Block Spin Transformation of 2D O(N) sigma model, Toward solving a Millennium Problems

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2015 Mar 06, Courant Inst.

From HomePage of Clay Institute

1. Construction of 4D YM Field Theory (Jaffe, Witten)
2. Solution of Navier-Stokes Equation (Feffermann)

What kind of Analysis do we need in these problems ?

Difficulties in 4D LGT, 2D Sigma and NvS Eq

1. The system is non-linear. Difficult to find linear part (or Gaussian part)
2. There appear relevant terms (increasing coupling constants by naive scaling or by BST)

History of 2D Spin Systems

2D O(N)Spin Model is simple, but hard to analyze.

1. 2D Ising spin, existence of spontaneous magnetization, R.Peierls (1936), L.Onsager (1944)
2. Kosterlitz-Thouless Transition in 2D XY model, J.Fröhlich and T.Spencer (1982)
3. non-existence of phase transition in the Heisenberg model with large N ($\sim$ quark confinement in YM_4) (this talk)

The Model

The 2D $O(N)$ Heisenberg model, $\phi(x) \in S^{N-1}$:

$$\langle \cdots \rangle = \int (\cdots) \exp[\sum_{n.n.} \phi_x \phi_y] \prod_x \delta(\phi^2(x) - N\beta) d^N \phi_x$$

$$= \int (\cdots) \exp[\sum_{n.n.} \phi_x \phi_y - \frac{g_0}{2N} \sum_x \left(\phi^2(x) - N\beta\right)^2] \prod_{x \in \Lambda} d^N \phi_x$$

where $\phi(x) = (\phi_1(x), \cdots, \phi_N(x))$ and x, y are lattice points $x, y \in \Lambda \subset Z^2$. Typical double-well potential $\left(\phi^2(x) - N\beta\right)^2$:

Gibbs measure:

$$\langle f(\phi)\rangle = \int f(\phi)\exp[-W_0(\phi)]\prod_x d^N\phi(x)$$

$$W_0 = \frac{1}{2}\langle \phi, (-\Delta + m_0^2)\phi\rangle + \frac{g_0}{2N}\langle :\phi^2:_G, :\phi^2:_G\rangle$$

$$:\phi^2:_G(x) = \sum_{i=1}^N \phi_i^2(x) - NG(0), \quad \beta = G(0)$$

$$(-\Delta)_{xy} = 4\delta_{xy} - \delta_{1,|x-y|}, \quad \text{Lattice Laplacian}$$

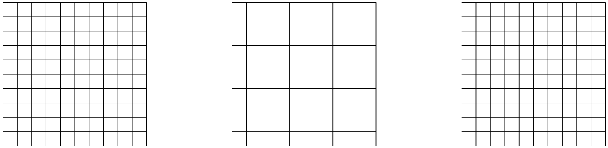
Here $G(0) = \beta$ means $m_0^2 \sim 32e^{-4\pi\beta}$:

$$G(x) = \frac{1}{-\Delta + m_0^2}(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ipx}}{m_0^2 + 2\sum(1-\cos p_i)}\prod\frac{dp_i}{2\pi}$$

Set

$$
\begin{aligned}
G_0(x,y) &= \frac{1}{-\Delta + m_0^2}(x,y) \\
G_n(x,y) &= \frac{1}{L^4}\sum_{\zeta,\xi\in\Delta_0} G_{n-1}(Lx+\zeta, Ly+\xi) \\
\phi_n(x) &= (C\phi_{n-1})(x) = \frac{1}{L^2}\sum_{\zeta\in\Delta_0}\phi_{n-1}(Lx+\zeta)
\end{aligned}
$$

C =Block Spin Operator
= Arithmetic average ($L^{-2}\sum$) +scaling ($Lx \to x$):



