positive C^1-function and Z is a normalization constant. The transition is based on the transition of the near-edge behaviour of the eigenvalue distribution. Let μ_{fc} be The limiting distribution of the eigenvalues of H. It is well-known that μ_{fc} is supported on a compact interval. Denoting by κ_E the distance to the endpoints of the support of μ_{fc}, i.e.,

$$\kappa_E := \min\{|E - L_-|, |E - L_+|\}\,, \qquad E \in \mathbb{R}\,, \tag{1.3}$$

we say that the distribution μ_{fc} exhibits the square root behaviour if there exists $C \ge 1$ such that

$$C^{-1}\sqrt{\kappa_E} \le \mu_{fc}(E) \le C\sqrt{\kappa_E}\,, \qquad E \in [L_-, L_+]\,. \tag{1.4}$$

The following lemma is proved in [21].

Lemma 1.1. *Let μ be a Jacobi measure; see* (1.2)*. Then, for any $\lambda \in \mathbb{R}$, there are $-\infty < L_- < 0 < L_+ < \infty$, such that* supp $\mu_{fc} = [L_-, L_+]$*. Moreover,*

1. *for $-1 < \mathfrak{a}, \mathfrak{b} \le 1$, for any $\lambda \in \mathbb{R}$, μ_{fc} exhibits the square root behaviour* (1.4)*;*
2. *for $1 < \mathfrak{a}, \mathfrak{b} < \infty$, there exists $\lambda_- \equiv \lambda_-(\mu) > 1$ and $\lambda_+ \equiv \lambda_+(\mu) > 1$ such that*
   (a) *for $|\lambda| < \lambda_-$, $|\lambda| < \lambda_-$, μ_{fc} exhibits the square root behaviour at both endpoints;*
   (b) *for $|\lambda| < \lambda_-$, $|\lambda| > \lambda_+$, μ_{fc} exhibits the square root behaviour at the lower endpoint of the support (i.e., for $E \in [L_-, 0]$), but there is $C \ge 1$, such that*

$$C^{-1}(L_+ - E)^{\mathfrak{b}} \le \mu_{fc}(E) \le C(L_+ - E)^{\mathfrak{b}}\,, \qquad E \in [0, L_+]\,. \tag{1.5}$$

Analogue statements hold for $|\lambda| > \lambda_-$, $|\lambda| < \lambda_+$, etc..

Depending on whether the measure μ_{fc} exhibits the square root behaviour, we have the following dichotomy:

1. if μ_{fc} exhibits the square root behaviour at the upper edge (Case 1. and Case 2.(a)), then there are N-independent constants $\hat{L}_+ \equiv \hat{L}_+(\mu, \lambda)$ and $a \equiv a(\mu, \lambda)$, such that

$$\lim_{N\to\infty} \mathbb{P}(N^{1/2}(\hat{L}_+ - \mu_1) \le x) = \Phi_a(x)\,, \quad \mathfrak{b} > 1\,, \quad |\lambda| < \lambda_+\,, \tag{1.6}$$

for the largest largest eigenvalue μ_1 of H, where Φ_a denotes the cumulative distribution function of a centered Gaussian law with variance a.

2. if μ_{fc} does not exhibit the square root behaviour at the upper edge (Case 2.(b)), then the largest eigenvalue μ_1 of H, satisfies

$$\lim_{N\to\infty} \mathbb{P}(N^{1/(\mathfrak{b}+1)}(L_+ - \mu_1) \leq x) = G_{\mathfrak{b}+1}(x)\,, \quad \mathfrak{b} > 1\,, \quad |\lambda| > \lambda_+\,, \tag{1.7}$$

where $G_{\mathfrak{b}+1}$ is a Weibull distribution with parameter $\mathfrak{b}+1$.

We remark that the appearance of the Weibull distribution in the model (1.1) is indeed expected in case λ grows sufficiently fast with N, since in this case the diagonal matrix dominates the spectral properties of H. However, it is quite surprising that the Weibull distributions already appear for λ order one, since the local behaviour of the eigenvalues in the bulk in the deformed model mainly stems from the Wigner part, and the contribution from the random diagonal part is limited to macroscopic fluctuations of the eigenvalues; see [21].

Having identified two possible limiting distribution of the largest eigenvalues, it is natural to ask about a behaviour of the associated eigenvectors. Before considering the deformed model, we recall that the eigenvectors of Wigner matrices with subexponential decay are completely delocalized, as was proved by Erdős, Schlein and Yau [11, 12].

In this paper, we show that the eigenvectors of the largest eigenvalues are, in case we have the edge behaviour (1.7), *partially localized*. More precisely, we prove that one component of the (ℓ^2-normalized) eigenvectors associated to eigenvalues at the extreme edge carries a weight of order one, while the other components carry a weight of order $o(1)$ each. If, however, the edge behaviour (1.6) emerges, all eigenvectors are completely delocalized. Although we do not prove it explicitly, we claim that the bulk eigenvectors of the model (1.1) with (1.2) for the choice of μ, are completely delocalized (for any choice of $\lambda \sim 1$). This can be proved with the very same methods as in [21].

The phenomenology described above is quite reminiscent to the one found for 'heavy tailed' Wigner matrices, e.g., real symmetric Wigner matrices, whose distribution function of the entries decays as a power law, i.e., the entries h_{ij} satisfy

$$\mathbb{P}(|h_{ij}| > x) = L(x)x^{-\alpha}\,, \qquad (1 \leq i,j \leq N)\,, \tag{1.8}$$

for some slowly varying function $L(x)$. It was proved by Soshnikov [28] that the linear statistics of the largest eigenvalues is Poissonian for $\alpha < 2$, in particular the largest eigenvalue has a Fréchet limit distribution. Later, Auffinger, Ben Arous and Péché [1] showed that the same conclusions hold for $2 \leq \alpha < 4$ as well. Recently, it was proved by Bodernave and Guionnet [7] that the eigenvectors of models satisfying (1.8) are weakly delocalized for $1 < \alpha < 2$. For $0 < \alpha < 1$, it is conjectured that there is a sharp 'metal-insulator' transition. In [7] it is proved that the eigenvectors of sufficiently large eigenvalues for are weakly localized, for $0 < \alpha < 2/3$.

To clarify the terminology 'partial localization' we remark that it is quite different from the usual notion of localization for random Schrödinger operators. The telltale signature of localization for random Schrödinger operators is exponential decay of off-diagonal Green function entries: it implies absence of diffusion, spectral localization etc.. For the Anderson model in dimensions $d \geq 3$ such an exponential decay was first obtained by Fröhlich and Spencer [18] using a multiscale analysis. Later, a similar bound was presented by Aizenman and Molchanov [2] using fractional moments. Due to the mean-field nature of the Wigner matrix W, there is no notion of distance for the deformed model (1.1) and we attain only a moderate decay, which coincides with what the first order perturbation theory predicts.

Yet, there are some similarities with the Anderson model in $d \geq 3$: In the Anderson model localization occurs where the density of states is (exponentially) small [18], this is known to happen close to the spectral edges or for large disorder. Further, it is strongly believed that the Anderson model admits extended states, i.e., the generalized eigenvectors in the bulk are expected to be delocalized. Moreover, it was proven by Minami [23] that the local eigenvalue statistics of the Anderson model can be described by a Poisson point process in the strong localization regime and it is also conjectured that the local eigenvalue statistics in the bulk is given by the GOE statistics, respectively GUE statistics in case time-reversal symmetry is broken.

