Rigorous numerics of global orbits for fast-slow systems

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Fast-slow system

\( \dot{x} = f(x, y, \epsilon) \)
\( \dot{y} = \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1 \)

\( x \in \mathbb{R}^n : \text{fast}, \quad y \in \mathbb{R}^k : \text{slow}, \quad t \in \mathbb{R} : \text{time} \)

ex. FitzHugh-Nagumo

\( u_t = \delta u_{xx} + f(u) - \lambda \)
\( \lambda_t = \epsilon (u - \gamma \lambda) \)
\( u(x, t) \mapsto u(x - \theta t) \)

\( \dot{u} = v \)
\( \dot{v} = \delta^{-1}(\theta v - f(u) + \lambda) \)
\( \dot{\lambda} = \epsilon \theta^{-1}(u - \gamma \lambda) \)

Multiscale Problems in e.g. Materials Science, Life Science.
Fast-slow system

ex. FitzHugh-Nagumo

\[ \dot{u} = v \]
\[ \dot{v} = \delta^{-1}(\theta v - f(u) + \lambda) \]
\[ \dot{\lambda} = \epsilon \theta^{-1} u \]

- \( f \) : cubic nonlinearity
- s.t. \( f(0) = f(1) = 0 \)

\( \epsilon = 0 \) : \( \{(u,v,\lambda)| v = 0, \theta v - f(u) + \lambda = 0\} \) is a family of equilibria (nullclines)

\( \epsilon > 0 \) : \((0,0,0)\) is the only equilibrium.

\( \epsilon = 0 \) : heteroclinic orbits and critical manifolds by nullclines

\( \epsilon > 0 \) : Sufficiently Small homoclinic orbits
Fast-slow system

ex. FitzHugh-Nagumo

\[ \dot{u} = v \]
\[ \dot{v} = \delta^{-1}(\theta v - f(u) + \lambda) \]
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\( \epsilon > 0 : (0,0,0) \) is the only equilibrium.
Goal: Produce the validation method for the existence of global orbits for \( \text{given } \epsilon \) as the continuation of singular limit orbits for fast-slow systems.

\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) \\
\dot{y} &= \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1
\end{align*}
\]

1. Slow Dynamics
2. Fast Dynamics
3. Matching

Key: Solve each scaled problem independently and match them.
Preceding works (examples)

Connecting Orbits + Rigorous Numerics

Rigorous numerics of horseshoes, Shi’lnikov orbits and N-pulse solutions via covering relations

Rigorous numerics of connecting orbits via Radii Polynomials + Parametrization

Singular Perturbation + Rigorous Numerics


Singularly perturbed Conley index → horseshoes in fast-slow systems
("sufficiently close ε")

Examples of interval arithmetics libraries: INTLAB, PROFIL, CAPD
1. Slow Dynamics
2. Fast Dynamics
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)
1. Slow Dynamics
2. Fast Dynamics
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)
**Slow manifold**

\( \varepsilon = 0 \)

\[
\begin{align*}
\dot{x} &= f(x, y, 0) \\
\dot{y} &= 0
\end{align*}
\]

\[
M_0 \subset \{ f(x, y, 0) = 0 \}
\]

(invariant)

\( \varepsilon \in (0, \varepsilon_0] \)

\[
\begin{align*}
\dot{x} &= f(x, y, \varepsilon) \\
\dot{y} &= \varepsilon g(x, y, \varepsilon)
\end{align*}
\]

\[
M_\varepsilon
\]

(locally invariant)
Slow manifold

\( \epsilon \in (0, \epsilon_0] \)

\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) \\
\dot{y} &= \epsilon g(x, y, \epsilon)
\end{align*}
\]

**Expression of Stable and Unstable Manifolds**

\[
\begin{align*}
\lim_{t \to -\infty} x(t; \lambda) &= p, \\
\lim_{t \to +\infty} x(t; \lambda) &= q.
\end{align*}
\]

How can we verify the infinite-time behavior mathematically with finitely many memories?

