On the Notion of Generalized Solutions of Viscous Incompressible Two-Phase Flows

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Abstract

In this article we review different approaches to the existence of solutions of a two-phase flow of two viscous, incompressible fluids globally in time. We compare the known results in the cases with and without surface tension. In particular, we discuss properties of the free surface/ the interface between the two fluids.

Key words: Two-phase flow, free boundary value problems, varifold solutions, measure-valued solutions, surface tension
AMS-Classification: 35Q30, 35Q35, 76D27, 76D45, 76T99

1 Introduction

We study the flow of two incompressible, viscous and immiscible fluids like oil and water inside a bounded domain $\Omega \subseteq \mathbb{R}^d$ or in $\Omega = \mathbb{R}^d$, $d = 2, 3$. For simplicity we assume that the densities of both fluids are the same and equal to one. The fluids fill disjoint domains $\Omega^+(t)$ and $\Omega^-(t)$, $t > 0$, and the interface between both fluids is denoted by $\Gamma(t) = \partial\Omega^\pm(t)$. Hence $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$. The flow is described using the velocity $v\colon \Omega \times (0, \infty) \to \mathbb{R}^d$ and the pressure $p\colon \Omega \times (0, \infty) \to \mathbb{R}$ in both fluids in Eulerian coordinates. We assume the fluids to be of a generalized Newtonian type, i.e., the stress tensors are of the form $T^\pm(v, p) = 2\nu^\pm(|Dv|)Dv - pI$ with viscosities $\nu^\pm$ depending on the shear rate $|Dv|$ of the fluid, $2Dv = \nabla v + \nabla v^T$. Moreover, we consider the cases with and without surface tension at the interface. Precise

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assumptions are made below. Under suitable smoothness assumptions, the flow is obtained as solution of the system

$$\partial_t v + v \cdot \nabla v - \operatorname{div} T^{\pm}(v,p) = 0 \qquad \text{in } \Omega^{\pm}(t), t > 0, \tag{1.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega^{\pm}(t), t > 0, \tag{1.2}$$

$$n \cdot T^{+}(v,p) - n \cdot T^{-}(v,p) = \kappa H n \qquad \text{on } \Gamma(t), t > 0, \tag{1.3}$$

$$V = n \cdot v \qquad \text{on } \Gamma(t), t > 0, \tag{1.4}$$

$$v = 0 \qquad \text{on } \partial\Omega, t > 0, \tag{1.5}$$

$$v|_{t=0} = v_0 \qquad \text{in } \Omega, \tag{1.6}$$

together with $\Omega^+(0) = \Omega_0^+$. Here V and H denote the normal velocity and mean curvature of $\Gamma(t)$, resp., taken with respect to the exterior normal n of $\partial\Omega^+(t)$, and $\kappa \geq 0$ is the surface tension constant ($\kappa = 0$ means no surface tension present). Equations (1.1)-(1.2) describe the conservation of linear momentum and mass in both fluids, (1.3) is the balance of forces at the boundary, (1.4) is the kinematic condition that the interface is transported with the flow of the mass particles, and (1.5) is the non-slip condition at the boundary of Ω. Moreover, it is assumed that the velocity field v is continuous along the interface.

Most publications on the mathematical analysis of free boundary value problems for viscous incompressible fluids study quite regular solutions and often deal with well-posedness locally in time or global existence close to equilibrium states, cf. e.g. Solonnikov [23, 24], Beale [4, 5], Tani and Tanaka [26], Shibata and Shimizu [21] or Abels [1]. These approaches are a priori limited to flows, in which the interface does not develop singularities and the domain filled by the fluid does not change its topology. In the present contribution we discuss certain classes of generalized solutions, which allow singularities of the interface and which exist globally in time for general initial data. For this purpose, we need a suitable weak formulation of the system above. Testing (1.1) with a divergence free vector field φ and using in particular the jump relation (1.4), we obtain

$$\begin{aligned} -(v, \partial_t\varphi)_Q - (v_0, \varphi|_{t=0})_\Omega + (v \cdot \nabla v, \varphi)_Q & \\ +(S(\chi, Dv), D\varphi)_Q = \kappa \int_0^\infty \left\langle H_{\Gamma(t)}, \varphi(t) \right\rangle dt & \end{aligned} \tag{1.7}$$

for all $\varphi \in C_{(0)}^\infty(\Omega \times [0,\infty))^d$ with $\operatorname{div} \varphi = 0$, where $Q = \Omega \times (0,\infty)$, $\chi = \chi_{\Omega^+}$, $S(1, Dv) = 2\nu^+(|Dv|)Dv$, $S(0, Dv) = 2\nu^-(|Dv|)Dv$, and

$$\left\langle H_{\Gamma(t)}, \varphi(t) \right\rangle := \int_{\Gamma(t)} H n \cdot \varphi(x,t)\, d\mathcal{H}^{d-1}(x). \tag{1.8}$$

Here $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and χ_A denotes the characteristic function of a set A.

Now the aim is to construct generalized solutions in a class of functions determined by the energy estimate. If v and $\Gamma(t)$ are sufficiently smooth, then choosing $\varphi = v\chi_{[0,T]}$ in (1.7) one obtains the *energy inequality*

$$\begin{aligned}\frac{1}{2}\|v(T)\|^2_{L^2(\Omega)} &+ \kappa\mathcal{H}^{d-1}(\Gamma(T)) \\ &+ \int_0^T \int_\Omega S(\chi, Dv) : Dv\, dx\, dt \leq \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \kappa\mathcal{H}^{d-1}(\Gamma_0) \end{aligned}\tag{1.9}$$

for all $T > 0$ (even with equality), where $\Gamma_0 = \partial\Omega_0^+$. – Note that

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} HV\, d\mathcal{H}^{d-1} = -\langle H_{\Gamma(t)}, v(t)\rangle \tag{1.10}$$

due to (1.4), cf. [11, Equation 10.12]. Now assuming that

$$\nu^\pm(|Dv|) \geq c|Dv|^{q-2}$$

for some $q > 1$ the equality above gives a uniform bound of

$$v \in L^\infty(0, \infty; L^2_\sigma(\Omega)) \quad \text{and} \quad Dv \in L^q((0, \infty) \times \Omega). \tag{1.11}$$

Here $L^p(M)$, $1 \leq p \leq \infty$, denotes the usual Lebesgue space, $L^p_{loc}(M)$ its local and $L^p(M; X)$ its vector-valued analog for a given Banach space X. Moreover, if $A \subset \mathbb{R}$, then $L^p(M; A)$ consists of all $f \in L^p(M)$ with $f(x) \in A$ for a.e. $x \in M$. Finally, $L^p_\sigma(\Omega)$ is the set of all divergence free vector fields $f \in L^p(\Omega)^d$.