Then

$$\langle \phi_n(x)\phi_n(y)\rangle = G_n(x,y)$$

We find matrices A_{n+1} and Q (by Gaw-Kupi) such that

$$\begin{aligned} \phi_n(x) &= \underbrace{A_{n+1}\phi_{n+1}}_{\text{averaged spin}}(x) + \underbrace{Q\xi_n}_{\text{zero-average fluct.}}(x) \\ \langle \phi_n, G_n^{-1}\phi_n\rangle_{\Lambda_n} &= \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1}\rangle_{\Lambda_{n+1}} + \langle \xi, Q^+ G_n^{-1} Q\xi\rangle_{\Lambda_n} \end{aligned}$$

where $\Lambda_n = L^{-n} \cap \Lambda$ and

$$CA_{n+1} = 1, \quad CQ = 0$$
$$A_{n+1} = G_n C^+ G_{n+1}^{-1} : R^{\Lambda_{n+1}} \longrightarrow R^{\Lambda_n}$$
$$Q : R^{\Lambda_n'} \rightarrow R^{\Lambda_n}$$

Thus we decompose $\langle\phi,(-\Delta+m_0^2)\phi\rangle$ into many Gaussians $\{z_n\}$ with covariances $\Gamma_n = Q^+G_n^{-1}Q$:

$$\begin{aligned}
&\langle\phi,(-\Delta+m_0^2)\phi\rangle = \langle\phi, G_0^{-1}\phi\rangle \\
&= \langle\phi_1, G_1^{-1}\phi_1\rangle + \langle z_0, \underbrace{Q^+G_0^{-1}Q}_{\Gamma_0^{-1}} z_0\rangle \\
&= \langle\phi_2, G_2^{-1}\phi_2\rangle_{\Lambda_2} + \langle z_1, \underbrace{Q^+G_1^{-1}Q}_{\Gamma_1^{-1}} z_1\rangle_{\Lambda_1} + \langle z_0, \underbrace{Q^+G_0^{-1}Q}_{\Gamma_0^{-1}} z_0\rangle_{\Lambda_0}
\end{aligned}$$

where

$$\Gamma_n^{-1} \equiv Q^+G_n^{-1}Q^+ > O(1) \text{ on } QR^{\Lambda_n'}$$

The zero-ave fluctuationa are of short range:

$$Qz_n = Q\Gamma_n^{1/2}\xi_n = \text{block average zero fluctuations}$$
$$\text{where } \xi_n \text{ obeys } \mathrm{N}(1,0)$$

where

$$\Gamma_n(x,y) \sim \exp[-c|x-y|]$$

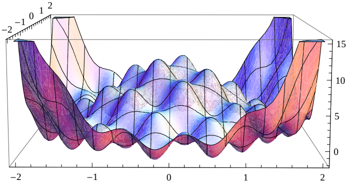
This kills long-range spin waves:

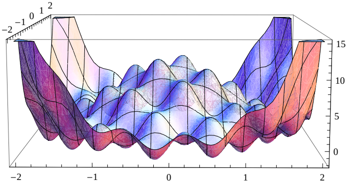
$$G_n(x,y) \sim \beta_n - \log|x-y|$$

Fluctuations $z_n = Q\xi_n$ influenced by Double-Wells

$$\exp\left[-\frac{g}{N}((\phi_{n+1}+z_n)^2 - N\beta_n)^2\right] \text{ with } |\phi_{n+1}^2 - N\beta_{n+1}| = O(1)$$

means z_n is $\perp \phi_{n+1}$:





Fluctuations $\xi_n(x)$ are strongly influenced by block spins. I.e., they can live only on the bottom of bottles, and $\xi_n(x)$ propagate along the direction orthogonal to ϕ_{n+1}.

BST=Perturbation around the Gaussians, but not in the present case since ϕ_{n+1} changes:

C leaves the fundamental Gaussian measures invariant.

$$G_n(x,y) = (CG_{n-1}C^{+})(x,y) \sim G_0(x,y)$$

Can we expect W_n keeps its main terms invariant under the influence of domain walls ?