Where is the slow manifold?

Is it really perturbed from \( M_0 \)?

Which is the direction of (un)stable manifolds?
Validation of slow manifolds

Invariant Manifold Theorem [Fenichel, 1979]

If the critical manifold $M_0$ is normally hyperbolic at $\varepsilon = 0$, then for sufficiently small $\varepsilon$, $W^u(M_\varepsilon)$ and $W^s(M_\varepsilon)$ can be defined by graphs of smooth functions $b = h_u(a, y, \varepsilon)$ and $a = h_s(b, y, \varepsilon)$, respectively ($a$ : fast unstable var., $b$ : fast stable var.).

Diagonalize at a point

$$\begin{align*}
\dot{a} &= Aa + F_1(a, b, y, \varepsilon) & \text{Spec}(A) \subset \{\text{Re}\lambda > 0\}, & \text{Spec}(B) \subset \{\text{Re}\lambda < 0\} \\
\dot{b} &= Bb + F_2(a, b, y, \varepsilon) & F_1, F_2 &= o(|a|, |b|) \\
\dot{y} &= \varepsilon g(a, b, y, \varepsilon)
\end{align*}$$

$K \subset \mathbb{R}^k$ : cpt, convex

$B = B_1 \times B_2 \subset \mathbb{R}^n$ : cpt, convex

s.t.

$f(x, y, \varepsilon) \cdot \nu_{\partial B_1} > 0$ on $\partial B_1 \times B_2 \times K \times [0, \epsilon_0]$, $f(x, y, \varepsilon) \cdot \nu_{\partial B_2} < 0$ on $B_1 \times \partial B_2 \times K \times [0, \epsilon_0]$

(Fast-saddle-type Block. $a$ : unstable coord., $b$ : stable coord.)
Validation of slow manifolds

\[ K \subset \mathbb{R}^k : \text{cpt, convex} \]
\[ B = B_1 \times B_2 \subset \mathbb{R}^n : \text{cpt, convex} \]


Define Maximal Singular Values of matrices:

\[ \sigma^s_{A_1} : A_1(z) = \left( \frac{\partial F_1}{\partial a}(z) \right), \quad \sigma^s_{A_2} : A_2(z) = \left( \frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right), \]
\[ \sigma^s_{B_1} : B_1(z) = \left( \frac{\partial F_2}{\partial a}(z) \right), \quad \sigma^s_{B_2} : B_2(z) = \left( \frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z) \right) \]
\[ \sigma^s_{g_1} : g_1(z) = \left( \frac{\partial g}{\partial a}(z) \right), \quad \sigma^s_{g_2} : g_2(z) = \left( \frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z) \right) \]

Assume the following inequalities (stable cone conditions):

\[ \inf \text{Spec}(A) - \left( \sup \sigma^s_{A_1} + \sup \sigma^s_{A_2} \right) > 0, \]
\[ \inf \text{Spec}(A) + \inf |\text{Spec}(B)| - \left\{ \sup \sigma^s_{A_1} + \sup \sigma^s_{A_2} + \sup \sigma^s_{B_1} + \sup \sigma^s_{B_2} + \epsilon_0 \left( \sup \sigma^s_{g_1} + \sup \sigma^s_{g_2} \right) \right\} > 0, \]

Then for all \( \epsilon \in [0, \epsilon_0] \) \( W^s(M_\epsilon) \cap (B \times K) \) can be represented by the graph of a Lipschitz function on \( B_2 \times K \). The similar statement holds for \( W^u(M_\epsilon) \cap (B \times K) \).

The slow manifold \( M_\epsilon \) is the k-dimensional submanifold in \( B \times K \) can be represented by their intersection. In particular, \( M_0 \) is normally hyperbolic.
Validation of slow manifolds

Fast-saddle-type blocks:
Slow manifold exists somewhere in the block.
The size of this block corresponds to the rigorous error between approximate and rigorous slow manifolds.