As will be shown below, if $\kappa > 0$, then (1.9) yields an a priori bound of

$$\chi \in L^\infty(0, \infty; BV(\Omega)),$$

where $BV(\Omega) = \{f \in L^1(\Omega) : \nabla f \in \mathcal{M}(\Omega)\}$ denotes the space of functions with bounded variation, cf. e.g. [3, 9] and $\mathcal{M}(\Omega) = C_0(\Omega)'$ is the space of finite Radon measures. In the case without surface tension, i.e., $\kappa = 0$, we only obtain that $\chi \in L^\infty(Q)$ is a priori bounded by one. This motivates to look for weak solutions (v, χ) lying in the function spaces above, satisfying (1.9) with a suitable substitute of (1.8), such that (v, χ) solve (1.7) as well as the transport equation

$$\partial_t\chi + v \cdot \nabla\chi = 0 \qquad \text{in } Q, \tag{1.12}$$

$$\chi|_{t=0} = \chi_0 \qquad \text{in } \Omega \tag{1.13}$$

for $\chi_0 = \chi_{\Omega_0^+}$ in a suitable weak sense. Note that (1.12) is a weak formulation of (1.4), cf. [15, Lemma 1.2].

In the following we will discuss the known mathematical results for the case with and without surface tension. Throughout the paper we make the following assumption:

Assumption 1.1 *We assume that $\kappa > 0$ and $\Omega = \mathbb{R}^d$ or that $\kappa = 0$ and either $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary or $\Omega = \mathbb{R}^d$, where $d = 2, 3$. Moreover, let $q > 1$ and let $\nu(j, s)$, $j = 0, 1$, be twice continuously differentiable for $s > 0$ such that $\nu(j, s)s^2$ is continuous at 0 and $\nu(j, s)$ satisfy*

$$c_0 s^{q-2} \leq \nu(j,s) \leq C_0 s^{q-2}, \quad \frac{d}{ds}(\nu(j,s)s) > 0, \quad \frac{d^2}{ds^2}(\nu(j,s)s^2) > 0 \quad (1.14)$$

for some constants $c_0, C_0 > 0$. Finally, we set $S(\theta, A) = \theta\nu(1, |A|)A + (1 - \theta)\nu(0, |A|)A$ for every $A \in \mathbb{R}^{d\times d}_{sym}$, $\theta \in [0, 1]$, and $V_q(\Omega) = W^1_{q,0}(\Omega)^d \cap L^q_\sigma(\Omega)$ if Ω is a bounded domain and $V_q(\mathbb{R}^d) = \{v \in L^q_{loc}(\mathbb{R}^d)^d : \nabla v \in L^q(\mathbb{R}^d), \operatorname{div} v = 0\}$.

Note that (1.14) imply that

$$F(v) = \int_\Omega S(\chi, Dv) : Dv\, dx, \qquad v \in W^1_q(\Omega)^d,$$

is a bounded, coercive, and strictly convex functional on $W^1_q(\Omega)$ for every $\chi \in L^\infty(\Omega; \{0, 1\})$ and that the mapping $A \mapsto S(\theta, A)$ is strictly monotone.

2 Two-Phase Flow Without Surface Tension

Throughout this section we assume that $\kappa = 0$, i.e., no surface tension is present. Then the two-phase flow consists of a coupled system of the Navier-Stokes equation with variable viscosities and a transport equation for the characteristic function $\chi(t) = \chi_{\Omega^+(t)}$. In the Newtonian case, i.e., $q = 2$ and $\nu(j, s) = \nu_j$, this is a special case of the so-called density-dependent Navier-Stokes equation, cf. f.e. Desjardins [6] and references given there. For given χ it is easy to construct a weak solution of the Navier-Stokes equation (1.7), $q = 2$, with the aid of a suitable approximation scheme (f.e. Galerkin approximation). No difficulties arise due to non-linear mean curvature term $\langle H_\gamma, .\rangle$.

For the coupled system (1.7) together with (1.12)-(1.13) there are two different approaches. The essential difference is in which sense the transport equation is solved. One approach is due to Giga and Takahashi [10], who solved (1.12)-(1.13) in the sense of viscosity solutions, where the characteristic functions $(\chi(t), \chi_0)$ are replaced by continuous level-set functions $(\psi(t), \psi_0)$ such that

$$\Omega_0^{\pm} = \{x \in \Omega : \psi_0(x) \gtrless 0\}\,.$$

For simplicity they consider periodic boundary conditions, i.e., $\Omega = \mathbb{T}^d$. Since v is in general not Lipschitz continuous, the existence of a viscosity solution of (1.12)-(1.13) with (χ, χ_0) replaced by continuous level-set functions (ψ, ψ_0) is not known. There are only a least super-solution $\psi^+(t)$ and a largest sub-solution $\psi^-(t)$ of the transport equation. Then one defines

$$\Omega^{\pm}(t) = \left\{x \in \Omega : \psi^{\pm}(x,t) \gtrless 0\right\}.$$

With this definition $\Omega^{\pm}(t)$ are disjoint open sets but the "boundary" $\Gamma(t) = \mathbb{T}^d \setminus (\Omega^+(t) \cup \Omega^-(t))$ might have interior points and might have positive Lebesgues measure. Giga and Takahashi call this possible effect "boundary fattening". With this definition they construct weak solutions of a two-phase Stokes flow, i.e., $q = 2$ and the convective term $v \cdot \nabla v$ is neglected in (1.7), assuming that the viscosity difference $|\nu^+ - \nu^-|$ is sufficiently small; see [10] for details. This approach was adapted to the case of a Navier-Stokes two-phase flow by Takahashi [25] and to a one-phase flow for an ideal, irrotational and incompressible fluid by Wagner [28].