$$\begin{aligned} W_n(\phi_n,\psi_n) &= \frac{1}{2}\langle\phi_n, G_n^{-1}\phi_n\rangle + \frac{g_n}{2N}\langle\phi_n^2 :_{G_n}, \phi_n^2 :_{G_n}\rangle \\ &\quad + \text{correction} \\ G_n(0) &= \beta_n \sim \beta_0 - \text{const.} n \end{aligned}$$

$$\begin{aligned} D(\phi_n) &= \text{Large and/or non-smooth configuration of } \phi_n \\ &= D_w(\phi_{n+1}) \cup R(z_n = Q\xi_n) \\ &= \text{Long Domain Walls + Short Domain walls} \end{aligned}$$

Domain walls D_w =STRONG SPIN ROTATION REGION

$$|\phi_n(x)\phi_n(y) - N\beta_n| > N^{1/2+\varepsilon} \exp[(c/10)|x-y|]$$

$$\forall x \in D_w, \exists y \in D_w$$

$1/2$ is the central limit theorem for $\sum : \xi_i^2$ ∴ Outside of D_w,

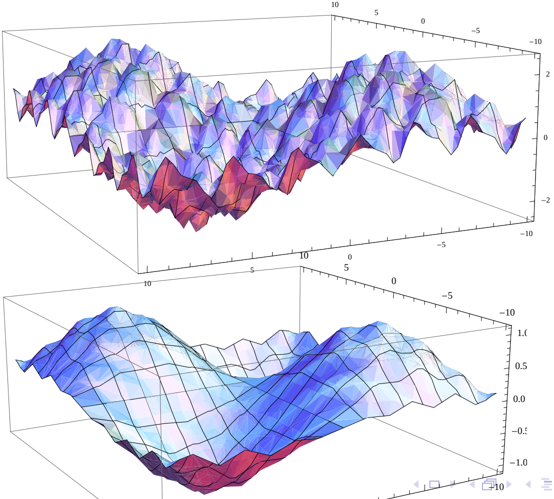
$$|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon} \exp[(c/10)|x-y|]$$

$$\forall x \in D_w^c, \forall y \in D_w^c$$

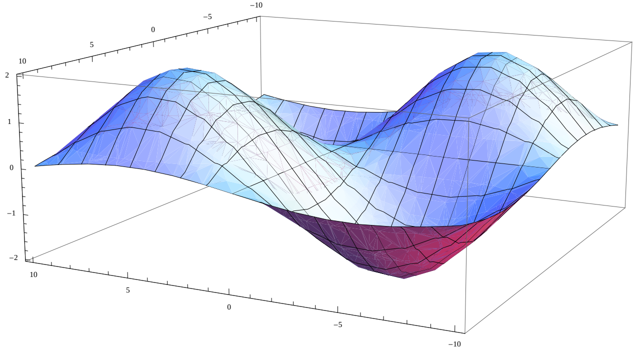
Thus $\phi_n(x)\phi_n(y) = NG_n(x,y)$ on $(D_w)^c$

Configuration of $\phi_n = A_{n+1}\phi_{n+1} + Qz_n$

small waves Qz_n on domain-walls (tsunami=$A_{n+1}\phi_{n+1}$)



Block Spin=Trimming short waves



Fluctuations $\xi_n(x)$ perpenficular to $\phi_n(x)$ have $N-1$ degrees of freedom of gaussian fields.

RG=Contraction Map on Banach Space $\mathcal{H}$

Namely we consider of Flow of Space $\mathcal{K}_n$ of Spin Configurations

$$\mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_n$$

$\mathcal{K}_n$ =smoothly propagating spin waves on the surfaces of balls

1. no domain walls
   $|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon} \exp[(c/10)|x-y|]$
   $\forall x, y \in K$
2. $|\phi_n(x)^2 - N\beta_n| < N^{1/2+\varepsilon}$
3. $|\nabla\phi_n(x)| < N^{1/2+\varepsilon}$

Serious difficulty is

$$\phi_n(x) = A_n \phi_{n+1} + Q\xi(x) \sim \phi_{n+1}([x/L]) + Q\xi(x)$$

Namely $\phi_n(x), |x| < L/2$ contain L^2 of $\phi_{n+1}([x/L])$. Thus

$$\sum_x \phi_n^2(x) \sim L^2 \sum_x \phi_{n+1}^2(x)$$

$$\sum_x (:\phi_n^2:_{G_n}(x))^2 \sim L^2 \sum_x (:\phi_{n+1}^2(x):_{G_{n+1}})^2$$

ϕ^4 term increases exponentially in n, i.e. relevant term.