Cone conditions:
(Un)stable manifolds of slow manifolds have graph representations on (un)stable coordinates in blocks.
Exit contains a point of unstable manifolds.
Enterance contains a point of stable manifolds.

Rigorous bound of manifolds can be explicitly estimated via rigorous numerics!
Requirements: inner product and singular values.
Towards rigorous numerics

Key. Fast-saddle-type block, Cone condition

Blocks: Zgliczynski-Mischaikow (FoCM, 2001)
Cone condition, construction of Lyapunov functions:
Ref.: Zgliczynski (2009), M. (NOLTA, 2013)

Lyapunov function + Implicit Function Theorem → normal hyperbolicity
1. Slow Dynamics

2. Fast Dynamics


4. m-cones

5. Towards Validation -- overview (FitzHugh-Nagumo)
Covering relations

Def. [h-sets, Zgliczynski-Gidea (2002)]

**h-set** is the 4-tuple of the following:

\[ N \subset \mathbb{R}^n : \text{A compact set} \]
\[ u(N), s(N) \in \mathbb{Z}_{\geq 0} \text{ s.t. } u(N) + s(N) = n \]
\[ c_N : \mathbb{R}^n \to \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} : \text{A homeomorphism s.t.} \]

\[ c_N(N) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}. \]

- **Ex.**: \( u(N)=1, \ s(N)=2 \)

\[ N_c := \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \]
\[ N^-_c := \partial B_{u(N)} \times \overline{B_{s(N)}}, \]
\[ N^+_c := \overline{B_{u(N)}} \times \partial B_{s(N)}, \]
\[ N^- := c_N^{-1}(N^-_c), \quad N^+ := c_N^{-1}(N^+_c). \]

- **Ex.**: \( u(N)=2, \ s(N)=1 \)
Covering relations

Def. [Covering Relation, Zgliczynski-Gidea (2002)]

\[ N, M : \text{h-sets}, \quad f : N \to \mathbb{R}^{\dim M} \quad u(N) = u(M) \]

Define \( N \overset{f}{\rightarrow} M \) (\( N \) f-covers \( M \)) by

1. There is a homotopy \( h : [0, 1] \times N_c \to \mathbb{R}^{\dim M} \) such that
   \[ h_0 = f_c, \quad f_c := c_M \circ f \circ c_N^{-1}, \]
   \[ h([0, 1], N^-) \cap M_c = \emptyset, \]
   \[ h([0, 1], N) \cap M_c^+ = \emptyset, \]
2. There is a linear map \( A : \mathbb{R}^u \to \mathbb{R}^u \) such that
   \[ h_1(p, q) = (A(p), 0), \]
   \[ A(\partial B_u(0, 1)) \subset \mathbb{R}^u \setminus \overline{B_u(0, 1)} \]
Covering relations


Let \( \{M_k\}_{k=1}^n \) : sequence of h-sets, \( u(M_1) = u(M_2) = \cdots = u(M_k) \)

\( f_k : M_k \to \mathbb{R}^{\dim M_{k+1}} \) : continuous

Assume \( M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} M_k \).

Then

\[ \exists x \in M_1 \text{ s.t. } f_{i \circ \cdots \circ f_1}(x) \in \text{int} M_{i+1}, \quad i = 1, \cdots, k-1. \]
Covering relations


Let \( \{M_k\}_{k=1}^n \) : sequence of h-sets, \( u(M_1) = u(M_2) = \cdots = u(M_k) \)

\[ f_k : M_k \rightarrow \mathbb{R}^{\dim M_{k+1}} \] : continuous

Assume \( M_1 \xRightarrow{f_1} M_2 \xRightarrow{f_2} \cdots \xRightarrow{f_{k-1}} M_k \).