The other approach was established by Nouri and Poupaud [15] and Nouri et. al. [16] and is based on the results of DiPerna and Lions [7] on renormalized solutions of the transport equation (1.12)-(1.13) for a velocity field v with bounded divergence. Here $\chi \in L^\infty(Q)$ is called a *renormalized solution* of (1.12)-(1.13) if for all $\beta \in C^1(\mathbb{R})$ which vanish near 0 $\beta(\chi)$ solves (1.12)-(1.13) with initial values $\beta(\chi_0)$, cf. [7] for details. In particular, this implies that $\chi(t,x) \in \overline{\{\chi_0(x) : x \in \Omega\}}$ for almost all $t > 0, x \in \Omega$. Due to [7, Theorem II.3], for every $\chi_0 \in L^\infty(\mathbb{R}^d)$ there is a *unique* renormalized solution of (1.12)-(1.13) under general conditions on v, which are weaker than the condition (1.11). Based on this notion the following result for the two-phase flow without surface tension holds true:

THEOREM 2.1 *Let Assumption 1.1 hold. Moreover, we assume that* $q \geq \frac{2d}{d+2} + 1$ *or that* $q = 2$ *and* $\nu(j,s) = \nu_j$, $j = 1,2$. *Then for every* $v_0 \in L^2_\sigma(\Omega)$, $\chi_0 \in L^\infty(\Omega; \{0,1\})$, *and* $f \in L^{q'}(0,\infty; V_q(\Omega)')$ *there are* $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^q(0,\infty; V_q(\Omega))$ *and* $\chi \in L^\infty(Q; \{0,1\})$ *that are a weak solution of the two-phase flow without surface tension in the sense that*

$$-(v, \partial_t \varphi)_Q - (v_0, \varphi(0))_\Omega + (v \cdot \nabla v, \varphi)_Q + (S(\chi, Dv), D\varphi)_Q = \langle f, \varphi \rangle \quad (2.1)$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,\infty))^d$ *with* $\operatorname{div}\varphi = 0$, χ *is the unique renormalized solution of the transport equation of (1.12)-(1.13), and (1.9) holds for almost all* $t > 0$ *with* $\kappa = 0$.

Notes on the proof: The Newtonian case, $q = 2$ and $\nu(j,s) = \nu_j$, was proved by Nouri and Poupaud [15] for the case of a bounded domain Ω with Lipschitz boundary. The generalized Newtonian case $q \geq \frac{2d}{d+2} + 1$ is proved in [2].

In order to prove the latter theorem, a key step is to show strong compactness of the sequence χ_k in $L^p(Q_T)$, $1 \leq p < \infty$, where $Q_T = \Omega \times (0,T), T > 0$, and (v_k, χ_k) is a suitably constructed approximation sequence. This is done by using the fact that

$$\|\chi_k(t)\|^p_{L^p(\Omega)} = \int_\Omega \chi_k(t,x)\,dx = \int_\Omega \chi_0(x)\,dx$$

if χ_k are solutions of (1.12)-(1.13) with v replaced by v_k and $\operatorname{div} v_k = 0$. Using that

$$\begin{aligned} \chi_k &\rightharpoonup^*_{k\to\infty} \chi && \text{in } L^\infty(Q), \\ \nabla v_k &\rightharpoonup^*_{k\to\infty} \nabla v && \text{in } L^q(Q) \end{aligned}$$

for a suitable subsequence one shows that χ solves (1.12)-(1.13), cf. [2, Lemma 5.1]. Here $\rightharpoonup^*$ denotes the weak-$*$ convergence. Therefore

$$\|\chi(t)\|^p_{L^p(\Omega)} = \int_\Omega \chi(t,x)\,dx = \int_\Omega \chi_0(x)\,dx = \|\chi_k(t)\|^p_{L^p(\Omega)}.$$

This implies strong convergence $\chi_k \to_{k\to\infty} \chi$ in $L^p(Q_T)$, $1 \leq p < \infty$, for every $T > 0$. Based on this, one can pass to the limit in all terms in (1.7) using the Minty-Browder method for $S(\chi, Dv)$.

Remark 2.2 Using the solution of Theorem 2.1, we can define the sets $\Omega^+(t) = \{x \in \Omega : \chi(t) = 1\}$ and $\Omega^-(t) = \{x \in \Omega : \chi(t) = 0\}$. Then we know that $|\Omega^+(t)| = |\Omega^+_0|$ and $\Omega \setminus (\Omega^+(t) \cup \Omega^-(t))$ has Lebesgue measure zero. But, since only $\chi \in L^\infty(Q)$ is known, it is not clear whether $\Omega^\pm(t)$ have interior points. In particular, it is not excluded that $\overline{\Omega^+(t)} = \Omega$ and $\operatorname{int}\Omega^+(t) = \emptyset$. Therefore it is not immediately clear what the "interface" between both fluids should be. If one naively defines the interface as $\Gamma(t) = \partial\Omega^+(t)$, then $\Gamma(t)$ can have positive Lebesgue measure as in the result by Giga and Takahasi.

It seems that by neglecting surface tension in the two phase flow, on looses a "good control" of the interface between both fluids.

Remark 2.3 In the case of a two-dimensional periodic domain $\Omega = \mathbb{T}^2$ and $\nu(0,s) = \nu(1,s) \equiv const.$ it can be shown that there is a weak solution of the two-phase flow with $v \in L^2(0,T;H^2(\Omega))$. Based on this result Desjardin [6] showed that the associated flow map defined by

$$\begin{aligned} \frac{d}{dt}X(t,x) &= v(X(t,x),t), \qquad 0<t<T, \\ X(0,x) &= x \end{aligned}$$

satisfies $X \in L^\infty(0,T;C^\alpha(\mathbb{T}^2))$ with $\alpha \in (0,1)$ and $T > 0$ arbitrary. In particular, the Hausdorff dimension of $\Gamma(t) = \partial\Omega^+(t)$, $t > 0$, is not greater than 1 if $\partial\Omega_0^+$ has Hausdorff dimension 1.

3 Case With Surface Tension: Varifold Solutions

As discussed in the previous section, a deficit of the two-phase flow without surface tension is that there is no good information on the properties of the interface. As mentioned in the introduction, if $\kappa > 0$, the energy equality (1.9) for sufficiently smooth solutions provides an a priori estimate of the interface:

$$\sup_{0\le t<\infty} \mathcal{H}^{d-1}(\Gamma(t)) \le \left(\frac{1}{2\kappa}\|v_0\|_2^2 + \mathcal{H}^{d-1}(\Gamma_0)\right). \tag{3.1}$$