Eventually, we mention that the localization result we prove in this paper also differs from that for random band matrices, where all the eigenvectors are localized, even in the bulk. We refer to [25, 10] for more discussion on the localization/delocalization in the random band matrices.

2 Definition and Results

In this section, we define our model and state our main results.

2.1 Free convolution

As first shown in [24] the limiting spectral distribution of the interpolating model (1.1) is given by the *(additive) free convolution measure* of μ, the limiting distribution of the entries of λV, and μ_{sc}, the semicircular measure. In a more general setting, the free convolution measure, $\mu_1 \boxplus \mu_2$, of two probability measures μ_1 and μ_2, is defined as the distribution of the sum of two freely independent non-commutative random variables, having distributions μ_1, μ_2 respectively. The (additive) free convolution may also be described in terms of the Stieltjes transform: Let μ be a probability measure on $\mathbb{R}$, then we define the Stieltjes transform of μ by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z}\,, \qquad z \in \mathbb{C}^+\,. \tag{2.1}$$

Note that $m_\mu(z)$ is an analytic function in the upper half plane, satisfying $\lim_{y\to\infty} \mathrm{i}y m_\mu(\mathrm{i}y) = 1$. As shown in [31, 6], the free convolution has the following property: Denote by m_{μ_1}, m_{μ_2}, $m_{\mu_1\boxplus\mu_2}$, the Stieltjes transforms of μ_1, μ_2, $\mu_1 \boxplus \mu_2$, respectively. Then there exist two analytic functions ω_1, ω_2, from $\mathbb{C}^+$ to $\mathbb{C}^+$, satisfying $\lim_{y\to\infty} \omega_i(\mathrm{i}y)/\mathrm{i}y = 1$, $(i = 1, 2)$, such that

$$\begin{aligned} m_{\mu_1\boxplus\mu_2}(z) &= m_{\mu_1}(\omega_1(z)) = m_{\mu_2}(\omega_2(z))\,, \\ \omega_1(z) + \omega_2(z) &= z - \frac{1}{m_{\mu_1\boxplus\mu_2}(z)}\,, \end{aligned} \tag{2.2}$$

for $z \in \mathbb{C}^+$. The functions ω_i are referred to as subordination functions. Note that (2.2) also shows that $\mu_1 \boxplus \mu_2 = \mu_2 \boxplus \mu_1$. It was pointed out in [4] that the system (2.2) may be used as an alternative definition of the free convolution. In particular, given μ_1, μ_2, the system (2.2) has a unique solution $(m_{\mu_1\boxplus\mu_2}, \omega_1, \omega_2)$.

In case we choose the measure μ_2 as the standard semicircular law $\mathrm{d}\mu_{sc}(E) = \frac{1}{2\pi}\sqrt{(4-E^2)_+}\mathrm{d}E$. A simple computation reveals that the Stieltjes transform $m_{\mu_{sc}} \equiv m_{sc}$ satisfies

$$m_{sc}(z) = -\frac{1}{z + m_{sc}(z)}\,, \qquad z \in \mathbb{C}^+\,.$$

Using this information, we can reduce the system (2.2), to the self-consistent equation

$$m_{fc}(z) = \int \frac{\mathrm{d}\mu(x)}{x - z - m_{fc}(z)}\,, \qquad z \in \mathbb{C}^+\,, \tag{2.3}$$

with $\lim_{y\to\infty} \mathrm{i}y\, m_{fc}(\mathrm{i}y) = 1$, where we have abbreviated $\mu \equiv \mu_1$. Equation (2.3) is often called the *Pastur relation*. A slightly modified version of the functional equation (2.3) is the starting point of the analysis in [24] and also of the present paper.

The (unique) solution of (2.3) has first been studied in details in [5]. In particular, it has been shown that $\limsup_{\eta\searrow 0} |\mathrm{Im}\, m_{fc}(E+\mathrm{i}\eta)| < \infty$, $E \in \mathbb{R}$, and hence the free convolution measure $\mu_{fc} \equiv \mu \boxplus \mu_{sc}$ is absolutely continuous (for simplicity we denote the density also with μ_{fc}) and we conclude from the Stieltjes inversion formula that

$$\mu_{fc}(E) = \lim_{\eta\searrow 0} \mathrm{Im}\, m_{fc}(E + \mathrm{i}\eta)\,, \qquad E \in \mathbb{R}\,.$$

Moreover, it was shown in [5] that the density μ_{fc} is analytic in the interior of the support of μ_{fc}. We refer to, e.g., [3] for further results on the regularity of the free convolution.

2.2 Notations and Conventions

To state our main results, we need some more notations and conventions. For high probability estimates we use two parameters $\xi \equiv \xi_N$ and $\varphi \equiv \varphi_N$: We assume that

$$a_0 < \xi \leq A_0 \log\log N\,, \qquad \varphi = (\log N)^C\,, \tag{2.4}$$

for some fixed constants $a_0 > 2$, $A_0 \geq 10$, $C \geq 1$. They only depend on θ and C_0 in (2.5) and will be kept fixed in the following.

Definition 2.1. We say an event Ω has (ξ, ν)-high probability, if

$$\mathbb{P}(\Omega^c) \leq \mathrm{e}^{-\nu(\log N)^\xi}\,,$$

for N sufficiently large.

Similarly, for a given event Ω_0 we say an event Ω holds with (ξ, ν)-high probability on Ω_0, if

$$\mathbb{P}(\Omega_0 \cap \Omega^c) \leq \mathrm{e}^{-\nu(\log N)^\xi}\,,$$

for N sufficiently large.

For brevity, we occasionally say an event holds with high probability, when we mean (ξ, ν)-high probability. We do not keep track of the explicit value of ν in the following, allowing ν to decrease from line to line such that $\nu > 0$. From our proof it becomes apparent that such reductions occur only finitely many times.

We define the resolvent, or *Green function*, $G(z)$, and the averaged Green function, $m(z)$, of H by

$$G(z) = (G_{ij}(z)) := \frac{1}{\lambda V + W - z}\,, \qquad m(z) := \frac{1}{N}\operatorname{Tr} G(z)\,, \qquad z \in \mathbb{C}^+\,.$$

Frequently, we abbreviate $G \equiv G(z)$, $m \equiv m(z)$, etc.

We use the symbols $\mathcal{O}(\,\cdot\,)$ and $o(\,\cdot\,)$ for the standard big-O and little-o notation. The notations $\mathcal{O}$, o, $\ll$, $\gg$, always refer to the limit $N \to \infty$. Here $a \ll b$ means $a = o(b)$. We use c and C to denote positive constants that do not depend on N, usually with the convention $c \leq C$. Their value may change from line to line. Finally, we write $a \sim b$, if there is $C \geq 1$ such that $C^{-1}|b| \leq |a| \leq C|b|$, and, occasionally, we write for N-dependent quantities $a_N \lesssim b_N$, if there exist constants $C, c > 0$ such that $|a_N| \leq C(\varphi_N)^{c\xi}|b_N|$.

2.3 Assumptions

We define the model (1.1) in details and list our main assumptions.

