A Calculable Model for a Cavity and a Atomic Beam

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ABSTRACT

We consider a simple model for the physical system consists of a cavity and a beam of atoms which pass the cavity successively.

The Hamiltonian contains time-dependent (piecewise constant) term describing interaction between the cavity and the atom in the beam which is passing the cavity at the prescribed moment. We deal with the model in which the radiation field inside the cavity and each atoms of the beam is modeled by simple harmonic oscillators.

We calculate the time evolution of the density matrix of the system and the asymptotic behavior of the cavity, both in the Hamiltonian dynamics and the Markovian dynamics of Kossakowski-Lindblad-Davies type. We also discuss the entropies and evolution of the reduced density matrix for subsystems near the cavity.

This talk is based on the joint work with Prof. V.A. Zagrebnov. The detailed description of the subject will be given in the forth coming paper.[TZ]

1 The Model

Let a, a^* be the annihilation and the creation operators living in the one mode Fock space $\mathscr{F}$:

$$[a, a^*] = 1, \quad [a, a] = 0, \quad [a^*, a^*] = 0.$$

$$\mathscr{F} = \overline{\mathscr{F}_{\text{fin}}}$$

$$\mathscr{F}_{\text{fin}} = \text{ algebraic span of } \{\Omega, a^*\Omega, \cdots, a^{*k}\Omega, \cdots\}.$$

Let $\mathscr{H}_n$ $(n = 0, 1, \cdots, N)$ be copies of $\mathscr{F}$ for a arbitrary but finite $N \in \mathbb{N}$.

On the Hilbert space tensor product

$$\mathscr{H} = \bigotimes_{n=0}^{N} \mathscr{H}_n,$$

we define the operators

$$\begin{aligned} b_0 &= a \otimes 1 \otimes \cdots \otimes 1, & b_0^* &= a^* \otimes 1 \otimes \cdots \otimes 1, \\ b_1 &= 1 \otimes a \otimes 1 \otimes \cdots \otimes 1, & b_1^* &= 1 \otimes a^* \otimes 1 \otimes \cdots \otimes 1, \\ b_2 &= 1 \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1, & b_2^* &= 1 \otimes 1 \otimes a^* \otimes 1 \otimes \cdots \otimes 1, \end{aligned}$$

and so on. The operators $b_j, b_j^* \quad (j = 0, 1, 2, \cdots, N)$ satisfy CCR

$$[b_i, b_j^*] = \delta_{ij}, \quad [b_i, b_j] = [b_i^*, b_j^*] = 0.$$

Remark : We regard $\mathscr{H}_0$ as the state space for the photon inside the cavity and $\mathscr{H}_n$ $(n = 1, 2, \cdots)$ for internal states of atoms. So, b_0^*, b_0 are creation and annihilation operators of photon and b_j^*, b_j are raising and lowering operator of the level of the j-th atom.

Remark : We consider the case $N < \infty$, for simplicity.

Let H_n be the self-adjoint Hamiltonian in $\mathscr{H}$ defined by

$$H_n = Eb_0^*b_0 + \epsilon \sum_{k=1}^{N} b_k^*b_k + \eta b_0^*b_n + \eta b_n^*b_0, \qquad (n = 1, 2, \cdots, N)$$

where $E > 0$ is the photon energy, $\epsilon > 0$ the energy level spacing of the atoms and $\eta > 0$ the interaction between the photon and atoms. We assume that η is small enough so that H_n is bounded below. (We understand that all operators like H_n are taken to be closed.) We regard that H_n is the Hamiltonian during the time interval $[(n-1)\tau, n\tau)$ when the n-th atom is passing inside the cavity.