This implies an a priori bound of χ in the space $BV(\Omega)$ as follows: Note that, if $\Gamma(t) = \partial\Omega^+(t)$ is sufficiently smooth, Gauss' theorem yields

$$-\langle\nabla\chi(t),\varphi\rangle = \int_{\Omega(t)} \operatorname{div}\varphi(x)\,dx = \int_{\Gamma(t)} n\cdot\varphi(x)\,d\mathcal{H}^{d-1}(x)$$

for all $\varphi \in C_0^\infty(\Omega)^d$. Hence the distributional gradient $\nabla\chi(t)$ is a finite Radon measure and

$$\|\nabla\chi(t)\|_{\mathcal{M}(\Omega)} = \mathcal{H}^{d-1}(\Gamma(t)).$$

Thus, if $\kappa > 0$, then $\chi(t) \in BV(\Omega)$ for all $t > 0$ and (3.1) gives an a priori estimate of

$$\chi \in L^\infty(0,\infty;BV(\Omega)).$$

Conversely, if $\chi(t) = \chi_E \in BV(\Omega)$ for some set $E = E(t)$, then E is said to be of *finite perimeter* and the following characterisation holds, cf. [9, Section 5.7, Theorem 2]:

$$-\langle\nabla\chi(t),\varphi\rangle = \int_{\partial^* E} \nu_E\cdot\varphi(x)\,d\mathcal{H}^{d-1}(x),$$

where $\partial^* E$ is the *reduced boundary of* E, cf. [9, Definition 5.7], and $\partial^* E$ is *countably* $(d-1)$*-rectifiable* in the sense that

$$\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N,$$

where K_k are compact subsets of C^1-hypersurfaces S_k, $k \in \mathbb{N}$, $\mathcal{H}^{d-1}(N) = 0$, and $\nu_E|_{S_k}$ is normal to S_k. Moreover, by [9, Section 5.8, Lemma 1] $\partial_* E \subseteq \partial^* E$ and $\mathcal{H}^{d-1}(\partial^* E \setminus \partial_* E)$, where $\partial_* E$ is the *measure theoretic boundary* of E consisting of all $x \in \Omega$ such that

$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\cap E)}{r^d} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\setminus E)}{r^d} > 0,$$

where $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$.

Based on these properties, one can define the mean curvature functional of a set of finite perimiter E as

$$\langle H_{\partial^* E}, \varphi\rangle \equiv \langle H_{\chi_E}, \varphi\rangle := -\int_{\partial^* E} \operatorname{Tr}(P_\tau \nabla\varphi)\, d\mathcal{H}^{d-1}, \qquad \varphi \in C_0^1(\Omega)^d, \quad (3.2)$$

where $P_\tau = I - \nu_E(x) \otimes \nu_E(x)$. Note that $\operatorname{Tr}(P_\tau \nabla\varphi)$ corresponds to the divergence of φ along the "surface" $\partial^* E$ and that by integration by parts (3.2) coincides with the usual definition if $\partial^* E$ is a C^2-surface, cf. f.e. Giusti [11, Chapter 10].

Motivated by the considerations above, we define weak solutions of the two-phase flow in the case of surface tension as follows:

Definition 3.1 (Weak solutions)
Let $\kappa > 0$ and let Assumption 1.1 hold. Then $v \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^d)) \cap L^q(0,\infty; V_q(\mathbb{R}^d))$, $\chi \in L^\infty(0,\infty; BV(\mathbb{R}^d;\{0,1\}))$, are called a weak solution of the two-phase flow *for initial data $v_0 \in L^2_\sigma(\mathbb{R}^d)$, $\chi_0 = \chi_{\Omega_0^+}$ for a bounded domain $\Omega_0^+ \Subset \mathbb{R}^d$ of finite perimeter if the following conditions are satisfied:*

1. *(1.7) holds for all $\varphi \in C^\infty_{(0)}(\mathbb{R}^d \times [0,\infty))^d$ with $\operatorname{div}\varphi = 0$, where $H_{\Gamma(t)}$ is replaced by $H_{\chi(t)}$ defined as in (3.2).*

2. *The energy inequality*

$$\frac{1}{2}\|v(t)\|_2^2 + \kappa\|\nabla\chi(t)\|_{\mathcal{M}} + \int_{Q_t} S(\chi, Dv) : Dv\, d(x,\tau) \le \frac{1}{2}\|v_0\|_2^2 + \kappa\|\nabla\chi_0\|_{\mathcal{M}} \quad (3.3)$$

holds for almost all $t \in (0,\infty)$.

Unfortunately, the existence of weak solutions as defined above is open. The reason are possible oscillation and concentration effects related to the interface, which cannot be excluded so far. This prevents us from passing to the limit in the mean curvature functional (3.2) during an approximation procedure used to construct weak solutions.

In order to demonstrate these effects, let E_k be a sequence of sets of finite perimeter such that $\chi_k \equiv \chi_{E_k}$ is bounded in $BV(\Omega)$ and let $\Omega = \mathbb{R}^d$. Then after passing to a suitable subsequence, we can assume that

$$\begin{aligned} \chi_k &\to_{k\to\infty} \chi && \text{in } L^1_{loc}(\mathbb{R}^d), \\ \nabla\chi_k &\rightharpoonup^*_{k\to\infty} \nabla\chi && \text{in } \mathcal{M}(\mathbb{R}^d), \\ |\nabla\chi_k| &\rightharpoonup^*_{k\to\infty} \mu && \text{in } \mathcal{M}(\mathbb{R}^d). \end{aligned}$$

But then the question arises how $|\nabla\chi|$ and μ are related and whether

$$\lim_{k\to\infty} \langle H_{\chi_{\varepsilon_k}}, \psi\rangle = \langle H_\chi, \psi\rangle \tag{3.4}$$

holds. The continuity result due Reshetnyak, cf. [3, Theorem 2.39], gives a sufficient condition for (3.4): If

$$\lim_{k\to\infty} |\nabla\chi_k|(\mathbb{R}^d) = |\nabla\chi|(\mathbb{R}^d), \tag{3.5}$$

then (3.4) holds. But in general (3.5) will not hold for example because of the following oscillation/concentration effects at the reduced boundary of E:

1. Several parts of the boundary $\partial^* E_k$ might meet.
2. Oscillations of the boundary might reduce the area in the limit.
3. There might be an "infinitessimal emulsification".

These effects are sketched in Figure 1.