Then

\[ \exists x \in M_1 \text{ s.t. } f_i \circ \cdots \circ f_1(x) \in \text{int}M_{i+1}, \quad i = 1, \cdots, k - 1. \]
Towards rigorous numerics

ODE Solver: Lohner method.


Solve ODE with the red face as the initial data

⇔

“Computation of unstable manifold”

Covering relations for verifying orbits
“Matching”

Is there a point in a neighborhood of heteroclinic orbits, near slow manifolds and another fast jump?

Mathematically known:
Exchange Lemma (Jones-Kopell 1994, etc.)
1. Slow Dynamics
2. Fast Dynamics
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)
Covering-Exchange property

\[ (*)_{\epsilon} \]
\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) \\
\dot{y} &= \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1
\end{align*}
\]

\[ x \in \mathbb{R}^n : \text{fast, } y \in \mathbb{R}^k : \text{slow, } t \in \mathbb{R} : \text{time} \]

From now on assume the following:

\[ \dot{y} = \epsilon g(x, y, \epsilon) \text{ can be represented by} \]

\[ y = (w, \theta_1, \cdots, \theta_{k-1}) \in \mathbb{R}^k, \]
\[ \dot{w} = \epsilon g_1(x, y, \epsilon), \]
\[ \dot{\theta}_i = 0. \]
Covering-Exchange property

Def. (Covering-Exchange)

\[ N \subset \mathbb{R}^{u+s+k} : h\text{-set}, \ M \subset \mathbb{R}^{u+s+k} : (u + s + k)\text{-dim. } h\text{-set} \]

We say that \( N \) satisfies the covering-exchange property (CE) with respect to \( M \) for \((\ast)_\epsilon\) if

1. \( M \) is a fast-saddle-type block.
2. \( M \) satisfies stable and unstable cone conditions.
3. For \( q \in \{ \pm 1 \} \)
   \[ q \cdot g_1(x, y, \epsilon) > 0 \text{ in } M. \]
4. Letting \( \varphi_\epsilon \) be the flow of \((\ast)_\epsilon\), for some \( T > 0 \)
   \[ N \xrightarrow{\varphi_\epsilon(T, \cdot)} M. \]

We say the pair \((N, M)\) a covering-exchange pair.
Covering-Exchange property

Dynamics of Covering-Exchange pairs

1. M is a fast-saddle-type block.
2. M satisfies stable and unstable cone conditions.
3. For \( q \in \{ \pm 1 \} \), \( q \cdot g_1(x, y, \epsilon) > 0 \) in \( M \).
4. Letting \( \varphi_\epsilon \) be the flow of \((\ast)_\epsilon\), for some \( T > 0 \), \( N \xrightarrow{\varphi_\epsilon(T, \cdot)} M \).

It comes from \( N \).

Topologically describes orbits colored by red.

After sufficiently long time...
Fast-exit face and admissibility

Def. (Fast-exit face)

Define a **fast-exit face** of a fast-saddle-type block $M$ by

$$M^a := c_M^{-1} \left( \{a\} \times \overline{B}_s \times (w^-, w^+) \times \prod_{i=2}^{k} [-1, 1] \right), \quad a \in \partial B_u.$$ 

where

$$M_c = \overline{B}_u \times \overline{B}_s \times [-1, 1] \times \prod_{i=2}^{k} [-1, 1]$$

Def. (admissibility)

$\tilde{M} \subset M$ : h-set satisfying 1~3 of (CE) and $M_0 \subset M$ : a fast-exit face are said to be **admissible in** $M$ if

$$M_0 \cap \tilde{M} = \emptyset, \quad u(M_0) = u(\tilde{M}),$$

The $u(M_0)$ -component of $M_0$ contains $w$-coordinate.

If $q=+1$, $\inf \pi_w(M_0)_c - \sup \pi_w(\tilde{M})_c > 0$.