2 Hamiltonian Dynamics

In this section, we consider the time evolution of the system governed by the time dependent (piecewise constant) Hamiltonian:

$$H(t) = \sum_{n=1}^{N} \chi_{[(n-1)\tau, n\tau)}(t) H_n.$$

The commutation relations

$$[H_n, b_0] = -Eb_0 - \eta b_n, \quad [H_n, b_j] = -\epsilon b_j - \delta_{jn}\eta b_0,$$

$$[H_n, b_0^*] = Eb_0^* + \eta b_n^*, \quad [H_n, b_j^*] = \epsilon b_j^* + \delta_{jn}\eta b_0^*$$

hold for $j = 1, \cdots, N$ and yield the following lemma.

Lemma 2.0.1 *For $j = 0, 1, 2, \cdots, N$ and $n = 1, 2, \cdots, N$,*

$$e^{-i\tau H_n} b_j e^{i\tau H_n} = \sum_{k=0}^{N} (U_n)_{jk} b_k, \quad e^{-i\tau H_n} b_j^* e^{i\tau H_n} = \sum_{k=0}^{N} \overline{(U_n)_{jk}} b_k^*,$$

$$e^{i\tau H_n} b_j e^{-i\tau H_n} = \sum_{k=0}^{N} (U_n^*)_{jk} b_k, \quad e^{i\tau H_n} b_j^* e^{-i\tau H_n} = \sum_{k=0}^{N} \overline{(U_n^*)_{jk}} b_k^*$$

hold. Here U_n and V_n are $(N+1)\times(N+1)$ matrices given by $U_n = e^{i\tau\epsilon} V_n$ and

$$(V_n)_{jk} = \begin{cases} gz\,\delta_{k0} + gw\,\delta_{kn} & (j=0) \\ -g\bar{w}\,\delta_{k0} + g\bar{z}\,\delta_{kn} & (j=n) \\ \quad\delta_{jk} & (\textit{otherwise}) \end{cases}$$

with

$$g = e^{i\tau(E-\epsilon)/2}, \qquad w = \frac{2i\eta}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin\tau\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}$$

$$z = \cos\tau\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} + \frac{i(E-\epsilon)}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin\tau\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}.$$

Note that $|z|^2 + |w|^2 = 1$ and that

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

is a unitary matrix. And so are V_n's and U_n's.

e.g.

$$U_1 = e^{i\tau\epsilon} V_1 = e^{i\tau\epsilon} \begin{pmatrix} gz & gw & 0 & 0 & 0 & \cdots \\ -g\bar{w} & g\bar{z} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

$$U_2 = e^{i\tau\epsilon} V_2 = e^{i\tau\epsilon} \begin{pmatrix} gz & 0 & gw & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ -g\bar{w} & 0 & g\bar{z} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}.$$

2.1 Time evolution of Product states

For $y \in \mathbb{C}$,

$$w(y) = e^{i(\bar{y}a + ya^*)}$$

denotes the Weyl operator over $\mathscr{F}$. We consider the Weyl algebra $\mathscr{A}(\mathscr{F})$ over $\mathscr{F}$ generated by $\{w(y)\}_{y\in\mathbb{C}}$ and the algebra $\mathscr{A}(\mathscr{H})$ over $\mathscr{H}$ which is generated by

$$W(\zeta) = \bigotimes_{k=0}^{N} w(\zeta_k), \qquad (\zeta = \{\zeta_k\}_{k=0}^{N}). \tag{2.1}$$

Using sesquilinear form notation

$$\langle \zeta, b \rangle = \sum_{j=0}^{N} \bar{\zeta}_j b_j, \qquad \langle b, \zeta \rangle = \sum_{j=0}^{N} \zeta_j b_j^*,$$

$W(\zeta)$ can be written as

$$W(\zeta) = \exp[i\big(\langle \zeta, b\rangle + \langle b, \zeta\rangle\big)].$$

Let ρ_k be a normalized self-adjoint non-negative trace class operator on $\mathscr{F}$ for $k = 0, 1, 2, \cdots, N$. It can be regarded as a state on $\mathscr{A}(\mathscr{F})$. We use the notation

$$C_k(y) = \mathrm{Tr}_{\mathscr{F}}[w(y)\rho_k].$$

Similarly, we consider the operator

$$\rho = \bigotimes_{k=0}^{N} \rho_k$$

as a state on $\mathscr{A}(\mathscr{H})$:

$$\omega_\rho(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[W(\zeta)\rho] = \prod_{k=0}^{N} C_k(\zeta_k).$$