So far it is not known how to exclude such kind of oscillation/concentration effects. – This might even not be possible in general since our model might not describe the behavior of both fluids appropriately when f.e. a lot of small scale drops are forming. – One way out of this problem is to define so-called *varifold solution* of a two-phase flow, which was first done by Plotnikov [18]. Here a general (oriented) varifold V on a domain Ω is simply a non-negative measure in $\mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. By disintegration, cf. [3, Theorem 2.28], a varifold V can be decomposed in a non-negative measure $|V| \in \mathcal{M}(\Omega)$ and a family of probability measures $V_x \in \mathcal{M}(\mathbb{S}^{d-1})$, $x \in \Omega$, such that

$$\langle V, \psi\rangle = \int_\Omega \int_{\mathbb{S}^{d-1}} \psi(x,s)\, dV_x(s)\, d|V|(x) \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1}).$$

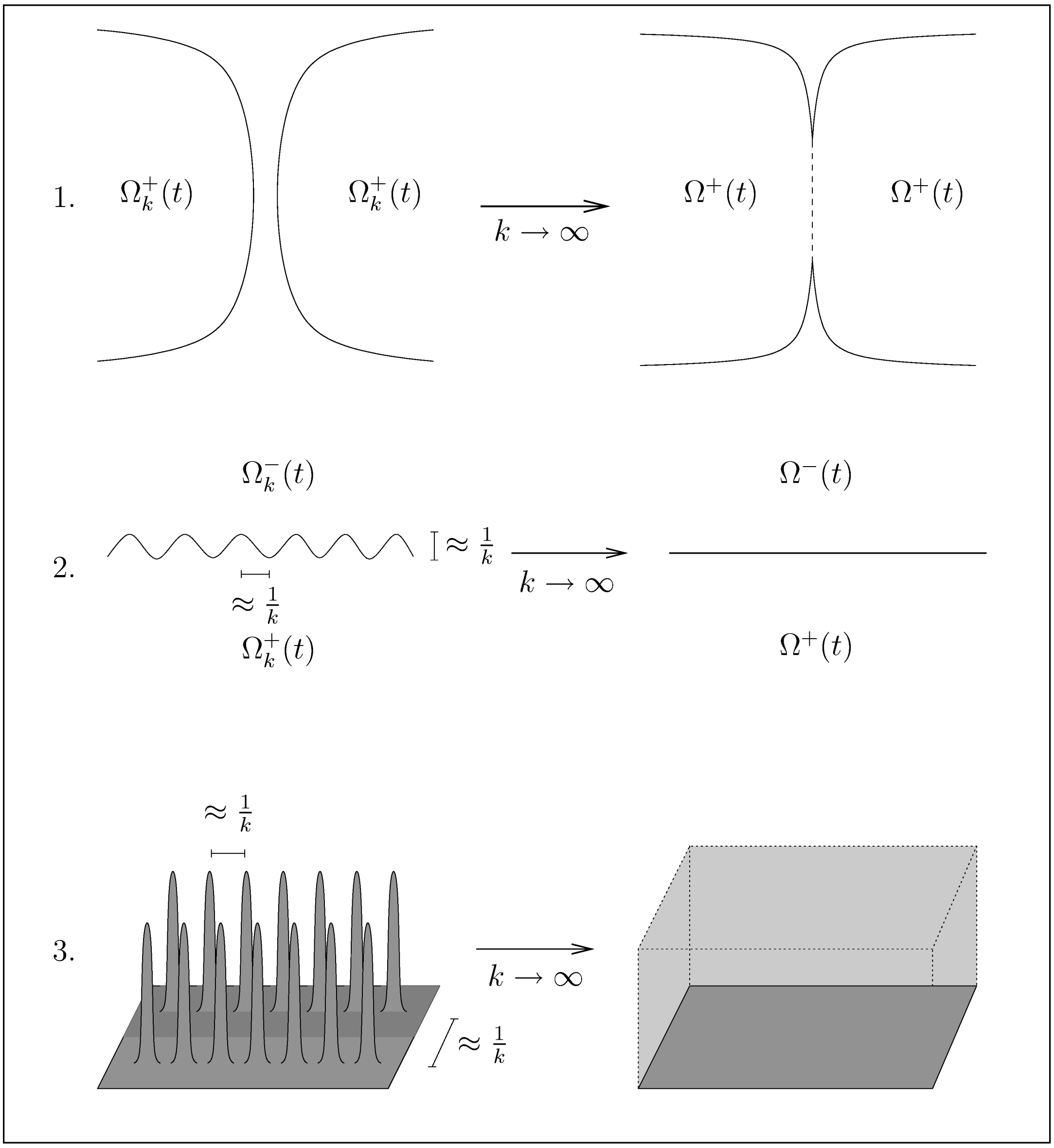

Figure 1: Some possible oscillation/concentration effect

Here $|V|$ corresponds to the measure of "area of the interface" and V_x defines a probability for the "normal at the interface".

The reduced boundary $\partial^* E$ of a set of finite perimeter induces naturally a varifold by setting $|V| = |\nabla \chi_E|$ and $V_x = \delta_{\nu_E(x)}$ for $x \in \partial^* E$, where δ_ν denotes the Dirac measure at $\nu \in \mathbb{S}^{d-1}$. Hence the associated varifold V_E is

$$\langle V_E, \psi \rangle = \int_\Omega \psi(x, \nu_E(x))\, d|V|(x) \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1}).$$

Now let E_k be a sequence of sets of finite perimeter as above. Then by the weak-$*$ compactness of $\mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, there is a limit varifold $V \in \mathcal{M}(\Omega \times \mathbb{S}^{d-1})$ such that

$$\langle V, \psi \rangle = \lim_{k\to\infty} \langle V_{E_k}, \psi \rangle \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1})$$

for a suitable subsequence. Hence using $\psi(s,x) = \mathrm{Tr}((I - s\otimes s)\nabla\varphi(x))$ for $\varphi \in C_0^1(\Omega)^d$ we conclude that

$$\lim_{k\to\infty} \langle H_{\chi_{E_k}}, \psi \rangle = \int_{\Omega\times\mathbb{S}^{d-1}} \mathrm{Tr}((I - s\otimes s)\nabla\varphi(x))\, dV(s,x) =: -\langle \delta V, \varphi \rangle \tag{3.6}$$

for all $\varphi \in C_0^1(\Omega)^d$. Here $\delta V \in C_0^1(\Omega;\mathbb{R}^d)'$ defined as above is called the *first variation* of the generalized varifold V. Moreover,

$$\begin{aligned} -\langle \nabla\chi_E, \varphi \rangle &= -\lim_{k\to\infty} \langle \nabla \chi_{E_k}, \varphi \rangle \\ &= \lim_{k\to\infty} \int_\Omega \nu_E(x)\cdot\varphi\, d|V_{E_k}|(x) = \int_{\Omega\times\mathbb{S}^{d-1}} s\cdot\varphi(x)\, dV(x,s). \end{aligned}$$

Hence V can be used to describe the limit of $H_{\chi_{E_k}}$ as well as the limit of $\nabla\chi_{E_k}$.