If $q=-1$, $\inf \pi_w(\tilde{M})_c - \sup \pi_w(M_0)_c > 0$. 
Singular limit connecting orbits and their continuation


For the fast-slow system \((\ast)_\epsilon\) assume that, for given \(\epsilon_0 > 0\) and \(\rho \in \mathbb{N}\) there is an \(\epsilon (\in [0, \epsilon_0])\)-parameter family of the following sets:

\(\mathcal{S}_\epsilon^j : (j=0, \ldots, \rho)\) fast-saddle-type block which forms a covering-exchange pair with \(\mathcal{F}_{\epsilon}^{j-1}\) (\(\mathcal{F}_{\epsilon}^{\rho}\) if \(j = 0\)).

\(\tilde{\mathcal{S}}_\epsilon^j : (j=0, \ldots, \rho)\) fast-saddle-type block which forms a covering-exchange pair with \(\mathcal{F}_{\epsilon}^{j-1}\) and the pair \((\tilde{\mathcal{S}}_\epsilon^j, \mathcal{F}_{\epsilon}^j)\) forms an admissible pair in \(\mathcal{S}_\epsilon^j\).

\(\mathcal{F}_{\epsilon}^j : (j=0, \ldots, \rho)\) a fast-exit face of \(\mathcal{S}_\epsilon^j\).

Then for all \(\epsilon \in (0, \epsilon_0]\) there is a periodic orbit for \((\ast)_\epsilon\) which passes all \(\mathcal{S}_\epsilon^j\).
Singular limit connecting orbits and their continuation


For the fast-slow system \((*)_\varepsilon\) assume that, for given \(\varepsilon_0 > 0\) and \(\rho \in \mathbb{N}\) there is an \(\varepsilon (\in [0, \varepsilon_0])\)-parameter family of the following sets:

\[ S^j_\varepsilon : (j=0, \ldots, \rho) \text{ fast-saddle-type block} \]

\((j=1, \ldots, \rho-1)\) fast-saddle-type block which forms a CE pair with \(F^{j-1}_\varepsilon\).

\((i=0, \rho)\) invariant sets \(S_{\varepsilon,u}, S_{\varepsilon,s}\) are contained there, respectively.

\[ \tilde{S}^j_\varepsilon : (j=0, \ldots, \rho) \text{ fast-saddle-type block} \]

\((j=1, \ldots, \rho)\) fast-saddle-type block which forms a CE pair with \(F^{j-1}_\varepsilon\) and the pair \((\tilde{S}^j_\varepsilon, F^j_\varepsilon)\) forms an admissible pair in \(S^j_\varepsilon\).

\[ F^j_\varepsilon : (j=0, \ldots, \rho-1) \text{ a fast-exit face of } S^j_\varepsilon \]

\((j=0)\) there is an intersection with \(W^u(S_{\varepsilon,u})\).

Then for all \(\varepsilon \in (0, \varepsilon_0]\) there is a heteroclinic orbit for \((*)_\varepsilon\) connecting \(S_{\varepsilon,u}\) and \(S_{\varepsilon,s}\) which passes all \(S^j_\varepsilon\).
Singular limit connecting orbits and their continuation

**Idea of the proof** (in the case of Periodic orbits)

\[ \Pi := (\tilde{S}_\epsilon^0)_c \times (\mathcal{F}_\epsilon^0)_c \times (\tilde{S}_\epsilon^1)_c \times (\mathcal{F}_\epsilon^1)_c \times \cdots \times (\tilde{S}_\epsilon^\rho)_c \times (\mathcal{F}_\epsilon^\rho)_c \]
\[ \subset \mathbb{R}^{d_s^0} \times \mathbb{R}^{d_f^0} \times \mathbb{R}^{d_s^1} \times \mathbb{R}^{d_f^1} \times \cdots \times \mathbb{R}^{d_s^\rho} \times \mathbb{R}^{d_f^\rho}. \]