Let us consider the time evolution of ρ by $H(t) \quad (0 \leqslant t \leqslant N\tau)$:

$$\rho(N\tau) := e^{-i\tau H_N} \cdots e^{-i\tau H_1} \rho e^{i\tau H_1} \cdots e^{i\tau H_N}.$$

Lemma 2.1.1

$$\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \cdots U_N \zeta)) = \prod_{k=0}^{N} C_k((U_1 \cdots U_N \zeta)_k)$$

holds, where

$$(U_1 \cdots U_N \zeta)_0 = e^{iN\tau\epsilon}\big((gz)^N \zeta_0 + \sum_{j=1}^{N} gw(gz)^{j-1}\zeta_j\big)$$

and

$$(U_1 \cdots U_N \zeta)_k = e^{iN\tau\epsilon}\big(- g\bar{w}(gz)^{N-k}\zeta_0 + g\bar{z}\zeta_k - \sum_{j=k+1}^{N} g^2|w|^2(gz)^{j-k-1}\zeta_j\big)$$

for $k > 0$.

2.2 The product Gibbs state

For the product Gibbs state

$$\rho = \bigotimes_{k=0}^{N} \rho_k \quad \text{with} \quad \rho_0 = e^{-\beta_0 a^* a}/Z(\beta_0), \quad \rho_j = e^{-\beta a^* a}/Z(\beta) \quad (j = 1, \cdots, N),$$

where $\beta, \beta_0 > 0$ and $Z(\beta) = (1 - e^{-\beta})^{-1}$, we have

Lemma 2.2.1 *The state corresponding to the density matrix (2.3) satisfies*

$$\omega_\rho(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[W(\zeta)\rho] = \exp\Big[-\frac{|\zeta_0|^2}{2}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\zeta,\zeta\rangle}{2}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]$$

and

$$S(\rho) = -\mathrm{Tr}_{\mathscr{H}}[\rho \log \rho] = Ns(\beta) + s(\beta_0),$$

where $s(\beta) := \beta(e^\beta - 1)^{-1} - \log(1 - e^{-\beta})$.

The time evolution of the density matrix

$$\rho(N\tau) = e^{-i\tau H_N} e^{-i\tau H_{N-1}} \cdots e^{-i\tau H_1} \rho e^{i\tau H_1} \cdots e^{i\tau H_N}.$$

satisfies the following properties:

Lemma 2.2.2

$$\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \cdots U_N \zeta)) =$$

$$\exp\Big[-\frac{|(U_1 \cdots U_N\zeta)_0|^2}{2}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\zeta,\zeta\rangle}{2}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big],$$

and

$$S(\rho(N\tau)) = Ns(\beta) + s(\beta_0) = S(\rho).$$

The relative entropy of $\rho(N\tau)$ w.r.t. ρ is:

Lemma 2.2.3

$$S(\rho(N\tau)|\rho) = -\mathrm{Tr}[\rho(N\tau)(\log\rho(N\tau) - \log\rho)] = -\mathrm{Tr}[\rho(\log\rho - \log\rho(-N\tau))]$$

$$= -\frac{(\beta_0 - \beta)(e^{\beta_0} - e^\beta)}{(e^{\beta_0} - 1)(e^\beta - 1)}(1 - |z|^{2N}).$$

Remark The relative entropy is non-positive generally. In this case, it decreases monotonically as $N \to \infty$ and converges to the limit:

$$\lim_{N\to\infty} S(\rho(N\tau)|\rho) = -\frac{(\beta_0 - \beta)(e^{\beta_0} - e^\beta)}{(e^{\beta_0} - 1)(e^\beta - 1)}.$$