Now we define a varifold solution of the two-phase flow as follows:

Definition 3.2 (Varifold solutions)
Let $\kappa > 0$ and let Assumption 1.1 hold. Then

$$\begin{aligned} &v \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^d)) \cap L^q(0,\infty; V_q(\mathbb{R}^d)), \\ &\chi \in L^\infty(0,\infty; BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d;\{0,1\})), \\ &V \in L^\infty_\omega(0,\infty; \mathcal{M}(\Omega\times\mathbb{S}^{d-1})) \end{aligned}$$

are called a varifold solution of the two-phase flow *for initial data $v_0 \in L^2_\sigma(\mathbb{R}^d)$ and $\chi_0 = \chi_{\Omega_0^+}$ for a bounded domain $\Omega_0^+ \Subset \mathbb{R}^d$ of finite perimeter if the following conditions are satisfied:*

1. *(1.7) holds for all* $\varphi \in C^\infty_{(0)}(\mathbb{R}^d \times [0,\infty))^d$ *with* $\operatorname{div} \varphi = 0$, *where* $H_{\Gamma(t)}$ *is replaced by* $-\delta V(t)$ *where*

$$\langle \delta V(t), \varphi \rangle = \int_{\Omega \times \mathbb{S}^{d-1}} \operatorname{Tr}((I - s \otimes s)\nabla \varphi(x))\, dV(s,x), \quad \varphi \in C^1_0(\Omega)^d.$$

2. *The modified energy inequality*

$$\frac{1}{2}\|v(t)\|_2^2 + \kappa \|V(t)\|_{\mathcal{M}} + \int_{Q_t} S(\chi, Dv) : Dv\, d(x,\tau) \le \frac{1}{2}\|v_0\|_2^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}} \tag{3.7}$$

holds for almost all $t \in (0,\infty)$.

3. *The compatibility condition*

$$-\langle \nabla \chi(t), \varphi \rangle = \int_{\Omega \times \mathbb{S}^{d-1}} s \cdot \varphi(x)\, dV(x,s), \quad \varphi \in C_0(\Omega)^d, \tag{3.8}$$

holds for almost all $t > 0$.

Here $L^\infty_\omega(0,T;X')$ denotes the space of weak-$*$ measurable essentially bounded functions $f\colon (0,T) \to X'$.

Remark 3.3 1. Let $(V_x(t), |V(t)|)$, $x \in \mathbb{R}^d$, denote the disintegration of $V(t) \in \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ as described above. Then (3.8) implies that $|\nabla \chi(t)|(A) \le |V(t)|(A)$ for all open sets A and almost all $t \in (0,\infty)$. Hence $|\nabla \chi(t)|$ is absolutely continuous with respect to $|V(t)|$ and

$$\int_{\mathbb{R}^d} f(x)\, d|\nabla \chi(t)| = \int_{\mathbb{R}^d} f(x) \alpha_t(x)\, d|V(t)|, \quad f \in C_0(\mathbb{R}^d),$$

for a $|V(t)|$-measurable function $\alpha_t\colon \mathbb{R}^d \to [0,\infty)$ with $|\alpha_t(x)| \le 1$ almost everywhere. In particular, this implies $\operatorname{supp} \nabla \chi_t \subseteq \operatorname{supp} V(t)$ and $\|\nabla \chi(t)\|_{\mathcal{M}} \le \|V(t)\|_{\mathcal{M}}$ for almost all $t \in (0,\infty)$. Hence every varifold solution satisfies the energy inequality (3.3) for almost every $t > 0$.

Moreover, if $E(t) = \{x \in \mathbb{R}^d : \chi(x,t) = 1\}$, $t > 0$, then (3.8) yields the relation

$$\int_{\mathbb{S}^{d-1}} s\, dV_x(t)(s) = \begin{cases} \alpha_t(x) \nu_{E(t)}(x) & \text{if } x \in \partial^* E_t \\ 0 & \text{else} \end{cases}$$

for $|V(t)|$-almost every $x \in \mathbb{R}^d$ and almost every $t > 0$. – In other words, the expectation of $V_x(t)$ is proportional to the normal n on the interface described by $\nabla \chi$ and zero away from it.

2. In general, it is an open problem whether $V(t)$ is a so-called countably $(d-1)$-rectifiable varifold, which implies that up to orientation $V_x(t)$ is a Dirac measure for $|V(t)|$-almost every x. Then $V(t)$ can naturally be identified with a countably $(d-1)$-rectifiable set – a "surface" – equipped with a density $\theta_t \geq 0$. So far we can only give a sufficient condition for the rectifiability of $V(t)$ in terms of the first variation $\delta V(t)$, cf. Section 4 below.

3. As noted above, the existence of weak solutions to the two-phase flow with surface tension is open. But a general porperty of varifold solutions is that a varifold solution is a weak solution if the energy equality holds, i.e., (3.3) holds with equality for almost every $t > 0$. See [2, Proposition 1.5] for details.

THEOREM 3.4 (Existence of Varifold Solutions)
Let $\kappa > 0$, $d = 2, 3$, and let Assumption 1.1 hold. Moreover, assume that $q = 2$ and $\nu(j, s) = \nu_j > 0$ for $j = 0, 1$, or assume that $q > d = 2$. Then for every $v_0 \in L^2_\sigma(\Omega)$ and $\chi_0 = \chi_{\Omega_0^+}$ where $\Omega_0^+ \Subset \mathbb{R}^d$ is a bounded C^1-domain there is a varifolds solution of the two-phase flow with surface tension $\kappa > 0$ in the sense of Definition 3.2.

Remark 3.5 For $q > d = 2$ existence of varifold solutions was proven by Plotnikov [18]. But his definition of varifold solutions is different from Definition 3.2, cf. [2, Remark 1.7] for details. Moreover, we refer to [2, Theorem 1.6] for further properties, which can be shown for the constructed varifold.

Remark 3.6 Generalized solutions for the two-phase flow with surface tension were also constructed by Salvi [20]. But in the latter work the meaning of the mean curvature functional is not specified and can be chosen arbitrarily within in a certain function space. Moreover, we note that a Bernoulli free boundary problem with surface tension was discussed by Wagner [27].