Let W be an $N \times N$ random matrix, whose entries, (w_{ij}), are independent, up to the symmetry constraint $w_{ij} = \overline{w}_{ji}$, centered, complex random variables with variance N^{-1} and subexponential decay, i.e.,

$$\mathbb{P}\left(\sqrt{N}|w_{ij}| > x\right) \leq C_0\,\mathrm{e}^{-x^{1/\theta}}\,, \tag{2.5}$$

for some positive constants C_0 and $\theta > 1$. In particular,

$$\mathbb{E}w_{ij} = 0\,, \qquad \mathbb{E}|w_{ij}|^p \leq C\frac{(\theta p)^{\theta p}}{N^{p/2}} \qquad (p \geq 3)\,, \tag{2.6}$$

and,

$$\mathbb{E}w_{ii}^2 = \frac{1}{N}\,, \qquad \mathbb{E}|w_{ij}|^2 = \frac{1}{N}\,, \qquad \mathbb{E}w_{ij}^2 = 0 \qquad (i \neq j)\,. \tag{2.7}$$

Remark 2.2. We remark that all our methods also apply to *symmetric* Wigner matrices, i.e., when (w_{ij}) are centered, real random variables with variance N^{-1}, with subexponential decay. In this case, (2.7) gets replaced by

$$\mathbb{E}w_{ii}^2 = \frac{2}{N}\,, \qquad \mathbb{E}w_{ij}^2 = \frac{1}{N} \qquad (i \neq j)\,. \tag{2.8}$$

Let V be an $N \times N$ diagonal random matrix, whose entries (v_i) are real, centered, i.i.d. random variables, independent of $W = (w_{ij})$, with law μ. More assumptions on μ will be stated below. Without loss of generality, we assume that the entries of V are ordered,

$$v_1 \geq v_2 \geq \cdots \geq v_N. \tag{2.9}$$

For $\lambda \in \mathbb{R}$, we consider the random matrix

$$H = (h_{ij}) := \lambda V + W\,. \tag{2.10}$$

We choose for simplicity μ as a Jacobi measure, i.e., μ is described in terms of its density

$$\mu(v) = Z^{-1}(1+v)^{\mathfrak{a}}(1-v)^{\mathfrak{b}}d(v)\mathbb{1}_{[-1,1]}(v)\,, \tag{2.11}$$

where $\mathfrak{a}, \mathfrak{b} > -1$, $d \in C^1([-1,1])$ such that $d(v) > 0$, $v \in [-1,1]$, and Z is an appropriately chosen normalization constant such that μ is a probability measure. We will assume, for simplicity of the arguments, that μ is centered, but this condition can easily be relaxed. We remark that the measure μ has support $[-1,1]$, but we observe that varying λ is equivalent to changing the support of μ. Since μ is absolutely continuous, we may assume that (2.9) holds with strict inequalities. Finally, since we assume that μ is centered, we may choose $\lambda \geq 0$ in the following.

We remark that, as one can see from (2.5),

$$|w_{ij}| \leq \frac{(\varphi_N)^{\xi}}{\sqrt{N}}\,, \tag{2.12}$$

with (ξ,ν)-high probability, whereas $v_i \in [-1,1]$, almost surely.

3 Results

In this section we state our main results.

Since we choose the measure μ to be centered, we may assume that $\lambda \geq 0$, without loss of generality in the following. Fix some $\lambda_0 > 0$, then we assume that the perturbation parameter λ is in the domain

$$\mathcal{D}_{\lambda_0} := \{\lambda \in \mathbb{R}^+ \,:\, |\lambda| \leq \lambda_0\}\,.$$

We define the spectral parameter $z = E + \mathrm{i}\eta$, with $E \in \mathbb{R}$ and $\eta > 0$. Let $E_0 \geq 3 + \lambda_0$ and define the domain

$$\mathcal{D}_L := \{z = E + \mathrm{i}\eta \in \mathbb{C} \,:\, |E| \leq E_0\,, (\varphi_N)^L \leq N\eta \leq 3N\}\,, \tag{3.1}$$

with $L \equiv L(N)$, such that $L \geq 12\xi$. Here, we chose E_0 bigger than $3 + \lambda$, since we know that the spectrum of W lies in the set $\{E \in \mathbb{R} \,:\, |E| \leq 3\}$ with high probability. Thus spectral perturbation theory implies that the spectrum of H is contained in $\{E \in \mathbb{R} \,:\, |E| \leq 3 + \lambda\}$, with high probability. Recall the definition of κ_E, the distance to the endpoints of the support of μ_{fc}. In the following, we often abbreviate $\kappa \equiv \kappa_E$.

3.1 Delocalization regime

The first theorem shows that a modified local semicircle law, which we will also call a deformed local law, holds when μ_{fc} exhibits a square root behaviour.

Theorem 3.1. [Strong local law] *Assume that the limiting distribution μ_{fc} for H in* (2.10) *exhibits a square root behaviour at the both edges of the spectrum. Let*

$$\xi = \frac{A_0 + o(1)}{2} \log \log N\,. \tag{3.2}$$

Then there are constants $\nu > 0$ and c_1, depending on the constants A_0, E_0, λ_0, θ, C_0 in (2.5) *and the measure μ, such that for $L \geq 40\xi$, the events*

$$\bigcap_{\substack{z\in\mathcal{D}_L\\ \lambda\in\mathcal{D}_{\lambda_0}}} \left\{ |m(z) - m_{fc}(z)| \leq (\varphi_N)^{c_1\xi} \left(\min\left\{ \frac{\lambda^{1/2}}{N^{1/4}}, \frac{\lambda}{\sqrt{\kappa+\eta}}\frac{1}{\sqrt{N}} \right\} + \frac{1}{N\eta} \right) \right\} \tag{3.3}$$

and

$$\bigcap_{\substack{z\in\mathcal{D}_L\\ \lambda\in\mathcal{D}_{\lambda_0}}} \left\{ \max_{i\neq j} |G_{ij}| \leq (\varphi_N)^{c_1\xi} \left(\sqrt{\frac{\operatorname{Im} m_{fc}(z)}{N\eta}} + \frac{1}{N\eta} \right) \right\} \tag{3.4}$$

both have (ξ,ν)-high probability.

For $\lambda = 0$, we have $m_{fc} = m_{sc}$, where m_{sc} is the Stieltjes transform of the standard semicircle law. In this case stronger estimates have been obtained; see, e.g., [9]. Roughly speaking, in this situation we have the high probability bounds

$$|m(z) - m_{sc}(z)| \lesssim \frac{1}{N\eta} \qquad \text{and} \qquad |G_{ij}(z) - \delta_{ij} m(z)| \lesssim \sqrt{\frac{\operatorname{Im} m_{sc}(z)}{N\eta}} + \frac{1}{N\eta}\,, \tag{3.5}$$

(up to logarithmic corrections), within the range of admitted parameters.

This suggests that the bound on $G_{ij}(z)$, $(i \neq j)$, in (3.4) is optimal. However, for $\lambda \neq 0$, the individual diagonal resolvent entries $G_{ii}(z)$ do not concentrate around their mean $m(z)$, due to the fluctuations in the random variables (v_i). This becomes apparent from Schur's complement formula and one easily establishes that $|G_{ii}(z) - m(z)| = \mathcal{O}(\lambda) + o(1)$, with high probability.