2.3 Subsystems

In this section, we devide the system into 2 subsystems. At time $t = k\tau$, the objects are ordered as follows:

$$\text{the first atom}, \cdots, \text{the } k\text{-th atom}, \text{the cavity}, \text{the } k+1\text{-th atom}, \cdots, \text{the } N\text{-th atom}.$$

We regard the cavity and the n atoms ahead the cavity as the subsystem:

$$\mathscr{H} = \mathscr{H}_s \bigotimes \mathscr{H}_r,$$

where

$$\mathscr{H}_s = \mathscr{H}_0 \bigotimes \bigotimes_{j=1}^{n} \mathscr{H}_{k-j+1}, \qquad \mathscr{H}_r = \bigotimes_{j=1}^{k-n} \mathscr{H}_j \bigotimes \bigotimes_{j=n+1}^{N} \mathscr{H}_j$$

We want to re-number the atoms in the subsystem:

For $\theta = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{pmatrix} \in \mathbb{C}^{n+1},$ take

$$\zeta_\theta^{(k)} = \begin{pmatrix} \theta_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \theta_n \\ \theta_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \theta_2 \\ \theta_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow 0 \text{ -th} \\ \leftarrow 1 \text{ -th} \\ \\ \\ \\ \leftarrow k-n \text{ -th} \\ \leftarrow k-n+1 \text{ -th} \\ \leftarrow k-n+2 \text{ -th} \\ \\ \\ \\ \leftarrow k-1 \text{ -th} \\ \leftarrow k \text{ -th} \\ \leftarrow k+1 \text{ -th} \\ \\ \\ \\ \leftarrow N \text{ -th} \end{array} \in \mathbb{C}^{N+1}$$

to get

$$\theta_0 b_0^* + \bar{\theta}_0 b_0 + \sum_{j=1}^{n} (\theta_j b_{k-j+1}^* + \bar{\theta}_j b_{k-j+1}) = \sum_{j=0}^{N} (\zeta_{\theta,j}^{(k)} b_j^* + \bar{\zeta}_{\theta,j}^{(k)} b_j).$$

And consider the Weil operator on $\mathscr{H}_s$

$$\begin{aligned} W_s(\theta) &= \exp\big[i\big(\theta_0 b_0^* + \bar{\theta}_0 b_0 + \sum_{j=1}^{n} (\theta_j b_{k-j+1}^* + \bar{\theta}_j b_{k-j+1})\big)\big] \\ &= \exp\big[i\big(\theta_0 \tilde{b}_0^* + \bar{\theta}_0 \tilde{b}_0 + \sum_{j=1}^{n} (\theta_j \tilde{b}_j^* + \bar{\theta}_j \tilde{b}_j)\big)\big], \end{aligned}$$

where, $\tilde{b}_0 = b_0, \tilde{b}_j = b_{k-j+1}$. (We used abused notations: e.g., b_0 is not an operator in $\mathscr{H}_s$ but in $\mathscr{H}$, while $\tilde{b}_0$ in $\mathscr{H}_s$, etc.)

For the density matrix ρ, let ρ_s be the reduced density matrix of the sub-system i.e.,

$$\rho_s = \mathrm{Tr}_{\mathscr{H}_r} \rho. \tag{2.2}$$

Then, we get

$$\omega_{\rho_s}(W_s(\theta)) = \omega_\rho(W(\zeta_\theta)).$$

Now let us consider time evolution. The time evoluted density $\rho(k\tau)$ of the initial Gibbs state

$$\rho = \exp\big[-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^{N} b_j^* b_j\big] / (Z(\beta_0) Z(\beta)^N), \tag{2.3}$$

has the reduced density matrix given by

Lemma 2.3.1

$$\omega_{\rho(k\tau)_s}(W_s(\theta)) = \omega_{\rho(k\tau)}(W(\zeta_\theta))$$

$$= \exp\Big[-\frac{|(U_1\cdots U_k\zeta_\theta)_0|^2}{2}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\theta,\theta\rangle}{2}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big],$$