Remarks on the proof of Theorem 3.4: As usual varifold solutions are constructed by approximation with solutions to an approximative (smoothed) problem. This can be done by solving the system

$$\begin{aligned} &-(v_\varepsilon, \partial_t \varphi)_{Q_T} - (v_0, \varphi(0))_{\mathbb{R}^d} - (\Psi_\varepsilon v_\varepsilon \otimes \Psi_\varepsilon v_\varepsilon, \nabla \Psi_\varepsilon \varphi)_{Q_T} \\ &\qquad + (S(\chi_\varepsilon, Dv_\varepsilon), D\varphi)_{Q_T} = \kappa \int_0^T \langle H_{\chi_\varepsilon(t)}, \Psi_\varepsilon \varphi(t) \rangle \, dt \end{aligned} \tag{3.9}$$

for all $\varphi \in C^\infty_{(0)}(\mathbb{R}^d \times [0, T))^d$ with $\operatorname{div} \varphi = 0$, together with the transport equation

$$\partial_t \chi_\varepsilon + (\Psi_\varepsilon v_\varepsilon) \cdot \nabla \chi_\varepsilon = 0 \qquad \text{in } Q_T, \tag{3.10}$$

$$\chi_\varepsilon|_{t=0} = \chi_0 \qquad \text{in } \mathbb{R}^d. \tag{3.11}$$

Here $\varepsilon > 0$, $T = \varepsilon^{-1}$, $\Psi_\varepsilon f = \psi_\varepsilon * f$, and $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$ is a standard smoothing kernel with $\psi_\varepsilon(-x) = \psi_\varepsilon(x)$. Since $\Psi_\varepsilon v_\varepsilon$ is smooth, the transport equation (3.10)-(3.11) can be solved explicitely with the method of characteristics. Moreover, the boundary $\Gamma_\varepsilon(t)$ of the domain described by χ_ε is as smooth as $\partial\Omega_0^+$. Therefore the mean curvature functional $\langle H_{\chi_\varepsilon(t)}, .\rangle$ can be defined in the classical sense. Furthermore, solutions $(v_\varepsilon, \chi_\varepsilon)$ of the system (3.9)-(3.11) satisfy the energy inequality (1.9) with $\Gamma(t)$ replaced by $\Gamma_\varepsilon(t)$ because of

$$\int_0^T \langle H_{\chi_\varepsilon(t)}, \Psi_\varepsilon v_\varepsilon(t)\rangle \, dt = \int_0^T \int_{\Gamma_\varepsilon(t)} HV \, d\mathcal{H}^{d-1} \, dt$$

$$= \; -\int_0^T \frac{d}{dt}\mathcal{H}^{d-1}(\Gamma_\varepsilon(t)) \, dt = \mathcal{H}^{d-1}(\Gamma_0) - \mathcal{H}^{d-1}(\Gamma_\varepsilon(T))$$

where we have used that the normal velocity of $\Gamma_\varepsilon(t)$ is $\nu \cdot \Psi_\varepsilon v_\varepsilon|_{\Gamma_\varepsilon(t)}$ due (3.9). Therefore we have uniform bounds of

$$v_\varepsilon \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^d)), Dv_\varepsilon \in L^q(Q_T), \chi_\varepsilon \in L^\infty(0,T; BV(\mathbb{R}^d)).$$

Hence one can use similar arguments as in the proof of Theorem 2.1 to pass to the limit in the system (3.9)-(3.11). New arguments are only needed for the mean curvature term $\langle H_{\chi_\varepsilon(t)}, \Psi_\varepsilon \cdot\rangle$ and the non-linearity $S(\chi, Dv)$ in the non-Newtonian case, i.e., $\nu(j,s) \not\equiv \mathit{const.}$, $j = 0,1$. But as explained above the mean curvature term $\langle H_{\chi_\varepsilon(t)}, \cdot\rangle$ can be expressed by the first variation of $V_{\Omega_\varepsilon^+(t)}$, where $V_{\Omega_\varepsilon^+(t)}$is the varifold associated to the boundary of $\Omega_\varepsilon^+(t)$ if $\chi_\varepsilon(t) = \chi_{\Omega_\varepsilon^+(t)}$. Then

$$V_{\Omega_{\varepsilon_k}} \rightharpoonup^*_{k\to\infty} V \qquad \text{in } L^\infty_\omega(0,\infty; \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$$

for a suitable subsequence $\varepsilon_k \to_{k\to\infty} 0$. With this definition it easily follows that

$$\lim_{k\to\infty} \int_0^T \langle H_{\chi_{\varepsilon_k}(t)}, \Psi_{\varepsilon_k}\varphi(t)\rangle \, dt = -\int_0^\infty \langle \delta V(t), \varphi(t)\rangle \, dt$$

for all $\varphi \in C^\infty_{(0)}(\mathbb{R}^d \times [0,\infty))$ and that (3.8) holds. In the non-Newtonian case one also has to show

$$\lim_{k\to\infty} (S(\chi_{\varepsilon_k}, Dv_{\varepsilon_k}), D\varphi)_Q = (S(\chi, Dv_{\varepsilon_k}), D\varphi)_Q$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,\infty))^d$. By the strong convergence of χ_{ε_k} in $L^p(Q_T)$ for all $T > 0$, $1 \leq p < \infty$, this reduces to showing that

$$\lim_{k\to\infty} (S(\chi, Dv_{\varepsilon_k}), D\varphi)_Q = (S(\chi, Dv_{\varepsilon_k}), D\varphi)_Q \tag{3.12}$$

for all φ as above. Since $d = 2$ in that case, one can show that the sequence of interfaces $\Gamma_{\varepsilon_k(t)}$ converges to a set $\Gamma^*(t)$ in the Hausdorff distance and that $\mathcal{H}^1(\Gamma^*(t)) \leq C$. Using this convergence and the Minty-Browder trick for $(v_{\varepsilon_k}, \chi_{\varepsilon_k})$ in space-time zylinders $\Omega' \times (t_1, t_2)$ away from the interface one can show (3.12).

Remark 3.7 The existence of generalized solutions can be extended to arbitrary $q > \frac{2d}{d+2}$, $d = 2, 3$. But under these assumptions we were not able to verify (3.12). This means that we cannot exclude possible additional oscillation and concentration effect of Dv_ε. But modifying the definition of varifold solution by replacing $S(\chi, Dv)$ in (1.7) by

$$\int_{\mathbb{R}^{d\times d}_{sym}} S(\chi, \lambda)\, d\mu_{x,t}(\lambda),$$

where $\mu_{x,t} \in L^\infty_\omega(Q; \mathcal{M}(\mathbb{R}^{d\times d}_{sym}))$ is the Young measure generated by $Dv_\varepsilon(x,t)$ it is still possible to prove existence of *measure-valued varifold solutions*. We obmit the precise definitions and statements and refer to [2, Section 1] for more details. This combines varifold solutions with the notion of (Young-)measure-valued solution for non-Newtonian fluids, cf. e.g. Málek et. al. [12].