Comparing the estimate on $m - m_{fc}$ in (3.3) with the corresponding estimate in (3.5), one may suspect that the leading correction terms in (3.3) stem from fluctuations of the random variables (v_i). The next theorem asserts that this is indeed true, at least in the bulk of the spectrum: There are random variables $\zeta_0 \equiv \zeta_0^N(z)$, which depend on the random variables (v_i), but are independent of the random variables (w_{ij}), such that $|m(z) - m_{fc}(z) - \zeta_0(z)| \lesssim (N\eta)^{-1}$ with high probability in the bulk of the spectrum. Concerning the spectral edge, we remark that the estimate in (3.3) is optimal for $\lambda \ll N^{-1/6}$, but it is not known whether $\lambda^{1/2}N^{-1/4}$ is the optimal rate for $\lambda \gg N^{-1/6}$.

Next, let $\mu_1 \geq \cdots \geq \mu_N$ denote the eigenvalues of $H = \lambda V + W$, and let $u_1, \cdots, u_N$ denote the associated eigenvectors. We use the notation $u_\alpha = (u_\alpha(i))_{i=1}^N$ for the vector components. All eigenvectors are ℓ^2-normalized. The next theorem asserts that, with high probability, all eigenvectors of $H = \lambda V + W$ are completely delocalized:

Theorem 3.2. [Eigenvector delocalization] *Assume that the limiting distribution μ_{fc} for H in* (2.10) *exhibits a square root behaviour at the both edges of the spectrum. Then there is a constant $\nu > 0$, depending on A_0, E_0, λ_0, θ and C_0 in* (2.5) *and the measure μ, such that for any ξ satisfying* (2.4)*, we have*

$$\max_{1\leq\alpha\leq N} \max_{1\leq i\leq N} |u_\alpha(i)| \leq \frac{(\varphi_N)^{4\xi}}{\sqrt{N}}\,,$$

with (ξ,ν)-high probability.

Remark 3.3. In case the entries of $V = (v_i)$ are independent Gaussian random variables, the situation is more subtle: For any finite E_0, there exists a constant c_{E_0}, independent of N, and a constant ν, depending on A_0, E_0, θ and C_0 in (2.5), such that for any ξ satisfying (2.4),

$$\max_{1\le i\le N} |u_\alpha(i)| \le c_{E_0} \frac{(\varphi_N)^{4\xi}}{\sqrt{N}}\,, \tag{3.6}$$

with (ξ,ν)-high probability. However, $c_{E_0} \to \infty$ and $\nu \to 0$, as $E_0 \to \infty$.

In the delocalized regime, we can find a Gaussian fluctuation of the largest eigenvalue, which is explained in the following theorem.

Theorem 3.4. *Let μ be a centered Jacobi measure defined in* (2.11) *with $\mathfrak{b} > 1$. Let* $\operatorname{supp}\mu_{fc} = [\hat{L}_-, \hat{L}_+]$, *where $\hat{L}_-$ and $\hat{L}_+$ are random variables depending on (v_i). Then, if $\lambda < \lambda_+$, the rescaled fluctuation $N^{1/2}(\hat{L}_+ - L_+)$ converges to a Gaussian random variable with mean* 0 *and variance* $(1 - [m_{fc}(L_+)]^2)$ *in distribution, as $N \to \infty$.*

Remark 3.5. When $\mathrm{a} > 1$, the analogous statement to Theorem 3.4 holds at the lower edge.

For the proof of Theorem 3.4, see Appendix.

3.2 Localization regime

The first result of this subsection shows that the locations of the extreme eigenvalues are given by the order statistics of the diagonal elements.

Theorem 3.6. *Let n_0 be a fixed constant independent of N. Let μ_k be the k-th largest eigenvalue of $H = \lambda V + W$, where $1 \le k < n_0$. Fix some $\lambda > \lambda_+$. Then, the joint distribution function of the k largest rescaled eigenvalues*

$$\mathbb{P}\left(N^{1/(\mathfrak{b}+1)}(L_+ - \mu_1) \le s_1,\, N^{1/(\mathfrak{b}+1)}(L_+ - \mu_2) \le s_2,\, \cdots,\, N^{1/(\mathfrak{b}+1)}(L_+ - \mu_k) \le s_k\right)\,, \tag{3.7}$$

converges to the joint distribution function of the k largest rescaled order statistics,

$$\mathbb{P}\left(C_\lambda N^{1/(\mathfrak{b}+1)}(1 - v_1) \le s_1,\, C_\lambda N^{1/(\mathfrak{b}+1)}(1 - v_2) \le s_2,\, \cdots,\, C_\lambda N^{1/(\mathfrak{b}+1)}(1 - v_k) \le s_k\right)\,, \tag{3.8}$$

as $N \to \infty$, where $C_\lambda = \frac{\lambda^2 - \lambda_+^2}{\lambda}$. In particular, the cumulative distribution function of the rescaled largest eigenvalue $N^{1/(\mathfrak{b}+1)}(L_+ - \mu_1)$ converges to the Weibull distribution

$$G_{\mathfrak{b}+1}(z) := C_\mu s^{\mathfrak{b}} \exp\left(-\frac{C_\mu s^{\mathfrak{b}+1}}{(\mathfrak{b}+1)}\right)\,, \tag{3.9}$$

where

$$C_\mu := \left(\frac{\lambda}{\lambda^2 - \lambda_+^2}\right)^{\mathfrak{b}+1} \lim_{v\to 1} \frac{\mu(v)}{(1-v)^{\mathfrak{b}}}\,.$$

The second result in this subsection asserts that the eigenvectors associated with the extreme eigenvalues are 'partially localized'. We denote by $(u_k(j))_{j=1}^N$ the component of the eigenvector u_k associated to the eigenvalue μ_k. All eigenvectors are normalized as $\sum_{j=1}^N |u_k(j)|^2 = \|u_k\|_2^2 = 1$.

Theorem 3.7. *Let n_0 be a fixed constant independent of N. Let μ_k be the k-th largest eigenvalue of $H = \lambda V + W$ and $u_k(j)$ the j-th component of the associated (normalized) eigenvector, where $k \in [\![1, n_0 - 1]\!]$. Fix $\lambda > \lambda_+$. Then, there exist constants $\delta, \delta', \sigma > 0$, such*

$$\mathbb{P}\left(\left||u_k(k)|^2 - \frac{\lambda^2 - \lambda_+^2}{\lambda^2}\right| \ge N^{\delta}\right) \le N^{-\sigma}\,. \tag{3.10}$$

and, for any $j \neq k$,

$$\mathbb{P}\left(|u_k(j)|^2 > \frac{N^{\delta'}}{N}\frac{1}{\lambda^2|v_k - v_j|^2})\right) \leq N^{-\sigma}\,. \tag{3.11}$$

Remark 3.8. In [21], it was proved that all eigenvectors are completely delocalized when $\lambda < \lambda_+$. This also shows a sharp transition from the partial localization to the complete delocalization. Following the proof in [21], we can prove that the eigenvectors are completely delocalized in the bulk even when $\lambda > \lambda_+$.

Remark 3.9. Theorems 3.6 and 3.7 remain valid for deterministic potentials V, provided the entires (v_i) satisfy some suitable assumptions.

Remark 3.10. From (3.10), we find that, for $k \in [\![1, n_0 - 1]\!]$,

$$\sum_{j:j\neq k}^{N} |u_k(j)|^2 = \frac{\lambda_+^2}{\lambda^2} + o(1)\,,$$

which is in accordance with the fact that (3.11) holds and that, typically,

$$\frac{1}{N}\sum_{j:j\neq k}^{N} \frac{1}{\lambda^2|v_k - v_j|^2} = \frac{\lambda_+^2}{\lambda^2} + o(1)\,,$$

where we used (3.8). Considering, on a formal level, W as a perturbation of λV, Rayleigh-Schrödinger perturbation theory predicts that

$$|u_k(j)|^2 \simeq \frac{1}{N\lambda^2|v_k - v_j|^2}\,, \qquad (k \neq j)\,.$$

It might be possible to justify some of our results using asymptotic perturbation theory.