Now consider the limit $k \to \infty (N \to \infty)$ with n fixed. We get

Proposition 2.3.2 *$\rho(k\tau)_s$ converge to $\rho^{(\beta)}$ and*

$$\lim_{k\to\infty} S(\rho(k\tau)_s) = S(\rho^{(\beta)}),$$

where

$$\rho^{(\beta)} = \exp\big[-\beta b_0^* b_0 - \beta \sum_{j=1}^{n} b^*_{N-j+1} b_{N-j+1}\big] / Z(\beta)^{n+1}.$$

Remark The local entropy decreases or increases accordong to $\beta > \beta_0$ or $\beta < \beta_0$, respectively.

2.4 A scaling limit for product states

Here, we mention an asymptotic behavior of the state of the cavity under the influence of the beam where the state for the atoms is product of general type.

We assume that

(1) $\rho_1 = \rho_2 = \cdots = \rho_N$;

(2) $\mathrm{Tr}[a\rho_1] = \mathrm{Tr}[a^2\rho_1] = \mathrm{Tr}[a^*\rho_1] = \mathrm{Tr}[a^{*2}\rho_1] = 0$;

(3) $\mathrm{Tr}[(a^*a)^2\rho_1] < \infty$.

Proposition 2.4.1 *Under the limit $\tau \to 0$ and $N \to \infty$ subject to $\tau^2 N \to \infty$ and $\tau^3 N \to 0$ (e.g., $\tau = O(N^{-0.4})$),*

$$\lim \omega_{\rho(N\tau)_s}(w(\theta)) = \lim \omega_{\rho(N\tau)}(W(\zeta_\theta)) = e^{-\mathrm{Tr}[(a^*a+aa^*)\rho_1]|\theta|^2/2}$$

holds for $\theta \in \mathbb{C}^{0+1}$.

3 Markovian Evolution

We consider here the evolution of the system under the Kossakowski-Lindblad-Davies equation, which yields a behavior the system in a large reservoir:

$$\partial_t \rho(t) = L_\sigma(t)(\rho(t)) \ , \ \rho = \rho(t)\big|_{t=0} \in \mathfrak{C}_1(\mathscr{H}),$$

where

$$L_\sigma(t)(\rho(t)) := -i[H(t), \rho(t)] + \sigma_- \ b_0 \ \rho(t) \ b_0^* - \frac{\sigma_-}{2}\{b_0^* b_0, \rho(t)\}$$
$$+\sigma_+ \ b_0^* \ \rho(t) \ b_0 - \frac{\sigma_+}{2}\{b_0 b_0^*, \rho(t)\}. \tag{3.1}$$

To satisfy the complete positivity-preserving the parameters of non-Hamiltonian part of dynamics must satisfy inequality $\sigma_{\mp} \geqslant 0$. We also impose condition $\sigma_+ \leqslant \sigma_-$ for the boundedness of expectations in the state, see [NVZ].

We introduce the family $\{T_{t,t'}\}_{0\leq t'\leq t}$ of trace-preserving and complete-positive evolution mappings:

$$T^{\sigma}_{t,0} : \rho \mapsto \rho_{\sigma}(t) = T^{\sigma}_{t,0}(\rho(0)) \;\text{ with }\; T^{\sigma}_{t,0} = T^{\sigma}_{t,t'} T^{\sigma}_{t',0}, \;\; (0 \leqslant t' \leqslant t). \tag{3.2}$$

As in the Hamiltonian evolution, we consider tuned repeated interactions, when the Hamiltonian part of dynamics is piecewise constant. Then for $t \in [(k-1)\tau, k\tau)$, the generator has the form:

$$\begin{aligned} L_{\sigma,k}(\rho(t)) &:= -i[H_k, \rho(t)] + \sigma_- \; b_0 \; \rho(t) \; b_0^* - \frac{\sigma_-}{2}\{b_0^* b_0, \rho(t)\} \\ &+ \sigma_+ \; b_0^* \; \rho(t) \; b_0 - \frac{\sigma_+}{2}\{b_0 b_0^*, \rho(t)\} \;\; (k \geqslant 1). \end{aligned} \tag{3.3}$$