BUT THIS DOES NOT HAPPEN.

Theorem on the RG flow

The main part of W_n is represented by 3 terms and 4 parameters β_n, g_n, γ_n and m_n^2, :

$$
\begin{aligned}
W_n(\phi_n, \psi_n) &= \frac{1}{2}\langle \phi_n, G_n^{-1}\phi_n + \frac{g_n}{2N}\langle : \phi_n^2 :_{G_n}, : \phi_n^2 :_{G_n}\rangle \\
&\quad + \frac{1}{2}\gamma_n < \phi_n^2, E^{\perp} G_n^{-1} E^{\perp} \phi_n^2 >
\end{aligned}
$$

where

1. $G_n^{-1} = -\Delta + m_n^2$, $m_n^2 = L^{2n} m_0^2$
2. $\gamma_n = (N\beta_n)^{-1}$.
3. $g_n \to g^* = O(1) > 0$ (convergetnt to the fixed point)
4. $E^{\perp} =$ projection to$\mathcal{N}(C) = \{f; Cf = 0\}$

- the first two terms = marginal (main term)
- the last term is irrelevant. it fades e away.
- $(:\phi_n^2:)^2$ is relevant but g_n converges to a constant in the scaling region

The flow is described by three parameters

$$
\begin{aligned}
m_n^2 &= L^{2n} m_0^2 \sim \exp[-4\pi\beta + 2n \log L] \to O(1), \\
\beta_n &= \beta - \mathrm{const}.n \to O(1) \\
\gamma_n &= O((\beta_n N)^{-1}) \\
g_n &= O(1)
\end{aligned}
$$

All this means is that system goes to the single-well potential, and then absence of phase transitions follows.

Sketch of the Proof

Main Ideas and Theorems:

Set $\phi_n = A_{n+1}\phi_{n+1} + z_n$, $z_n = Q\xi_n$ so that

$$
\begin{aligned}
< \phi_n, G_n^{-1}\phi_n > &= < \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} > + < \xi_n, \Gamma_n^{-1}\xi_n >, \\
\Gamma_n^{-1} &= Q^+ G_n^{-1} \sim Q^+(-\Lambda)Q > O(1) \\
: \phi_n^2(x) :_{G_n} &= : \phi_{n+1}^2(x) :_{G_{n+1}} + q(x) \\
q(x) &= 2\phi_{n+1}(x) z_n(x) + : z(x)_n^2 :_{\Gamma_n}
\end{aligned}
$$

Calculate the distribution function of $q(\xi)_x$:

$$\begin{aligned} P(\varphi_{n+1}, p) &= \int \exp\left[\frac{i}{\sqrt{N}} \sum_x (p - q(\xi))_x \lambda_x\right] d\mu(\xi) \prod d\lambda_x \\ &= \exp\left[\frac{i}{\sqrt{N}} \langle (p - q(\xi)), \lambda \rangle\right] d\mu(\xi) \prod d\lambda_x \\ d\mu(\xi) &= \exp[-\langle \xi, \Gamma_n^{-1} \xi \rangle] \prod d\xi_x \end{aligned}$$

the distribution function of $q(\xi) = 2\phi_{n+1}(x) z_n(x) + : z(x)_n^2 :_{\Gamma_n}$ with respect to $d\mu(\xi)$

Thorem 1:

$$
\begin{aligned}
P(p,\varphi) &= \exp[-\frac{1}{4N}\langle p,\frac{1}{M}p\rangle] \\
M &= \Gamma_n^{\circ 2} + \frac{2}{N}(\phi_n\phi_n)\circ\Gamma_n \\
&= \Gamma_n^{\circ 2} + 2\beta_n\Gamma_n + \underbrace{:\phi_n\phi_n:}_{\text{domain wall term}} \circ\Gamma_n/N
\end{aligned}
$$

where

$$
\begin{aligned}
(\Gamma_n)(x,y) &= (QG_n^{-1}Q^+)(x,y) \sim \exp[-|x-y|] \\
((\phi\phi)\circ\Gamma)(x,y) &= (\phi(x)\phi(y))\Gamma(x,y) \sim NG(x,y)\Gamma(x,y) \\
\text{spec } M &= \{\underbrace{\kappa_0}_{O(1)>0}, \underbrace{\kappa_1,\cdots,\kappa_{L^2-1}}_{O(\beta_n)}\}
\end{aligned}
$$