In the next section, we introduce the main steps of the proof of Theorem 3.6. Proofs of other theorems in this section, as well as the detailed proof of Theorem 3.6, can be found in [21, 22].

4 Proof of Theorem 3.6

In this section, we outline the proof of Theorem 3.6. We first fix the diagonal random entries (v_i) and consider $\hat{\mu}_{fc}$, the deformed semicircle measure with fixed (v_i). The main tools we use in the proof are Lemma 4.2, where we obtain a linear approximation of m_{fc}, and Lemma 4.5, which estimates the difference between m_{fc} and $\hat{m}_{fc}$, the Stieltjes transform of $\hat{\mu}_{fc}$. Using Proposition 4.6 that estimates the eigenvalue locations in terms of $\hat{m}_{fc}$, we prove Theorem 3.6.

4.1 Definition of Ω_V

In this subsection we define an event Ω_V, on which the random variables (v_i) exhibit 'typical' behaviour. For this purpose we need some more notation:

Define the domain, $\mathcal{D}_\epsilon$, of the spectral parameter z by

$$\mathcal{D}_\epsilon := \{z = E + \mathrm{i}\eta \in \mathbb{C}^+ \,:\, -3-\lambda \leq E \leq 3+\lambda,\; N^{-1/2-\epsilon} \leq \eta \leq N^{-1/(\mathfrak{b}+1)+\epsilon}\}\,. \tag{4.1}$$

Using spectral perturbation theory, we find that the following a priori bound

$$|\mu_k| \leq \|H\| \leq \|W\| + \lambda\|V\| \leq 2 + \lambda + (\varphi_N)^{c\xi}N^{-2/3}\,, \tag{4.2}$$

holds with high probability; see, e.g., Theorem 2.1. in [16].

Further, denote by $\mathfrak{b}$ the constant

$$\mathfrak{b} := \frac{1}{2} - \frac{1}{\mathfrak{b}+1} = \frac{\mathfrak{b}-1}{2(\mathfrak{b}+1)} = \frac{\mathfrak{b}}{\mathfrak{b}+1} - \frac{1}{2}\,, \tag{4.3}$$

which only depends on $\mathfrak{b}$. Fix a sufficiently small $\epsilon > 0$ satisfying

$$\epsilon < \left(10 + \frac{\mathfrak{b}+1}{\mathfrak{b}-1}\right)\mathfrak{b}\,. \tag{4.4}$$

Finally, we define N-dependent constants κ_0 and η_0 as

$$\kappa_0 := N^{-1/(\mathfrak{b}+1)}, \qquad \eta_0 := \frac{N^{-\epsilon}}{\sqrt{N}}\,. \tag{4.5}$$

In most cases, the point $z = L_+ - \kappa + \mathrm{i}\eta$ we consider will satisfy $\kappa \lesssim \kappa_0$ and $\eta \geq \eta_0$.

Now, we are ready to give a definition of the 'good' event Ω_V:

Definition 4.1. Let $n_0 > 10$ be a fixed positive integer independent of N. We define Ω_V to be the event on which the following conditions hold for any $k \in [\![1, n_0 - 1]\!]$:

1. The k-th largest random variable v_k satisfies, for all $j \in [\![1, N]\!]$ with $j \neq k$,

$$N^{-\epsilon}\kappa_0 < |v_j - v_k| < (\log N)\kappa_0\,. \tag{4.6}$$

In addition, for $k = 1$, we have

$$N^{-\epsilon}\kappa_0 < |1 - v_1| < (\log N)\kappa_0\,. \tag{4.7}$$

2. There exists a constant c independent of N such that, for any $z \in \mathcal{D}_\epsilon$ satisfying

$$\min_{i \in [\![1,N]\!]} |\mathrm{Re}\,(z + m_{fc}(z)) - \lambda v_i| = |\mathrm{Re}\,(z + m_{fc}(z)) - \lambda v_k|\,, \tag{4.8}$$

we have

$$\frac{1}{N}\sum_{i:i\neq k}^{N} \frac{1}{|\lambda v_i - z - m_{fc}(z)|^2} < c < 1\,. \tag{4.9}$$

We remark that, together with (4.6) and (4.7), (4.8) implies

$$|\mathrm{Re}\,(z + m_{fc}(z)) - \lambda v_i| > \frac{N^{-\epsilon}\kappa_0}{2}\,, \tag{4.10}$$

for all $i \neq k$.

3. There exists a constant $C > 0$ such that, for any $z \in \mathcal{D}_\epsilon$, we have

$$\left|\frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda v_i - z - m_{fc}(z)} - \int \frac{\mathrm{d}\mu(v)}{\lambda v - z - m_{fc}(z)}\right| \leq \frac{CN^{3\epsilon/2}}{\sqrt{N}}\,. \tag{4.11}$$

It can be checked that

$$\mathbb{P}(\Omega_V) \geq 1 - C(\log N)^{1+2\mathfrak{b}}N^{-\epsilon}, \tag{4.12}$$

thus $(\Omega_V)^c$ is indeed a rare event. See Appendix I of [22] for more detail.

4.2 Definition of $\hat{m}_{fc}$

Recall that we assume that $v_1 > v_2 > \cdots > v_N$. We will mainly focus on the case where Ω_V holds, i.e., (v_i) are fixed and satisfy the conditions in Definition 4.1. Under such consideration, we let $\hat{\mu}$ be the empirical measure defined by

$$\hat{\mu} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda v_i} \tag{4.13}$$

and we set $\hat{\mu}_{fc} := \hat{\mu} \boxplus \mu_{sc}$, i.e., $\hat{\mu}_{fc}$ is the free convolution measure of the empirical measure $\hat{\mu}$ and the semicircular measure μ_{sc}. As in the case of m_{fc}, the Stieltjes transform $\hat{m}_{fc}$ of the measure $\hat{\mu}_{fc}$ is a solution to the equation

$$\hat{m}_{fc}(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda v_i - z - \hat{m}_{fc}(z)}\,, \qquad \operatorname{Im} \hat{m}_{fc}(z) \geq 0\,, \quad z \in \mathbb{C}^+\,. \tag{4.14}$$

We are going to show that $m_{fc}(z)$ is a good approximation of $\hat{m}_{fc}(z)$ on Ω_V for z in some subset of $\mathcal{D}_\epsilon$.

4.3 Properties of m_{fc} and $\hat{m}_{fc}$

Recall the definitions of m_{fc} and $\hat{m}_{fc}$. Let

$$R_2(z) := \int \frac{\mathrm{d}\mu(v)}{|\lambda v - z - m_{fc}(z)|^2}, \quad \hat{R}_2(z) := \frac{1}{N}\sum_{i=1}^{N} \frac{1}{|\lambda v_i - z - \hat{m}_{fc}(z)|^2}\,, \qquad z \in \mathbb{C}^+\,. \tag{4.15}$$

Since

$$\operatorname{Im} m_{fc}(z) = \int \frac{\operatorname{Im} z + \operatorname{Im} m_{fc}(z)}{|\lambda v - z - m_{fc}(z)|^2} \mathrm{d}\mu(v)\,,$$

we have that

$$R_2(z) = \frac{\operatorname{Im} m_{fc}(z)}{\operatorname{Im} z + \operatorname{Im} m_{fc}(z)} < 1\,.$$

Similarly, we also find that $\hat{R}_2(z) < 1$.