→ Prove that the **mapping degree** \( \operatorname{deg}(F_\epsilon, \Pi, 0) \) of the map below can be defined and is nonzero:

\[
F_\epsilon \left( \begin{array}{c}
(p_s^0, q_s^0) \\
(p_f^0, q_f^0) \\
(p_s^1, q_s^1) \\
(p_f^1, q_f^1) \\
\vdots \\
(p_s^\rho, q_s^\rho) \\
(p_f^\rho, q_f^\rho)
\end{array} \right) := \left( \begin{array}{c}
(p_f^0, q_f^0) - \pi^0 \circ (P_\epsilon^0)_c (p_s^0, q_s^0) \\
(p_s^0, q_s^1) - (\varphi_\epsilon(T^0, \cdot))_c (p_f^0, q_f^0, (\pi^0)_c \circ (P_\epsilon^0)_c (p_s^0, q_s^0)) \\
(p_f^1, q_f^1) - \pi^1 \circ (P_\epsilon^1)_c (p_s^1, q_s^1) \\
(p_s^1, q_s^2) - (\varphi_\epsilon(T^1, \cdot))_c (p_f^1, q_f^1, (\pi^1)_c \circ (P_\epsilon^1)_c (p_s^1, q_s^1)) \\
\vdots \\
(p_f^\rho, q_f^\rho) - \pi^\rho \circ (P_\epsilon^\rho)_c (p_s^\rho, q_s^\rho) \\
(p_s^\rho, q_s^0) - (\varphi_\epsilon(T^\rho, \cdot))_c (p_f^\rho, q_f^\rho, (\pi^\rho)_c \circ (P_\epsilon^\rho)_c (p_s^\rho, q_s^\rho))
\end{array} \right). \]

Components involving (un)stable manifolds are added in the case of heteroclinic orbits.
Towards rigorous numerics

**Key. Covering-Exchange**

- Blocks and Cone conditions: Already stated.
- Covering Relation: Already stated.
- Sign of vector fields: Easy!
- Fast-exit face + Admissibility: Easy!

Nothing new for rigorous numerics!
Practical Computations

\[ \dot{u} = v \]
\[ \dot{v} = 0.2(\theta v - f(u) + \lambda) \]
\[ \dot{\lambda} = \epsilon \theta^{-1} u \]

\[ f(u) = u(u - 0.2)(1 - u), \quad \theta \in [0.947, 0.948], \quad \epsilon \in [0, 10^{-5}] \]

Total orbit: \( dt = 0.001, \ t = 0 \sim 190 \)

- Blocks are chosen small in order to get a good estimate of manifolds.

- Rigorous numerics encloses the error of global orbits in each step and become bigger and bigger!

Left: Enclosure of orbits is already larger than the block!

Validations without any ideas are so crazy!
1. Slow Dynamics
2. Fast Dynamics
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)
m-cones

Extend (un)stable manifolds making sharp cones.

cone: $|x| > |y|

m-cone: $|x| > m|y|

Isolating blocks
- Very small in general.
- Where the unstable manifold extends? (cone: orange domain)
- Flow moves very slowly near fixed points
→ Increase of computation costs.

Cones, m-cones
- Unstable manifold is contained in cones
→ Be cones sharper and raise the accuracy of the unstable manifold.
- Away from equilibria.
- isolation is preserved.
m-cones

Cone condition for fast-slow system.