The solution of the corresponding Cauchy problem

$$\partial_t \rho(t) = L_{\sigma}(t)(\rho(t)) \; , \; \rho(t)\big|_{t=0} = \rho_0 \otimes \bigotimes_{k=1}^{N} \rho_k, \tag{3.4}$$

has a form:

$$\rho(N\tau) = T^{\sigma}_{N\tau,0}(\rho(0)) = e^{\tau L_{\sigma,N}} \, ... \, e^{\tau L_{\sigma,2}} e^{\tau L_{\sigma,1}}(\rho(0)).$$

Let us use the notation:

$$T^{\sigma}_{k} := T^{\sigma}_{k\tau,(k-1)\tau} = e^{\tau L_{\sigma,k}}. \tag{3.5}$$

And we consider evolution of the Weyl operators, which is dual to the evolution of states

$$\mathrm{Tr}_{\mathscr{H}}[T^{\sigma}_{N\tau,0}(\rho) W(\zeta)] = \mathrm{Tr}_{\mathscr{H}}[\rho \; T^{\sigma \, *}_{N\tau,0}(W(\zeta))]. \tag{3.6}$$

Note that

$$T^{\sigma}_{N\tau,0} = e^{\tau L_{\sigma,N}} \, ... \, e^{\tau L_{\sigma,2}} e^{\tau L_{\sigma,1}}$$

and its dual evolution

$$T^{\sigma \, *}_{N\tau,0} = e^{\tau L^*_{\sigma,1}} e^{\tau L^*_{\sigma,2}} \, ... \, e^{\tau L^*_{\sigma,N}} \; . \tag{3.7}$$

3.1 Evolution of Open System

First we establish a formula for the one-step mappings in (3.7) of the Weyl operators.

Lemma 3.1.1 *Let $k, l = 0, 1, 2, \ldots, N$ and $n = 1, 2, \ldots, N$. Let vector $\zeta = \{\zeta_k\}_{k=0}^{N} \in \mathbb{C}^{N+1}$ be as in (2.1). Then we obtain*

$$T_n^{\sigma *}(W(\zeta)) := e^{tL^*_{\sigma,n}}(W(\zeta)) = \Omega_t^{\sigma,n}(\zeta) W(U_n^{\sigma}(t)\zeta) \ , \tag{3.8}$$

where

$$\Omega_t^{\sigma,n}(\zeta) = \exp\Big[-\frac{1}{4} \ \frac{\sigma_+ + \sigma_-}{\sigma_+ - \sigma_-} \big(\langle U_n^{\sigma}(t)\zeta, U_n^{\sigma}(t)\zeta \rangle - \langle \zeta, \zeta \rangle \big) \Big] \ , \tag{3.9}$$

$$U_n^{\sigma}(t) = \exp\Big[it\Big(Y_n - i \ \frac{\sigma_+ - \sigma_-}{2} P_0 \Big) \Big] \ , \qquad (P_0)_{kl} = \delta_{k0}\delta_{l0}. \tag{3.10}$$

Remark The main difference between the mapping for $\sigma_{\mp} = 0$ and (3.8), (3.10) is that the energy parameter (Lemma 2.0.1) has the shift:

$$E \ \rightarrow \ E_{\sigma} := E - i \ \frac{\sigma_+ - \sigma_-}{2} \ .$$

Note that $\mathrm{Im}(E_{\sigma}) > 0$, if $\sigma_+ < \sigma_-$.