The following lemma shows that m_{fc} is approximately a linear function near the spectral edge.

Lemma 4.2. *Let $z = L_+ - \kappa + \mathrm{i}\eta \in \mathcal{D}_\epsilon$. Then,*

$$z + m_{fc}(z) = \lambda - \frac{\lambda^2}{\lambda^2 - \lambda_+^2}(L_+ - z) + \mathcal{O}\left((\log N)(\kappa+\eta)^{\min\{\mathfrak{b},2\}}\right)\,. \tag{4.16}$$

Similarly, if $z, z' \in \mathcal{D}_\epsilon$, then

$$m_{fc}(z) - m_{fc}(z') = \frac{\lambda_+^2}{\lambda^2 - \lambda_+^2}(z - z') + \mathcal{O}\left((\log N)^2 (N^{-1/(\mathfrak{b}+1)})^{\min\{\mathfrak{b}-1,1\}}|z - z'|\right)\,. \tag{4.17}$$

Proof. We only prove the first part of the lemma; the second part can be proved analogously. Since $L_+ + m_{fc}(L_+) = \lambda$, we can write

$$\begin{aligned} m_{fc}(z) - m_{fc}(L_+) &= \int \frac{\mathrm{d}\mu(v)}{\lambda v - z - m_{fc}(z)} - \int \frac{\mathrm{d}\mu(v)}{\lambda v - L_+ - m_{fc}(L_+)} \\ &= \int \frac{m_{fc}(z) - m_{fc}(L_+) + (z - L_+)}{(\lambda v - z - m_{fc}(z))(\lambda v - \lambda)} \mathrm{d}\mu(v)\,. \end{aligned} \tag{4.18}$$

If we let

$$T(z) := \int \frac{\mathrm{d}\mu(v)}{(\lambda v - z - m_{fc}(z))(\lambda v - \lambda)}\,, \tag{4.19}$$

we find

$$|T(z)| \le \left(\int \frac{\mathrm{d}\mu(v)}{|\lambda v - z - m_{fc}(z)|^2}\right)^{1/2} \left(\int \frac{\mathrm{d}\mu(v)}{|\lambda v - \lambda|^2}\right)^{1/2} \le \sqrt{R_2(z)}\frac{\lambda_+}{\lambda} < \frac{\lambda_+}{\lambda} < 1\,.$$

Hence, for $z \in \mathcal{D}_\epsilon$, we have

$$m_{fc}(z) - m_{fc}(L_+) = \frac{T(z)}{1 - T(z)}(z - L_+)\,, \tag{4.20}$$

which shows that

$$z + m_{fc}(z) = \lambda - \frac{1}{1 - T(z)}(L_+ - z)\,. \tag{4.21}$$

We also obtain from (4.21) that

$$|z + m_{fc}(z) - \lambda| \le \frac{\lambda}{\lambda - \lambda_+}|L_+ - z|\,.$$

We now estimate $T(z)$. Let $\tau = z + m_{fc}(z)$. We have

$$T(z) - \frac{\lambda_+^2}{\lambda^2} = \int \frac{\mathrm{d}\mu(v)}{(\lambda v - \tau)(\lambda v - \lambda)} - \int \frac{\mathrm{d}\mu(v)}{(\lambda v - \lambda)^2} = (\tau - \lambda)\int \frac{\mathrm{d}\mu(v)}{(\lambda v - \tau)(\lambda v - \lambda)^2}. \tag{4.22}$$

In order to find an upper bound on the integral on the very right side, we consider the following cases:

1. When $\mathfrak{b} \ge 2$, we have

$$\left|\int \frac{\mathrm{d}\mu(v)}{(\lambda v - \tau)(\lambda v - \lambda)^2}\right| \le C\int_{-1}^{1} \frac{\mathrm{d}v}{|\lambda v - \tau|} \le C\log N\,. \tag{4.23}$$

2. When $\mathfrak{b} < 2$, define a set $A \subset [-1, 1]$ by

$$A := \{v \in [-1, 1] : \lambda v < -\lambda + 2\,\mathrm{Re}\,\tau\}\,,$$

and $B := [-1, 1]\backslash A$. Estimating the integral in (4.22) on A we find

$$\left|\int_A \frac{\mathrm{d}\mu(v)}{(\lambda v - \tau)(\lambda v - \lambda)^2}\right| \le C\int_A \frac{\mathrm{d}\mu(v)}{|\lambda v - \lambda|^3} \le C|\lambda - \tau|^{\mathfrak{b}-2}\,, \tag{4.24}$$

where we have used that, for $v \in A$,

$$|\lambda v - \tau| > |\mathrm{Re}\,\tau - \lambda v| > \frac{1}{2}(\lambda - \lambda v)\,.$$

On the set B, we have

$$\left|\int_B \frac{\mathrm{d}\mu(v)}{(\lambda v - \tau)(\lambda v - \lambda)}\right| \le C\int_B \frac{|\lambda - \lambda v|^{\mathfrak{b}-1}}{|\lambda v - \tau|}\mathrm{d}v \le C|\lambda - \tau|^{\mathfrak{b}-1}\log N\,, \tag{4.25}$$

where we have used that, for $v \in B$,

$$|\lambda - \lambda v| \le 2(\lambda - \mathrm{Re}\,\tau) \le 2|\lambda - \tau|\,.$$

We also have

$$\left|\int_B \frac{\mathrm{d}\mu(v)}{(\lambda v-\lambda)^2}\right| \le C\int_B |\lambda v-\lambda|^{\mathfrak{b}-2}\mathrm{d}v \le C|\lambda-\tau|^{\mathfrak{b}-1}\,. \tag{4.26}$$

Thus, we obtain from (4.22), (4.25) and (4.26) that

$$\left|\int \frac{\mathrm{d}\mu(v)}{(\lambda v-\tau)(\lambda v-\lambda)^2}\right| \le C|\lambda-\tau|^{\mathfrak{b}-2}\log N\,. \tag{4.27}$$

We thus have proved that

$$T(z)=\frac{\lambda_+^2}{\lambda^2}+\mathcal{O}((\log N)|L_+-z|^{\min\{\mathfrak{b}-1,1\}})\,, \tag{4.28}$$

which, combined with (4.21), proves the desired lemma. □

Remark 4.3. Choosing in Lemma 4.2 $z_k=L_+-\kappa_k+\mathrm{i}\eta\in\mathcal{D}_\epsilon$ with

$$\kappa_k=\frac{\lambda^2-\lambda_+^2}{\lambda}(1-v_k)$$

we obtain

$$z_k+m_{fc}(z_k)=\lambda v_k+\frac{\lambda^2}{\lambda^2-\lambda_+^2}\eta+\mathcal{O}((\log N)N^{-\min\{\mathfrak{b},2\}/(\mathfrak{b}+1)+2\epsilon})\,. \tag{4.29}$$

To estimate $|\hat{m}_{fc}-m_{fc}|$, we consider the following subset of $\mathcal{D}_\epsilon$:

Definition 4.4. Let $A:=[\![n_0,N]\!]$. We define the domain $\mathcal{D}'_\epsilon$ of the spectral parameter z as

$$\mathcal{D}'_\epsilon=\left\{z\in\mathcal{D}_\epsilon\;:\;|\lambda v_a-z-m_{fc}(z)|>\frac{1}{2}N^{-1/(\mathfrak{b}+1)-\epsilon},\;\forall a\in A\right\}\,. \tag{4.30}$$

Eventually, we will show that $\mu_k+\mathrm{i}\eta_0\in\mathcal{D}'_\epsilon$, $k\in[\![1,n_0-1]\!]$, with high probability on Ω_V; see remark 4.7. We now prove an a priori bound on the difference $|\hat{m}_{fc}-m_{fc}|$ on $\mathcal{D}'_\epsilon$.

