# Super-Brownian motion in random environment and its duality

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**Abstract**

In [19], the author construct super-Brownian motion in random environment as the limit points of scaled branching random walks in random environment which are solutions of an SPDE. To see its convergence, we use the exponential dual process. In our case, the exponential dual process satisfies a certain SPDE.

We denote by $(\Omega, \mathcal{F}, P)$ a probability space. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}^* = \{1, 2, 3, \cdots\}$, and $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$. We denote by $\mathcal{M}_F(S)$ the set of finite Borel measures on $S$ with the topology by weak convergence. Let $C_K(S)$ be the set of continuous functions with support compact. If $F$ is a set of functions on $\mathbb{R}$, we write $F_+$ or $F^+$ for non-negative functions in $F$.

## 1 Introduction

Dawson and Watanabe independently introduced super-Brownian motion [5, 23] which was obtained as the limit of critical (or asymptotically critical) branching Brownian motions (or branching random walks). Also, It is known that super-Brownian motion appears as scaling limit of several models in physics or biology. There are many books for introduction of super-Brownian motion [7, 10] and dealing with several aspects of it [8, 9, 12, 20].

There are several ways to characterize SBM, the unique solutions of martingale problem, non-linear PDE, etc. Here, we characterize it as the unique solution of the martingale problem:

**Definition 1.1.** *We call a measure valued process $\{X_t(\cdot) : t \in [0, \infty)\}$ super-Brownian motion when $X_t$ is the unique solution of the martingale problem*

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \qquad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2} X_s\left(\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and} \\ \qquad \langle Z(\phi) \rangle_t = \int_0^t \gamma X_s(\phi^2) ds, \end{cases}$$

where $\gamma > 0$ is a constant.

We are interested in the path property of super-Brownian motion on which many researcher wrote papers. Here is one of them, absolute continuity and singularity with respect to Lebesgue measure.

**Theorem 1.2.** *[11, 20, 21] Assume $X$ is a Super-Brownian motion with $X_0 = \mu$, where $\mu \in \mathcal{M}_F(\mathbb{R}^d)$.*

*(i) ($d = 1$) There exists an adapted continuous $C_K(\mathbb{R})$-valued process $\{u_t : t > 0\}$ such that $X_t(dx) = u_t(x)dx$ for all $t > 0$ $P$-a.s. and $u$ satisfies the SPDE (defined on the larger probability space $(\Omega', \mathcal{F}', P')$)*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{\gamma u}\dot{W}, \ u_{0+}(dx) = \mu(dx), \tag{SPDE}$$

*where $W$ is an white noise defined on the larger probability space $(\Omega', \mathcal{F}', P')$.*

*(ii) ($d \geq 2$) $X_t(\cdot)$ is singular with respect to Lebesgue measure almost surely.*

**Remark:** There are some results on the detailed path properties for $d \geq 2$.

We focus on (SPDE). (SPDE) is generally expressed as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + a(u)\dot{W}, \tag{SPDE($a$)}$$

where $a(u)$ is $\mathbb{R}$-valued continuous function on $\mathbb{R}$.

There are some examples for (SPDE($a$)):

(a) If $a(u) = \lambda u$, then the solution of (SPDE($a$)) is the Cole-Hopf solution of KPZ equation.

(b) If $a(u) = \sqrt{u - u^2}$, then the solution of (SPDE($a$)) appears as the density of stepping-stone model.

Also, we constructed another example of (SPDE($a$)) in [19].

**Remark:** The existence of solutions for (SPDE($a$)) is studied in [14] with some assumptions on $a(\cdot)$ and the initial condition $\mu$.

In [19], we constructed some measure valued process as a limit points of some particle systems which satisfies an SPDE,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{\gamma u + \beta^2 u^2}\dot{W}.$$

In this article, we will give a review of the author's paper [19].

# 2 Super-Brownian motion in random environment

Super-Brownian motion in random environment was originally introduced by Mytnik [15]. He obtained super-Brownian motion in random environment as the scaling limit of branching Brownian motion in random environment, where random environment means that offspring distributions depends on time-space site.

## 2.1 Branching Brownian motion in random environment

Branching Brownian motion in random environment is defined by the following rule.