Corol.2: Assume

$$| : \phi_n(x)\phi_n(y) : \circ\Gamma_n(x,y)| < N^{1/2+\varepsilon} \times 1$$

Then

$$\begin{aligned} \langle p, \frac{1}{M} p \rangle &= \sum_{\text{blocks}:U\subset\Lambda_n} \langle p_U, \left(\frac{1}{\kappa_0} P_0 + \sum_i \frac{1}{\kappa_i} P_i \right) p_U \rangle \\ &\sim \sum_{\text{blocks}:U\subset\Lambda_n} \langle p_U, \left(\frac{1}{\kappa_0} P_0 \right) p_U \rangle \\ &= \sum_{\text{blocks}:U\subset\Lambda_n} \frac{1}{\kappa_0} (P_0 p_U)^2 \end{aligned}$$

where

$$\begin{aligned} P_0 &= \text{projection to block-wise cobstant functions} \\ P_i &= \text{projection to zero-average functions} \end{aligned}$$

Definition of Domain Wall

Domain walls are paved set such that

$$|\phi_n(x)\phi_n(y) - N\beta_n| > N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w, \exists y \in D_w$$

$1/2$ is the central limit theorem for $\sum : \xi_i^2 :$. Outside of D_w,

$$|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w^c, \forall y \in D_w^c$$

Thus $\quad \phi_n(x)\phi_n(y) = NG_n(x,y) \quad$ on $(D_w)^c$

Theorem 2

Domain Wall region D_w has high energy:

$$\int \exp[-\frac{1}{2}\langle \varphi_n, G_0^{-1}\varphi_n\rangle_{D_w}]d\mu(\xi) < \exp[-N^{2\varepsilon}|D_w|]$$

Outside of D_w, we can replace $\varphi\varphi$ by NG_n, and we have a Gaussian integral over p.

We integrate over ξ under the influence of long spin wave by p variables. Using $:\varphi_n^2:^2 = (:\varphi_{n+1}^2: + p)^2$, we replace ξ^4 by p^2:
Theorem 3

$$\int \exp[-\frac{g_n}{2N}\langle :\varphi_n^2:, :\varphi_n^2:\rangle + (.....)]d\mu(\xi)$$
$$= \int \exp[-\frac{g_n}{2N}\langle :\varphi_{n+1}^2: + p, :\varphi_{n+1}^2: + p\rangle]P(p,\varphi)\prod dp$$

$$P(p) = \exp[-\frac{1}{4N}\langle p, M^{-1}p\rangle]$$
$$= \exp[-\frac{1}{4N}\langle p, \left(\frac{1}{\kappa_0}P_0\right)p\rangle] = \exp[-\frac{1}{4N\kappa_0}\sum (P_0 p_U)^2]$$

where $P_0 p$ is block-wise constant spins (block spin type.)

Final Step:

Put

$$\frac{g_n}{2N}\sum_x (:\varphi_{n+1}^2: + p)^2 + \frac{1}{2N}\sum_x (P_0 p)^2$$
$$= \frac{g_n}{2N}\sum_x \left[(P_0(:\varphi_{n+1}^2: + p))^2 + ((1-P_0)(:\varphi_{n+1}^2: + p))^2\right]$$
$$+ \frac{1}{2N}\sum_x (P_0 p)^2$$

Integrate over $P_0 p$ and $(1-P_0)p$ apply steepest descent + perturbation. Since $P(p) = P_n(p)$ is a gaussian for all n,

Theorem 4 g_n converges in the scaling region: $g_n \to g^*$

This completes the proof. Thank you very much for your attension and patience!