Lemma 4.5. *For any $z\in\mathcal{D}'_\epsilon$, we have on Ω_V that*

$$|m_{fc}(z)-\hat{m}_{fc}(z)|\le\frac{N^{2\epsilon}}{\sqrt{N}}\,. \tag{4.31}$$

Proof. Assume that Ω_V holds. For given $z\in\mathcal{D}'_\epsilon$, choose $k\in[\![1,n_0-1]\!]$ satisfying (4.8), i.e., among (λv_i), λv_k is closest to $\mathrm{Re}\,(z+m_{fc}(z))$. Suppose that (4.31) does not hold. By definition, we obtain the following self-consistent equation for $(\hat{m}_{fc}-m_{fc})$:

$$\begin{aligned}\hat{m}_{fc}-m_{fc}&=\frac{1}{N}\sum_{i=1}^N\left(\frac{1}{\lambda v_i-z-\hat{m}_{fc}}-m_{fc}\right)\\&=\frac{1}{N}\sum_{i=1}^N\left(\frac{1}{\lambda v_i-z-\hat{m}_{fc}}-\frac{1}{\lambda v_i-z-m_{fc}}\right)+\left(\frac{1}{N}\sum_{i=1}^N\frac{1}{\lambda v_i-z-m_{fc}}-\int\frac{\mathrm{d}\mu(v)}{\lambda v-z-m_{fc}}\right)\\&=\frac{1}{N}\sum_{i=1}^N\frac{\hat{m}_{fc}-m_{fc}}{(\lambda v_i-z-\hat{m}_{fc})(\lambda v_i-z-m_{fc})}+\left(\frac{1}{N}\sum_{i=1}^N\frac{1}{\lambda v_i-z-m_{fc}}-\int\frac{\mathrm{d}\mu(v)}{\lambda v-z-m_{fc}}\right)\,.\end{aligned} \tag{4.32}$$

From the assumption (4.11), we find that the second term in the right hand side of (4.32) is bounded by $N^{-1/2+3\epsilon/2}$.

We want to estimate the first term in the right hand side of (4.32). For $i = k$, we have

$$|\lambda v_k - z - \hat{m}_{fc}| + |\lambda v_k - z - m_{fc}| \geq |\hat{m}_{fc}(z) - m_{fc}(z)| > \frac{N^{2\epsilon}}{\sqrt{N}},$$

which shows that either

$$|\lambda v_k - z - \hat{m}_{fc}| \geq \frac{N^{2\epsilon}}{2\sqrt{N}}, \quad \text{or} \quad |\lambda v_k - z - m_{fc}| \geq \frac{N^{2\epsilon}}{2\sqrt{N}}.$$

In either case, by considering the imaginary part, we find

$$\frac{1}{N}\left|\frac{1}{(\lambda v_k - z - \hat{m}_{fc})(\lambda v_k - z - m_{fc})}\right| \leq \frac{1}{N}\frac{2\sqrt{N}}{N^{2\epsilon}}\frac{1}{\eta} \leq CN^{-\epsilon}, \qquad z \in \mathcal{D}'_\epsilon.$$

For the other terms, we use

$$\frac{1}{N}\left|\sum_i^{(k)} \frac{1}{(\lambda v_i - z - \hat{m}_{fc})(\lambda v_i - z - m_{fc})}\right| \leq \frac{1}{2N}\sum_i^{(k)}\left(\frac{1}{|\lambda v_i - z - \hat{m}_{fc}|^2} + \frac{1}{|\lambda v_i - z - m_{fc}|^2}\right). \tag{4.33}$$

From (4.14), we have that

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{|\lambda v_i - z - \hat{m}_{fc}|^2} = \frac{\operatorname{Im}\hat{m}_{fc}}{\eta + \operatorname{Im}\hat{m}_{fc}} < 1. \tag{4.34}$$

We also assume in the assumption (4.9) that

$$\frac{1}{N}\sum_i^{(k)}\frac{1}{|\lambda v_i - z - m_{fc}|^2} < c < 1, \tag{4.35}$$

for some constant c. Thus, we get

$$|\hat{m}_{fc}(z) - m_{fc}(z)| < \frac{1+c}{2}|\hat{m}_{fc}(z) - m_{fc}(z)| + N^{-1/2+3\epsilon/2}, \qquad z \in \mathcal{D}'_\epsilon, \tag{4.36}$$

which implies that

$$|\hat{m}_{fc}(z) - m_{fc}(z)| < CN^{-1/2+3\epsilon/2}, \qquad z \in \mathcal{D}'_\epsilon.$$

Since this contradicts with the assumption that (4.31) does not hold, this proves the desired lemma. □

4.4 Proof of Theorem 3.6

The main result of this subsection is Proposition 4.8, which will imply Theorem 3.6. The key ingredient of the proof of Proposition 4.8 is an implicit equation for the largest eigenvalues (μ_k) of H, Equation (4.37) in Proposition 4.6 below, involving the Stieltjes transform $\hat{m}_{fc}$ and the random variables (v_k). Using the information on $\hat{m}_{fc}$ gathered in the previous subsections the Equation (4.37) can be solved approximately for (μ_k).

Proposition 4.6. *Let $n_0 > 10$ be a fixed integer independent of N. Let μ_k be the k-th largest eigenvalue of H, $k \in [\![1, n_0 - 1]\!]$. Suppose that the assumptions in Theorem 3.6 hold. Then, the following holds with $(\xi - 2, \nu)$-high probability on Ω_V:*

$$\mu_k + \operatorname{Re}\hat{m}_{fc}(\mu_k + \mathrm{i}\eta_0) = \lambda v_k + \mathcal{O}(N^{-1/2+3\epsilon}). \tag{4.37}$$

Remark 4.7. Since $|\lambda v_i - \lambda v_k| \geq N^{-\epsilon}\kappa_0 \gg N^{-1/2+3\epsilon}$, for all $i \neq k$, on Ω_V, we obtain from Proposition 4.6 that

$$|\mu_k + \mathrm{i}\eta_0 + \operatorname{Re}\hat{m}_{fc}(\mu_k + \mathrm{i}\eta_0) - \lambda v_i| \geq |\lambda v_i - \lambda v_k| - |\mu_k + \mathrm{i}\eta_0 + \operatorname{Re}\hat{m}_{fc}(\mu_k + \mathrm{i}\eta_0) - \lambda v_k| \geq \frac{N^{-\epsilon}\kappa_0}{2},$$

on Ω_V. Hence, we find that $\mu_k + \mathrm{i}\eta_0 \in \mathcal{D}'_\epsilon$, $k \in [\![1, n_0 - 1]\!]$, with high probability on Ω_V.

For the proof of Proposition 4.6, see Section 5 of [22], where Cauchy's interlacing property of eigenvalues of H and its minor $H^{(i)}$ is used. Combining the tools we developed in the previous subsection, we now prove the main result on the location of the eigenvalues.