(i) For each $N$, particles locate in $\{x_1, \cdots, x_{K_n}\} \subset \mathbb{R}^d$ at time 0.

(ii) Each particle at time $\dfrac{k}{n}$ independently performs Brownian motion up to time $\dfrac{k+1}{n}$ and then independently splits into two particles with probability $\dfrac{1}{2} + \dfrac{\xi_k^{(n)}(x)}{2n^{1/2}}$ or dies with probability $\dfrac{1}{2} - \dfrac{\xi_k^{(n)}(x)}{2n^{1/2}}$, where $x$ is the site which the particle reached at time $\frac{k+1}{n}$ and $\{\{\xi_k^{(n)}(x)\}_{x\in\mathbb{R}^d}, k \in \mathbb{N}\}$ is i.i.d. random field which is defined by $\xi_k^{(n)}(x) = (-\sqrt{n} \vee \xi_k(x)) \wedge \sqrt{n}$.

$\{\{\xi(k)\}_{x\in\mathbb{R}^d} : k \in \mathbb{N}\}$ is i.i.d. random field on $\mathbb{R}^d$ such that

$$E\left[|\xi_k(x)|^3\right] < \infty \qquad \text{for all } x \in \mathbb{R}^d \text{ and } k \in \mathbb{N}.$$
$$P(\xi_k(x) > z) = P(\xi_k(x) < -z) \qquad \text{for all } x \in \mathbb{R}^d,\ z \in \mathbb{R},\ \text{and } k \in \mathbb{N}.$$

Let $g_n(x,y)$ and $g(x,y)$ be the covariance functions of $x_k^{(n)}(\cdot)$ and $\xi_k(\cdot)$ respectively, that is

$$g_n(x,y) = E[\xi_k^{(n)}(x)\xi_k^{(n)}(y)]$$
$$g(x,y) = E[\xi_k(x)\xi_k(y)] \qquad x, y \in \mathbb{R}^d, k \in \mathbb{N}.$$

We assume that $g(x,y)$ is a continuous function with limit at infinity.

We identify the branching Brownian motion in random environment as the measure valued process by

$$X_t^{(n)}(A) = \frac{1}{n}\sharp\,\{\text{particles locate in } A \text{ at time } t\}$$

for any Borel set $A$.

Then, we have the following:

**Theorem 2.1.** *Assume that* $X_0^{(n)} \Rightarrow X_0$ *in* $\mathcal{M}_F(\mathbb{R}^d)$. *Then,* $X^{(n)} \Rightarrow X$, *where* $X \in C([0,\infty), \mathcal{M}_F(\mathbb{R}^d))$ *is the unique solution of the following martingale problem:*

$$\begin{cases} \text{For all } \phi \in C_b^2(\mathbb{R}^d), \\ \quad Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{1}{2}\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X \text{ continuous square integrable martingale such that } Z_0(\phi) = 0 \text{ and} \\ \quad \langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2) ds + \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} g(x,y)\phi(x)\phi(y) X_s(dx) X_s(dy) ds. \end{cases} \tag{2.1}$$

Also, Mytnik gave a remark that if $d = 1$ and $g(x,y)$" = "$\delta_0(x-y)$ and let $u$ be a solution of SPDE

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{u + u^2}\dot{W},$$

then $X_t(dx) = u(t,x)dx$ solves the martingale problem (2.1). Since $\delta_0(x-y)$ is not a continuous function any more, it is a "special case" of Mytnik's result. In [19], we obtain a measure valued process satisfying the special case as the scaling limit of some branching systems.

## 2.2 branching random walks in random environment

Although there are a lot of definition of branching random walks in random environment, ours is the one introduced in [2]. Let $N \in \mathbb{N}$ be large enough. We consider the system where particles move on $\mathbb{Z}$ and the process evolves according to the following rule:

(i) There are particles at $\{x_1, \cdots, x_{M_N}\}$ at time 0.