Define **Maximal Singular Values** of matrices:

\[
\sigma_{A_1}^s : A_1(z) = \left( \frac{\partial F_1}{\partial a}(z) \right), \quad \sigma_{A_2}^s : A_2(z) = \left( \frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right),
\]

\[
\sigma_{B_1}^s : B_1(z) = \left( \frac{\partial F_2}{\partial a}(z) \right), \quad \sigma_{B_2}^s : B_2(z) = \left( \frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z) \right),
\]

\[
\sigma_{g_1}^s : g_1(z) = \left( \frac{\partial g}{\partial a}(z) \right), \quad \sigma_{g_2}^s : g_2(z) = \left( \frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z) \right).
\]

Assume the following inequalities (**stable cone conditions**):

\[
\inf \text{Spec}(A) - \left( \sup \sigma_{A_1}^s + \sup \sigma_{A_2}^s \right) > 0,
\]

\[
\inf \text{Spec}(A) + \inf |\text{Spec}(B)| - \left\{ \sup \sigma_{A_1}^s + \sup \sigma_{A_2}^s + \sup \sigma_{B_1}^s + \sup \sigma_{B_2}^s + \epsilon_0 \left( \sup \sigma_{g_1}^s + \sup \sigma_{g_2}^s \right) \right\} > 0,
\]

Then for all \( \epsilon \in [0, \epsilon_0] \) \( W^s(M_\epsilon) \cap (B \times K) \) can be represented by the graph of a Lipschitz function on \( B_2 \times K \). The similar statement holds for \( W^u(M_\epsilon) \cap (B \times K) \). The slow manifold \( M_\epsilon \) is the k-dimensional submanifold in \( B \times K \) can be represented by their intersection. In particular, \( M_0 \) is normally hyperbolic.
m-cones

Stable m-cone condition for fast-slow system.

Thm. [M., cf. M.-Yamamoto]

Let B, K as above.

Define **Maximal** Singular Values of matrices:

\[
\sigma^{s,m}_{A_1}: A_1(z) = \left( \frac{\partial F_1}{\partial a}(z) \right), \quad \sigma^{s,m}_{A_2}: A_2(z) = m^{-1} \left( \frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right),
\]

\[
\sigma^{s,m}_{B_1}: B_1(z) = m \left( \frac{\partial F_2}{\partial a}(z) \right), \quad \sigma^{s,m}_{B_2}: B_2(z) = \left( \frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z) \right),
\]

\[
\sigma^{s,m}_{g_1}: g_1(z) = m \left( \frac{\partial g}{\partial a}(z) \right), \quad \sigma^{s,m}_{g_2}: g_2(z) = \left( \frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z) \right).
\]

Assume the following inequalities (**stable m-cone conditions**):

\[
\inf \text{Spec}(A) - (\sup \sigma^{s,m}_{A_1} + \sup \sigma^{s,m}_{A_2}) > 0,
\]

\[
\inf \text{Spec}(A) + \inf |\text{Spec}(B)| - \left\{ \sup \sigma^{s,m}_{A_1} + \sup \sigma^{s,m}_{A_2} + \sup \sigma^{s,m}_{B_1} + \sup \sigma^{s,m}_{B_2} + \sigma \left( \sup \sigma^{s,m}_{g_1} + \sup \sigma^{s,m}_{g_2} \right) \right\} > 0,
\]

Then the function \( M(t) := |\Delta a(t)|^2 - m^2 |\Delta \zeta(t)|^2 (\zeta = (b, y)) \) satisfies:

\( M'(t) > 0. \) holds on the set \( M(t) = 0 \) as long as orbits stay B×K.
with m-cones ...

\[ \dot{u} = v \]
\[ \dot{v} = 0.2(\theta v - f(u) + \lambda) \]
\[ \dot{\lambda} = \epsilon \theta^{-1} u \]

\[ f(u) = u(u - 0.2)(1 - u), \]
\[ \theta \in [0.947, 0.948], \epsilon \in [0, 10^{-5}] \]

\[ \lambda \in [-0.00242308, 0.00242308] \]

Total orbit: \( dt = 0.001, t = 0 \sim 190 \)

- Unstable m-cone: orbits leaves a neighborhood of slow manifolds in a short time.
  → prevent error accumulations

- Stable m-cone: blocks for verifying covering relations become larger.