Proposition 4.8. *Let $n_0 > 10$ be a fixed integer independent of N. Let μ_k be the k-th largest eigenvalue of $H = \lambda V + W$, where $k \in [\![1, n_0 - 1]\!]$. Then, there exist constants C and $\nu > 0$ such that we have, with $(\xi - 2, \nu)$-high probability on Ω_V,*

$$\left|\mu_k - \left(L_+ - \frac{\lambda^2 - \lambda_+^2}{\lambda}(1 - v_k)\right)\right| \leq C\frac{1}{N^{1/(\mathfrak{b}+1)}}\left(\frac{N^{3\epsilon}}{N^{\mathfrak{b}}} + \frac{(\log N)^2}{N^{1/(\mathfrak{b}+1)}}\right). \tag{4.38}$$

Proof of Theorem 3.6 and Proposition 4.8. It suffices to prove Proposition 4.8. Let $k \in [\![1, n_0 - 1]\!]$. From Lemma 4.5 and Proposition 4.6, we find that, with high probability on Ω_V,

$$\mu_k + \operatorname{Re} m_{fc}(\mu_k + \mathrm{i}\eta_0) = \lambda v_k + \mathcal{O}(N^{-1/2+3\epsilon}). \tag{4.39}$$

In Lemma 4.2, we showed that

$$\mu_k + \mathrm{i}\eta_0 + m_{fc}(\mu_k + \mathrm{i}\eta_0) = \lambda - \frac{\lambda^2}{\lambda^2 - \lambda_+^2}(L_+ - \mu_k) + \mathrm{i}C\eta_0 + \mathcal{O}\left(\kappa_0^{\min\{\mathfrak{b},2\}}(\log N)^2\right). \tag{4.40}$$

Thus, we obtain

$$\mu_k + \operatorname{Re} m_{fc}(\mu_k + \mathrm{i}\eta_0) = \lambda - \frac{\lambda^2}{\lambda^2 - \lambda_+^2}(L_+ - \mu_k) + \mathcal{O}\left(\kappa_0^{\min\{\mathfrak{b},2\}}(\log N)^2\right). \tag{4.41}$$

Therefore, we have with high probability on Ω_V that

$$\mu_k = L_+ - \frac{\lambda^2 - \lambda_+^2}{\lambda}(1 - v_k) + \mathcal{O}\left(\kappa_0^{\min\{\mathfrak{b},2\}}(\log N)^2\right) + \mathcal{O}(N^{-1/2+3\epsilon}), \tag{4.42}$$

completing the proof of Proposition 4.8. □

Remark 4.9. The constants in Proposition 4.8 depend only on λ, the distribution μ and the constant C_0 and θ in (2.5), but are otherwise independent of the detailed structure of the Wigner matrix W.

5 Appendix

In this appendix, we consider the Gaussian fluctuation of the largest eigenvalue in Theorem 3.4.

Proof of Theorem 3.4. Following the proof in [27, 21], we find that $\hat{L}_+$ be the solution to the equations

$$\hat{m}_{fc}(\hat{L}_+) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\lambda v_j - \hat{L}_+ - \hat{m}_{fc}(\hat{L}_+)}, \qquad \frac{1}{N}\sum_{j=1}^{N}\frac{1}{(\lambda v_j - \hat{L}_+ - \hat{m}_{fc}(\hat{L}_+))^2} = 1. \tag{5.1}$$

Let

$$\tau := L_+ + m_{fc}(L_+), \quad \hat{\tau} := \hat{L}_+ + \hat{m}_{fc}(\hat{L}_+).$$

From the condition $\lambda < \lambda_+$, we assume that

$$\int \frac{\mathrm{d}\mu(v)}{(\lambda v - \lambda)^2} > 1 + \delta, \quad \frac{1}{N}\sum_{j=1}^{N} \frac{1}{(\lambda v_j - \lambda)^2} > 1 + \delta \tag{5.2}$$

for some $\delta > 0$. Notice that the second inequality holds with high probability on V. From the assumption, we also find that $\tau, \hat{\tau} > \lambda$.

We first consider

$$\begin{aligned} 0 &= \frac{1}{N}\sum_{j=1}^{N} \frac{1}{(\lambda v_j - \hat{\tau})^2} - 1 = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{(\lambda v_j - \hat{\tau})^2} - \frac{1}{N}\sum_{j=1}^{N} \frac{1}{(\lambda v_j - \tau)^2} + \mathcal{O}(\varphi^{\xi} N^{-1/2}) \\ &= \frac{1}{N}\sum_{j=1}^{N} \frac{(-2\lambda v_j + \tau + \hat{\tau})(\tau - \hat{\tau})}{(\lambda v_j - \tau)^2(\lambda v_j - \hat{\tau})^2} + \mathcal{O}(\varphi^{\xi} N^{-1/2}), \end{aligned} \tag{5.3}$$

which holds with high probability. Since $\tau, \hat{\tau} > \lambda$, we have

$$-2\lambda v_j + \tau + \hat{\tau} \geq 0.$$

Moreover, with high probability, $|\{v_j : v_j < 0\}| > cN$ for some constant $c > 0$, independent of N. In particular,

$$\frac{1}{N}\sum_{j=1}^{N} \frac{-2\lambda v_j + \tau + \hat{\tau}}{(\lambda v_j - \tau)^2(\lambda v_j - \hat{\tau})^2} > c' > 0$$

for some constant c' independent of N. This shows that

$$\tau - \hat{\tau} = \mathcal{O}(\varphi^{\xi} N^{-1/2}).$$

We now consider

$$\begin{aligned} \hat{m}_{fc}(L_+) = \hat{\tau} - \hat{L}_+ &= \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda v_j - \hat{\tau}} = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda v_j - \tau} + \frac{1}{N}\sum_{j=1}^{N} \frac{\hat{\tau} - \tau}{(\lambda v_j - \tau)^2} + \mathcal{O}(\varphi^{2\xi} N^{-1}) \\ &= m_{fc}(L_+) + X + (\hat{\tau} - \tau) + \mathcal{O}(\varphi^{2\xi} N^{-1}), \end{aligned} \tag{5.4}$$

with high probability, where we define the random variable X by

$$X := \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda v_j - \tau} - \int \frac{\mathrm{d}\mu(v)}{\lambda v - \tau} = \frac{1}{N}\sum_{j=1}^{N} \left(\frac{1}{\lambda v_j - \tau} - \mathbb{E}\left[\frac{1}{\lambda v_j - \tau}\right] \right). \tag{5.5}$$

Notice that, by the central limit theorem, we have that X converges to the Gaussian random variable with mean 0 and variance $N^{-1}(1 - (m_{fc}(L_+))^2)$. Thus, we obtain that

$$L_+ - \hat{L}_+ = X + \mathcal{O}(\varphi^{2\xi} N^{-1}), \tag{5.6}$$

which proves the desired lemma. □

When (v_i) are fixed, we may follow the proof of Theorem 2.21 in [21] and get

$$|L_+ - \mu_1| \leq \varphi^{C\xi} N^{-2/3} \tag{5.7}$$

with high probability. Since $|\hat{L}_+ - L_+| \sim N^{-1/2}$, we find that the leading fluctuation of the largest eigenvalue comes from the Gaussian fluctuation we proved in Lemma 3.4. This also shows that there is a sharp transition from the order statistics to the Gaussian as λ changes.

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