(ii) If a particle locates at site $x \in \mathbb{Z}$ at time $n$, then it moves to a uniformly chosen hearest neighbor site and split into two particles with probability $\frac{1}{2} + \frac{\beta\xi(n,x)}{2N^{1/4}}$ or dies out with probability $\frac{1}{2} - \frac{\beta\xi(n,x)}{2N^{1/4}}$, where jump and branching system are independent of each particles, $\{\xi(n,x) : (n,x) \in \mathbb{N}\times\mathbb{Z}\}$ are $\{1,-1\}$-valued i.i.d. random variables with $P(\xi(n,x) = 1) = P(\xi(n,x) = -1) = \frac{1}{2}$, and $\beta > 0$ is constant.

**Remark:** In our model, random environment is given by branching mechanics which are updated for each site and each time.

**Remark:** $N$ is the scaling parameter which tends to infinity later. Also, we emphasize that the fluctuations of offspring distributions are different from the ones in [15].

We don't give the mathematically rigorous definition in this paper.

## 2.3 Super-Brownian motion in random environment

In this subsection, we introduce super-Brownian motion in random environment. Super-Brownian motion is obtained as the limit of scaled critical branching Brownian motions (branching random walks). When we look at our model, the mean number of offsprings from one particle is 1, so that we can regard our model as "critical" branching random walks in random environment in some sense. We will try to obtain the scaled limit process.

We denote by $B_{n,x}^{(N)}$ the number of particles at site $x$ at time $n$. We define $X_t^{(N)}(dx)$ by

$$X_0^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{M_N}\delta_{x_i}(dx),$$
$$X_t^{(N)}(dx) = \frac{1}{N}\sum_{y\in\mathbb{Z}} B_{\lfloor tn\rfloor,y}^{(N)}\delta_y(N^{1/2}dx).$$

More simply, we can express the definition of $X_t^{(N)}(\cdot)$ as follows: Let $A\in\mathcal{B}(\mathbb{R})$ be a Borel set in $\mathbb{R}$. Then,

$$X_t^{(N)}(A) = \frac{\sharp\{\text{particles locates in } N^{1/2}A \text{ at time } \lfloor Nt\rfloor\}}{N}.$$

In [19], we have the following result.

**Theorem 2.2.** *If $X_0^{(N)} \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R})$, then $\{X_\cdot^{(N)} : N\in\mathbb{N}^*\}$ is $C$-relatively compact. Moreover, if we denote by $\{X_t(\cdot)\}$ a limit point, then $X_t(\cdot)$ is absolutely continuous with respect to Lebesgue measure for all $t>0$ $P$-a.s. and its density $u(t,x)$ satisfies SPDE*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{u+2\beta^2u^2}\dot{W},\ u_{0+}dx = \delta_0(dx). \tag{2.2}$$

Formally, $\{X_t(\cdot) : t\geq 0\}$ is a solution of the following martingale problem:

$$\begin{cases}\text{For all } \phi\in\mathcal{D}(\Delta),\\ \qquad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \displaystyle\int_0^t \frac{1}{2}X_s(\Delta\phi)\,ds\\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and}\\ \qquad \langle Z(\phi)\rangle_t = \displaystyle\int_0^t X_s(\phi^2)ds + 2\beta^2\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\delta_{x-y}\phi(x)\phi(y)X_s(dx)X_s(dy)ds.\end{cases}$$

To be rigorous, $\{X_t(\cdot) : t\geq 0\}$ is a solution of the following martingale problem:

$$\begin{cases}\text{For all } \phi\in\mathcal{D}(\Delta),\\ \qquad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \displaystyle\int_0^t \frac{1}{2}X_s(\Delta\phi)\,ds\\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and}\\ \qquad \langle Z(\phi)\rangle_t = \displaystyle\int_0^t X_s(\phi^2)ds + 2\beta^2\int_0^t\int_{\mathbb{R}}\phi^2(x)X_s^2(x)dxds.\end{cases} \tag{2.3}$$

We shall call solutions of the above martingale problem super-Brownian motion in random environment.

Also, we are interested in uniqueness of solutions of (2.3). Before giving an answer, we introduce a notation. Let $C^+_{\text{rap}}(\mathbb{R})$ be the set of rapidly decreasing functions, that is

$$C^+_{\text{rap}}(\mathbb{R}) = \left\{ g \in C^+(\mathbb{R}) : |g|_p \equiv \sup_{x\in\mathbb{R}} e^{p|x|}|g(x)| < \infty, \forall p > 0 \right\}.$$

The following theorem gives us an answer.