Verifications become dramatically easy !!
1. Slow Dynamics
2. Fast Dynamics
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)
Homoclinic orbits of the FitzHugh-Nagumo system -- overview

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= 0.2(\theta v - f(u) + \lambda) \\
\dot{\lambda} &= \epsilon \theta^{-1} u \\
f(u) &= u(u - 0.2)(1 - u), \\
\theta &\in [0.947, 0.948], \ \epsilon \in [0, 10^{-5}]
\end{align*}
\]

Computation environment

Library: CAPD (http://capd.ii.uj.edu.pl) 3.0
CPU: 1.6GHz Intel Core i5 (Macbook Air 2011 model)
Memory: 4GB 1333 MHz DDR3

1. 1st branch

We can construct fast-saddle-type blocks satisfying cone conditions for 
\(\lambda \in [-0.0005, 0.1]\) around green branch.

2. 3rd branch

We can construct fast-saddle-type blocks satisfying cone conditions for 
\(\lambda \in [-0.0005, 0.1]\) around blue branch.
Homoclinic orbits of the FitzHugh-Nagumo system -- overview

\[ \dot{u} = v \]
\[ \dot{v} = 0.2(\theta v - f(u) + \lambda) \]
\[ \dot{\lambda} = \epsilon \theta^{-1} u \]

\[ f(u) = u(u - 0.2)(1 - u), \quad \theta \in [0.947, 0.948], \quad \epsilon \in [0, 10^{-5}] \]

Total orbit: \( dt = 0.001, \ t = 0 \sim 190 \)

3. Fast trajectory from \( (u, v, \lambda) \approx (0, 0, 0) \)

\[ \lambda \in [-0.00242308, 0.00242308] \]
\[ \lambda = -0.00242308 \]
\[ \lambda = +0.00242308 \]
Homoclinic orbits of the FitzHugh-Nagumo system -- overview

\[ \dot{u} = v \]
\[ \dot{v} = 0.2(\theta v - f(u) + \lambda) \]
\[ \dot{\lambda} = \epsilon \theta^{-1} u \]

\[ f(u) = u(u - 0.2)(1 - u), \]
\[ \theta \in [0.947, 0.948], \quad \epsilon \in [0, 10^{-5}] \]

Total orbit: \( dt = 0.001, \ t = 0 \sim 190 \)

4. Fast trajectory from \((u, v, \lambda) \approx (0.8, 0, 0.0955)\)

\[ \lambda \in [0.0929167, 0.0980833] \]
\[ \lambda = 0.0929167 \]
\[ \lambda = 0.0980833 \]
Homoclinic orbits of the FitzHugh-Nagumo system -- overview

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= 0.2(\theta v - f(u) + \lambda) \\
\dot{\lambda} &= \epsilon \theta^{-1} u
\end{align*}
\]

\[f(u) = u(u - 0.2)(1 - u), \quad \theta \in [0.947, 0.948], \quad \epsilon \in [0, 10^{-5}]\]

**Computer Assisted Result [M.]**

There exist the following trajectories of the FitzHugh-Nagumo system:

1. \(\epsilon = 0\): A singular homoclinic orbit consisting of two components of nullcline and two heteroclinic orbits connecting them.

2. \(\epsilon \in (0, 10^{-5}]\): homoclinic orbit of \((u, v, \lambda) = (0, 0, 0)\) as the continuation of the singular orbit obtained in 1.
Conclusion

- **Slow Dynamics**: proposed a sufficient condition for validating slow manifolds and dynamics around them.
- **Matching**: topologically described the matching of dynamics in different time scales.

→ Sample validation of singular perturbation problem.

*Periodic, Heteroclinic*: computing.

**Further directions**:

- Other examples (multi-slow variables)
- Slow manifolds containing non-hyperbolic points like **fold points**
- Transversality (via Exterior Algebra)

Ex.: Double-pulse in the FitzHugh-Nagumo sys.
Guckenheimer-Kuehn, SIADS(2009) →