Corollary 3.1.2

$$T^{\sigma \ *}_{N\tau,0}(W(\zeta)) = \exp\Big[-\frac{\sigma_+ + \sigma_-}{4(\sigma_+ - \sigma_-)} \big(\langle U_1^{\sigma}(\tau) \ldots U_N^{\sigma}(\tau) \ \zeta, U_1^{\sigma}(\tau) \ldots U_N^{\sigma}(\tau) \ \zeta \rangle - \langle \zeta, \zeta \rangle \big) \Big]$$
$$\times \ W(U_1^{\sigma}(\tau) \ldots U_N^{\sigma}(\tau) \ \zeta).$$

Combining the above Corollary and Lemma, we get the following theorem.

Theorem 3.1.3 *Let $\rho = \rho(t = 0)$ be initial condition which coincides with the Gibbs density matrix (**??**). Then we get*

$$\omega_{T^{\sigma}_{N\tau,0}\rho}(W(\zeta)) = \exp\Big[-\frac{1}{4} \langle \zeta, X^{\sigma}(N\tau)\zeta \rangle \Big],$$

where $X^{\sigma}(N\tau)$ is the $(N+1) \times (N+1)$ matrix given by

$$X^{\sigma}(N\tau) = U_N^{\sigma}(\tau)^* \ldots U_1^{\sigma}(\tau)^* \Big[\Big(\frac{\sigma_+ + \sigma_-}{\sigma_+ - \sigma_-} + \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) I + \Big(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) P_0 \Big]$$
$$\times U_1^{\sigma}(\tau) \ldots U_N^{\sigma}(\tau) \ - \ \frac{\sigma_+ + \sigma_-}{\sigma_+ - \sigma_-} I.$$

3.2 Limit of reduced density for the cavity

Hereafter, we use the notations:
$U_n^\sigma(t) = e^{it\epsilon}V_n^\sigma(t)$ and

$$(V_n^\sigma(t))_{jk} = \begin{cases} g^\sigma(t)z^\sigma(t)\,\delta_{k0} + g^\sigma(t)w^\sigma(t)\,\delta_{kn} & (j=0) \\ g^\sigma(t)w^\sigma(t)\,\delta_{k0} + g^\sigma(t)z^\sigma(-t)\,\delta_{kn} & (j=n) \\ \qquad\delta_{jk} & (\text{otherwise}) \end{cases}$$

with

$$g^\sigma(t) = e^{it(E_\sigma-\epsilon)/2}, \qquad w^\sigma(t) = \frac{2i\eta}{\sqrt{(E_\sigma-\epsilon)^2+4\eta^2}}\sin t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2}, \tag{3.11}$$

$$z^\sigma(t) = \cos t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2} + \frac{i(E_\sigma-\epsilon)}{\sqrt{(E_\sigma-\epsilon)^2+4\eta^2}}\sin t\sqrt{\frac{(E_\sigma-\epsilon)^2}{4}+\eta^2}. \tag{3.12}$$

Note the relation $z^\sigma(t)z^\sigma(-t) - w^\sigma(t)^2 = 1$ holds, but $z^\sigma(-t) \neq \overline{z^\sigma(t)}$ for $\sigma+ \neq \sigma-$.

We consider the system with initial product state (**??**)

$$\rho := \bigotimes_{k=0}^{N} \rho_k \quad \text{with} \quad \rho_1 = \rho_2 = \cdots = \rho_N, \tag{3.13}$$

where ρ_0, ρ_1 are density matrices on $\mathscr{F}$. We assume that ρ_1 is gaude invariant. For fixed ρ_1, we define one step evolution of the cavity state ρ_0 by

$$\mathcal{T}(\rho_0) = (T^\sigma_{\tau,0}\rho)_0, \tag{3.14}$$

where ρ is (3.13) and the subscript $(\ )_0$ in the righthand side represents the reduced density corresponding to the subsystem consists of the cavity only. The application of $\mathcal{T}$ can be expressed explicitly by the use of the expectation of the Weyl operator:

$$\omega_{\mathcal{T}(\rho_0)}(\hat{w}(\theta)) = \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_;}{\sigma_- - \sigma+}\big(1 - |g^\sigma(\tau)z^\sigma(\tau)|^2 - |g^\sigma(\tau)w^\sigma(\tau)|^2\big)\Big]$$

$$\times\omega_{\rho_0}\big(\hat{w}(e^{i\tau\epsilon}g^\sigma(\tau)z^\sigma(\tau)\theta\big)\omega_{\rho_1}\big(\hat{w}(e^{i\tau\epsilon}g^\sigma(\tau)w^\sigma(\tau)\theta)\big). \tag{3.15}$$