**Theorem 2.3.** *Solutions of martingale problem (2.3) is unique if* $X_0(dx) = u_0(x)dx$ *for* $u_0 \in C^+_{rap}(\mathbb{R})$*. Moreover, if* $X_0 \in \mathcal{M}_F(\mathbb{R})$*, then* $X^{(N)} \Rightarrow X$ *in* $C([0,\infty), \mathcal{M}_F(\mathbb{R}))$*, where* $X$ *is a solution of the martingale problem of (2.3).*

# 3 Uniqueness

Although there are several definition of the uniqueness for SPDE, we consider the uniqueness in law for our model. The readers can refer some papers on the uniqueness (in law or pathwise) of the solutions of (SPDE($a$)) [13, 16, 17, 18]. In most cases, Hölder continuity of $a(\cdot)$ influences on the uniqueness. Actually, the uniqueness in law holds when $a(u) = u^\gamma$, $\gamma \in [\frac{1}{2}, 1]$. In our case, the Hölder contiuity of $a(\cdot)$ is $\frac{1}{2}$ so that we can conjecture the uniqueness in law does hold.

We suppose that $X_t$ is a solution of (2.3).

The main idea to prove the uniqueness of solutions of the martingale problem (2.3) is to prove the existence of the "dual" process $\{Y_t : t \geq 0\}$, which is $C^+_{\text{rap}}(\mathbb{R})$-valued process and satisfies the equation

$$E\left[\exp\left(-\langle Y_t, X_0\rangle\right)\right] = E\left[\exp\left(-\langle \phi, X_t\rangle\right)\right] \tag{3.1}$$

for each $\phi \in C^+_{\text{rap}}(\mathbb{R})$, where $\langle \phi, \mu\rangle = \int_{\mathbb{R}} \phi(x)\mu(dx)$ for $\phi \in C_b(\mathbb{R})$ and $\mu \in \mathcal{M}_F(\mathbb{R})$.

In particular, a solution of the SPDE

$$\frac{\partial Y}{\partial t} = \frac{1}{2}\Delta Y_t - \frac{1}{2}Y_t^2 - \sqrt{2}|\beta| Y_t \dot{\dot{W}},\ Y_0(x) = \phi(x) \tag{3.2}$$

is a "dual" process of $\{X_t : t \geq 0\}$. Indeed, if $Y_t \in C^1_+(\mathbb{R})$ for all $t \geq 0$, then it follows from Ito's lemma that

$$\begin{aligned}\exp\left(-\langle Y_{t-s}, X_s\rangle\right) &= \exp\left(-\langle Y_t, X_0\rangle\right) - \int_0^s \left\langle \frac{1}{2}Y_{t-u}^2, X_u \right\rangle \exp\left(-\langle Y_{t-u}, X_u\rangle\right) du \\ &- \int_0^s \beta^2 \left\langle Y_{t-u}^2, X_u^2\right\rangle \exp\left(-\langle Y_{t-u}, X_u\rangle\right) du + \int_0^s \left\langle \frac{1}{2}\Delta Y_{t-u}, X_u \right\rangle \exp\left(-\langle Y_{t-u}, X_u\rangle\right) \\ &- \int_0^s \left\langle \frac{1}{2}\Delta Y_{t-u}, X_u \right\rangle \exp\left(-\langle Y_{t-u}, X_u\rangle\right)\end{aligned}$$

$$+\int_0^s \left\langle Y_{t-u}^2, \frac{1}{2}X_u + \beta^2 X_u^2 \right\rangle \exp\left(-\langle Y_{t-u}, X_u\rangle\right) du + (\text{martingale part}).$$