4 Discussion and Comparision

In the following it will be important to forget the orientation of the general varifold. This means that instead of $V(t)$ we consider the unoriented general varifold $\widetilde{V}(t) \in \mathcal{M}(\mathbb{R}^d \times G_{d-1})$ defined by

$$\langle \widetilde{V}, \varphi \rangle = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi(x, [s])\, dV(x,s), \quad \varphi \in C_0(\mathbb{R}^d \times G_{d-1}). \tag{4.1}$$

Here $G_{d-1} \cong \mathbb{S}^{d-1}/\{\nu \equiv -\nu\}$ denotes the space of all unoriented $(d-1)$-dimensional subspaces of $\mathbb{R}^d$ and $[s]$ denotes the $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ with s as normal. It is an open problem whether there are varifold solutions such that the unoriented general varifold $\widetilde{V}(t)$ is a $(d-1)$-rectifiable varifold for almost all $t > 0$, i.e., $\widetilde{V_x}(t) = \delta_{P(x,t)}$ and

$$\langle \widetilde{V_x}(t), \varphi \rangle = \int \varphi(x, P(x,t))\theta_t(x)\, d\mathcal{H}^{d-1}\lfloor M_t(x), \qquad \varphi \in C_0(\Omega \times G_{d-1}),$$

for some countably $(d-1)$-rectifiable set M_t and a $\mathcal{H}^{d-1}\lfloor M_t$-measurable positive function θ_t, cf. [22]. In particular, the case that $\theta_t(x)$ is a positive integer for almost all (x,t) would give a more satisfactory answer to the existence of varifold solutions.

As noted by Plotnikov [17], the major problem is that (1.7) with $H_{\Gamma(t)}$ replaced by $-\delta V(t)$ gives only information of $\langle \delta V, \psi\rangle$ for $\psi \in C_0^\infty(Q)^d$ with $\operatorname{div}\psi = 0$. But in order to apply techniques from geometric measure theory it is necessary to have a good estimate of $\langle \delta V, \psi\rangle$ for $\psi \in C_0^\infty(Q)^d$ with $\operatorname{div}\psi \neq 0$ or at least for suitable gradients.

Because of the rectifiability result by Luckhaus [14], it would be sufficent to show

$$\delta V \in L^1(0,\infty; W_s^{-1}(\mathbb{R}^d)) \qquad \text{for some } s > 1$$

and a $(d-1)$-density bound of $|V(t)|$ from below in order to prove the rectifiability of $\widetilde{V}(t)$ for almost every $t > 0$, cf. [2, Appendix A] for the details.

At this point let us note a crutial difference between the two-phase flow for incompressible viscous fluids and parabolic surface evolution problems as f.e. the mean curvature flow or the Stefan problem with Gibbs-Thomsen law. This concerns the a priori estimates for sufficiently smooth solutions. If $\Gamma(t) \subseteq \mathbb{R}^d$, $t \in [0,T)$, is a family of smooth embedded closed $(d-1)$-dimensional surfaces solving the mean curvature equation

$$V(t,x) = H(t,x) \qquad \text{for all } t \in (0,T), x \in \Gamma(t)$$

where $V(t,x), H(t,x)$ are the normal velocity and mean curvature of $\Gamma(t)$, resp., then due to the first equality in (1.10) the “energy equality”

$$\mathcal{H}^{d-1}(\Gamma(t)) + \int_0^t \int_{\Gamma(t)} |H(\tau)|^2 \, d\mathcal{H}^{d-1} \, d\tau = \mathcal{H}^{d-1}(\Gamma(0))$$

holds for all $t \in (0,T)$, cf. Ecker [8, Chapter 4]. Hence the mean curvature H and therefore the first variation of $\delta V(t)$ are a priori bounded in $L^2(0,T; L^2(\Gamma(t); d\mathcal{H}^{d-1}))$. This is fundamentally used in order to construct Brakke's varifold solutions to the mean curvature flow, cf. e.g. [8] for more details.

In the case that $u\colon \mathbb{R}^d \times (0,T) \to \mathbb{R}$ and $\Gamma(t) \subset \mathbb{R}^d$, $t \in (0,T)$, are a smooth solution of the Stefan problem with Gibbs-Thomsen law:

$$\partial_t u - \Delta u = 0 \qquad \text{in } \Omega^\pm(t) \times (0,T) \tag{4.2}$$

$$u = \kappa H \qquad \text{on } \Gamma(t) \times (0,T) \tag{4.3}$$

$$n \cdot \nabla u^+ - n \cdot \nabla u^- = V \qquad \text{on } \Gamma(t) \times (0,T) \tag{4.4}$$

$$(u, \Gamma)|_{t=0} = (u_0, \Gamma_0) \tag{4.5}$$

the identity

$$\frac{1}{2}\|u(t)\|_2^2 + \kappa\mathcal{H}^{d-1}(\Gamma(t)) + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \, d\tau = \frac{1}{2}\|u(t)\|_2^2 + \kappa\mathcal{H}^{d-1}(\Gamma(0))$$

holds. This estimate is similar to (1.9), but the crutial difference is that (4.3) implies an a priori bound of H as well, which was used by Röger [19] in order to show the rectifiability of the varifold in the construction of weak solution of the Stefan problem. See also Luckhaus [13].

In the case for the two-phase flow discussed in this paper, (1.9) does not imply an a priori bound of the mean curvature since in the equation (1.3) an estimate of the pressure p is missing. Moreover, as pointed out by Beale [5, p.312] the free boundary value problem for the Navier-Stokes equation with surface tension seems to be more of mixed hyperbolic-parabolic character. The interface is merely transported with the flow of fluids and the presence of surface tension only assures the boundedness of the total area of the interface. In particular, there is no dissipation term related to the interface in the energy inequality. – Note that in the absence of friction in the bulk, i.e., $\nu \equiv 0$, the energy of the system, consisting of kinetic energy and potential energy related to the interface $\kappa\mathcal{H}^{d-1}(\Gamma(t))$, is conserved for smooth solutions. – Therefore the author believes that it is more instructive to look at the evolution of the interface in the two-phase flow as a damped wave equation rather than a parabolic surface evolution equation. Hence new techniques are needed to get information on the possible oscillatory behaviour of the interface.

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