Then we get:

Proposition 3.2.1 *Suppose that*

$$E(\theta) := \prod_{k=0}^{\infty} \omega_{\rho_1}(\hat{w}(e^{i(k+1)\tau\epsilon} g^\sigma(\tau)^{(k+1)} z^\sigma(\tau)^k w^\sigma(\tau)\theta))$$
$$= \prod_{k=0}^{\infty} C_1(e^{i(k+1)\tau\epsilon} g^\sigma(\tau)^{(k+1)} z^\sigma(\tau)^k w^\sigma(\tau)\theta)$$

is convergent and continuous for all $\theta \in \mathbb{C}$. Then there is a unique state ρ_* on $\mathscr{A}(\mathscr{F})$ which satisfies

$$(i) \qquad \mathcal{T}(\rho_*) = \rho_*, \tag{3.16}$$
$$(ii) \qquad \forall \rho_0 \in \mathfrak{C}_1(\mathscr{F}) : \lim_{k\to\infty} \mathcal{T}^k(\rho_0) = \rho_*, \tag{3.17}$$
$$(iii) \qquad \lim_{N\to\infty} (T^\sigma_{N\tau,0}\rho)_s = (T^\sigma_{n\tau,0}\rho_{(*)})_s, \tag{3.18}$$

where in the third item, $\rho_{(*)}$ is (3.13) with $\rho_0 = \rho_*$ and the subscript s stands for the reduced density to the subsystem consists of the cavity and the n atoms near the cavity, see (2.2).

Moreover ρ_* have the expectation

$$\omega_{\rho_*}\big(\hat{w}(\theta)\big) = \exp\Big[-\frac{|\theta|^2}{4}\frac{\sigma_- + \sigma_;}{\sigma_- - \sigma+}\Big(1 - \frac{|g^\sigma(\tau)w^\sigma(\tau)|^2}{1-|g^\sigma(\tau)z^\sigma(\tau)|^2}\Big)\Big]E(\theta). \tag{3.19}$$

4 Summary

As a simple mathematical model for atomic beam passing through a cavity, we considered a system consists of harmonic oscillators.

We have studied the Hamiltonian evolution of the system by calculating the expectation values of Weyl operators, explicitly. For Gibbs initial states, we consider a relaxation phenomena of the sub-system arround the cavity. For initial product states, we saw the convergence to the Gibbsian density matrix in a certain scaling limit.

We also studied the Markovian evolution of the model. We gave a formula for the dual evolution of the Weyl operators, explicitly. For a certain initial product states, we gave the asysmptotic behavior of the states for subsystems around the cavity.

The detailed presentation of the subject and their proofs will be given in [TZ].

References

[AJP1] *Open Quantum Systems I, The Hamiltonian Approach*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1880**, Springer-Verlag, Berlin-Heidelberg 2006.

[AJPII] *Open Quantum Systems II, The Markovian Approach*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1881**, Springer-Verlag, Berlin-Heidelberg 2006.

[AJP3] *Open Quantum Systems III, Recent Developements*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1882**, Springer-Verlag, Berlin-Heidelberg 2006.

[BJM] L.Bruneau, A.Joye, and M.Merkli, Repeated interactions in open quantum systems, (May 14, 2013). Submitted to *J.Math.Phys.*

[NVZ] B. Nachtergaele, A. Vershynina, and V. A. Zagrebnov, Non-Equilibrium States of a Photon Cavity Pumped by an Atomic Beam, *Annales Henri Poincaré* (2013)

[TZ] H. Tamura and V.A.Zagrebnov, in preparation.