Then, taking expectation and letting $s = t$, $E\left[\exp(-\langle Y_0, X_t\rangle)\right] = E[\exp(-Y_t, X_0)]$. However, we find that $Y_t$ is not differentiable for any $x \in \mathbb{R}$ such that $Y_t(x) \neq 0$ so we need to approximate $Y_t$ by $Y_t^\varepsilon(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\varepsilon}} Y_t(x+y)\exp(-\frac{y^2}{2\varepsilon})dy$. We will omit a proof of the statement that

$$\lim_{\varepsilon\to 0} E\left[\exp\left(-\langle Y_t^\varepsilon, X_0\rangle\right)\right] = E\left[\exp(-\langle Y_t, X_0\rangle)\right] = E\left[\exp(-\langle Y_0, X_t\rangle)\right] \qquad (3.3)$$

for any $t \in [0,\infty)$ and $Y_0(x) = \phi(x) \in C_{\text{rap}}^+(\mathbb{R})$. We remark that when we prove (3.3), we have used estimates coming from branching random walks in random environment. It implies that we don't still prove the uniqueness of solutions of the martingale problem (2.3) for $X_0 \in \mathcal{M}_F(\mathbb{R})$. However, if $X_(dx) = \psi(x)dx$ for $\psi \in C_{\text{rap}}^+(\mathbb{R})$, then we can prove (3.3) directly by using the properties of $X_t$.

The existence of nonnegative solutions to (3.2) for the case where $Y_0 \in C_{rap}^+(\mathbb{R})$ follows from [22] by using Dawson's Girsanov theorem[6]. Indeed, the existence and the uniqueness of nonnegative solutions to

$$\tilde{Y}_0(x) = \phi(x),\ \frac{\partial}{\partial t}\tilde{Y}_t(x) = \frac{1}{2}\Delta\tilde{Y}_t(x) + \sqrt{2}|\beta|\tilde{Y}_t(x)\dot{\tilde{W}}(t,x).$$

has been already known, where $\tilde{W}$ is a time-space white noise independent of $X$[1, 22]. We denote by $P_{\tilde{Y}}$ the law of $\tilde{Y}$. Let $P_Y$ be the probability measure with Radon-Nikodym derivatives

$$\left.\frac{dP_Y}{dP_{\tilde{Y}}}\right|_{\mathcal{F}_t^{\tilde{Y}}} = \exp\left(\frac{\gamma}{2\sqrt{2}|\beta|}\int_0^t\int_{\mathbb{R}} \tilde{Y}_s(y)\tilde{W}(ds,dy) - \frac{\gamma^2}{16\beta^2}\int_0^t\int_{\mathbb{R}} \tilde{Y}_s^2(y)dyds\right).$$

Then, under $P_Y$, $\tilde{Y}$ satisfies (3.2) and $\tilde{Y}$ is also a $C_{rap}^+(\mathbb{R})$-valued process. Thus, we constructed a solution to (3.2). Especially, we remark that the solutions to (3.2) satisfy for $t \geq 0$

$$\begin{aligned} Y_t(x) = &\int_{\mathbb{R}} p_t(x+y)\phi(y)dy - \frac{\gamma}{2}\int_0^t\int_{\mathbb{R}} p_{t-s}(x+y)Y_s^2(y)dyds \\ &+ \sqrt{2}|\beta|\int_0^t\int_{\mathbb{R}} p_{t-s}(x+y)Y_s(y)\tilde{W}(ds,dy), \end{aligned} \qquad (3.4)$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$ for $t > 0$ and $x \in \mathbb{R}$.

The following lemma tells us that $(\{Y_t\}_{t\geq 0}, \mathcal{F}_t^Y, P_Y)$ is a solution to the martingale problem:

$$\begin{cases} \text{For all } \psi \in C_b^2(\mathbb{R}), \\ \qquad \tilde{Z}_t(\psi) = \langle Y_t, \psi\rangle - \langle Y_0, \psi\rangle + \frac{\gamma}{2}\int_0^t \langle Y_s^2, \psi\rangle ds - \int_0^t \langle Y_s, \frac{1}{2}\Delta\psi\rangle ds \\ \text{is an } \mathcal{F}_t^Y\text{-continuous square integrable martingale and} \\ \qquad \langle \tilde{Z}(\psi)\rangle_t = 2\beta^2\int_0^t \langle Y_s^2, \psi^2\rangle ds \end{cases}$$

**Lemma 3.1.** *Let $\phi \in C^+_{rap}(\mathbb{R})$. Let $(\{Y_t\}_{t\geq 0}, \mathcal{F}^Y, \{\mathcal{F}^Y_t\}_{t\geq 0}, P_Y)$ be a nonnegative solution to (3.2). Then, we have that*

$$E_Y\left[\int_{\mathbb{R}} Y_t(x)dx\right] \leq \int_{\mathbb{R}} \phi(x)dx, \tag{3.5}$$

*and*

$$E_Y\left[\int_0^t \int_{\mathbb{R}} Y_s^p(x)dxds\right] < \infty, \tag{3.6}$$

*for all $0 \leq t < \infty$ and $p \geq 1$.*

*Proof.* (3.5) is clear from (3.4). Let $0 \leq t \leq T$. Also, we have that

$$Y_t^p(x) \leq C(p,\beta)\left\{\left(\int_{\mathbb{R}} p_t(x+y)\phi(y)dy\right)^p + \left(\int_0^t \int_{\mathbb{R}} p_{t-s}(x+y)Y_s(y)\tilde{W}(ds,dy)\right)^p\right\}.$$

We define

$$T(\ell) = \inf\{t \geq 0 : \sup_x e^{|x|}|Y_t(x)| > \ell\}.$$

We remark that $T(\ell) \to \infty$ $P_Y$-a.s. as $\ell \to \infty$ since $Y_t \in C^+_{rap}(\mathbb{R})$ for all $t \geq 0$ $P_Y$-a.s. Then, we have by Hölder's inequality and Burkholder-Daivs-Gundy inequality that

$$\begin{aligned}
&E_Y\left[Y_t^p(x) : t \leq T(\ell)\right] \\
&\quad\leq C(p,\beta)E_Y\left[\left(\int_{\mathbb{R}} p_t(x+y)\phi(y)dy\right)^p + \left(\int_0^t \int_{\mathbb{R}} 1\{t \leq T(\ell)\}p^2_{t-s}(x+y)Y_s^2(y)dyds\right)^{\frac{p}{2}}\right] \\
&\quad\leq C(p,\beta)\left(\int_{\mathbb{R}} p_t(x+y)\phi(y)dy\right)^p \\
&\quad+ C(p,\beta)E_Y\left[\left(\int_0^t \int_{\mathbb{R}} 1\{t \leq T(\ell)\}p^2_{t-s}(x+y)Y_s^p(y)dyds\right)^{\frac{p}{2}}\left(\int_0^t \int_{\mathbb{R}} p^2_{t-s}(x+y)dyds\right)^{\frac{p}{2}-1}\right] \\
&\quad\leq C(p,\beta)\left(\int_{\mathbb{R}} p_t(x+y)\phi(y)dy\right)^p \\
&\quad+ C(p,\beta)t^{\frac{p-2}{4}}\int_0^t \int_{\mathbb{R}} (t-s)^{-\frac{1}{2}}p_{t-s}(x+y)E_Y\left[Y_s^p(y) : t \leq T(\ell)\right]dyds,
\end{aligned}$$

where we have used that $p_s^2(x) \leq Cs^{-\frac{1}{2}}p_s(x)$ and $\int_0^t \int_{\mathbb{R}} p_s^2(x)dxds \leq Ct^{\frac{1}{2}}$. Integrating on $x$ over $\mathbb{R}$ and letting $\nu(s,t,\ell,p) = \int_{\mathbb{R}} E_Y\left[Y_s^p(x) : t \leq T(\ell)\right]dx$, then $\nu(s,t,\ell,p) < \infty$ by definition and we have that

$$\nu(t,t,\ell,p) \leq C(p,\beta,T)\left(1 + \int_0^t (t-s)^{-\frac{1}{2}}\nu(s,t,\ell,p)ds\right),$$

where we have used $\sup_{t\le T}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} p_t(x+y)\phi(y)dy\right)^2 dx < \infty$. It follows from Lemma 4.1 in [14] that

$$\nu(t,t,\ell,p) \le C(p,\beta,T,Y_0)\exp\left(C(p,\beta,T,Y_0)t^{\frac{1}{2}}\right) \quad \text{for } t \le T.$$

Since the right hand side does not depend on $\ell$, it follows from the monotone convergence theorem that

$$\int_{\mathbb{R}} E_Y\left[Y_t^p(x)\right] dx \le C(p,\beta,T,Y_0)$$

and

$$\int_0^T \int_{\mathbb{R}} E_Y\left[Y_t^p(x)\right] dxdt \le C(p,\beta,T,Y_0)T.$$

□

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