

# Numerical approximation of viscoelastic Oldroyd-B flows in curved pipes

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## Abstract

The aim of this paper is to study a finite element numerical approximation of steady flows of an incompressible viscoelastic Oldroyd-B fluid in curved pipes of arbitrary cross-section and curvature ratio. Using rectangular toroidal coordinates, existence and uniqueness of approximated solutions are proved as well as a priori error estimates, under a natural restriction on the pipe curvature ratio.

## 1. Introduction

Fluids with complex microstructure such as inks, polymeric liquids, magma or biological fluids are such that the relation between the Cauchy stress and the strain tensor is non-linear and are called non-Newtonian fluids. The departure from the Navier-Stokes behavior manifests itself in a variety of ways [18]: non-Newtonian viscosity (shear-thinning or shear-thickening), stress relaxation, non-linear creeping, normal stresses and yield stress. Striking manifestations of the non-Newtonian phenomena have been observed experimentally, such as the Weissenberg or rod-climbing effect, extrudate swell or vortex growth in a contraction flow (see the monograph [7]).

In general terms, non-Newtonian viscoelastic fluids exhibit both viscous and elastic properties and can be classified as fluids of differential type, rate type and integral type ([18]). We refer to the monographs [6, 23, 26] for relevant issues related to non-Newtonian fluids behavior and modeling. Models of rate type such as Maxwell or Oldroyd-B fluids can predict stress relaxation and are used to describe flows in polymer processing.

Over the past twenty years, a significant progress has been made in the mathematical analysis, numerical approximation and simulations of the equations of motion of

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non-Newtonian viscoelastic fluids. Usually, the constitutive equations lead to highly non-linear systems of partial differential equations of a combined elliptic-hyperbolic type (or parabolic-hyperbolic, for unsteady flows) closed with appropriate boundary (or initial and boundary) conditions. The study of the behavior of their solutions in different geometries requires the use of specific techniques of non-linear analysis, such as fixed-point arguments associated to auxiliary linear sub-problems. We refer to [11] for an introduction to existence results in viscoelastic flows and to [19] for a description of some more recent mathematical developments in the area of non-Newtonian fluids.

The hyperbolic nature of the constitutive equations is responsible for many of the difficulties associated with the numerical analysis and simulation of viscoelastic flows. Some factors including singularities in the geometry, boundary layers in the flow and the dominance of the non-linear terms in the equations, result in numerical instabilities for high values of Weissenberg number (see [13, 15] and references cited therein). A variety of alternative numerical methods have been developed to overcome this difficulty, but many challenges still remain, in particular for viscoelastic flows in complex geometries. The numerical schemes used for solving these complex systems of PDEs must be based on a deep understanding of the mixed mathematical structure of the equations (elliptic-hyperbolic in the steady case), in order to prevent numerical instabilities on mathematically well-posed problems.

Steady fully developed viscous flows in curved pipes of circular, elliptical and annular cross-section of both Newtonian and non-Newtonian fluids, have been studied theoretically by several authors (see e.g. [10], [12], [16], [20], [21], [24], [25]) following the fundamental work of Dean [8] for Newtonian fluids in circular cross-section pipes. The great interest in the study of curved pipe flows is in particular due to its wide range of applications in engineering (e. g. hydraulic pipe systems related to corrosion failure) or in biofluid dynamics.

Our purpose in this paper is to study a finite element numerical approximation of steady flows of an incompressible viscoelastic Oldroyd-B fluid in curved pipes of arbitrary cross-section and curvature ratio. The governing equations, written in rectangular toroidal coordinates, are decoupled into a Stokes-like system and a tensorial transport equation, that are studied separately as two auxiliary problems. Existence and uniqueness of approximated solutions, as well as a priori error estimates, are established for both the Stokes-like system, discretized with the classical Hood-Taylor elements, and the transport equation, approximated by a discontinuous Galerkin method. Using a fixed-point argument we finally prove, under a natural restriction on the curvature ratio, the existence and uniqueness of an approximated solution to the coupled problem and give the corresponding error estimates.

In order to fix notation, the standard Sobolev spaces are denoted by  $W^{k,p}(\Omega)$  ( $k \in \mathbb{N}$  and  $1 < p < \infty$ ), and their norms by  $\|\cdot\|_{W^{k,p}}$ . We set  $W^{0,p}(\Omega) \equiv L^p(\Omega)$  and  $\|\cdot\|_{W^{0,p}} \equiv \|\cdot\|_{L^p}$ , and the norm in  $L^\infty$  is denoted  $\|\cdot\|_\infty$ . For  $p = 2$  and  $k \geq 0$ , we set  $W^{k,2}(\Omega) \equiv H^k(\Omega)$  and  $\|\cdot\|_{W^{k,2}} \equiv \|\cdot\|_k$ . Moreover, we use the following weighted norm  $\|\cdot\|_\alpha \equiv \|\alpha \cdot\|_0$ , for any real number  $\alpha$ .

## 2. Governing Equations

We are concerned with steady flows of incompressible viscoelastic Oldroyd-B fluids in a curved pipe  $\Omega \subset \mathbb{R}^3$ , with arbitrary (sufficiently smooth) cross-section  $\Sigma$ . For these fluids, the extra-stress tensor is related to the kinematic variables through

$$\mathbf{S} + \lambda_1 \overset{\nabla}{\mathbf{S}} = 2\mu(D\mathbf{u} + \lambda_2 \overset{\nabla}{D}\mathbf{u}), \quad (2.1)$$

where  $\mathbf{u}$  is the velocity field,  $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$  denotes the symmetric part of the velocity gradient,  $\mu > 0$  is the kinematic viscosity and  $\lambda_1, \lambda_2$  (with  $0 \leq \lambda_2 < \lambda_1$ ) are viscoelastic constants, representing the relaxation and retardation times, respectively, see e.g. [6, 18, 19, 23, 26]. The symbol  $\overset{\nabla}{\cdot}$  denotes the objective derivative of Oldroyd type defined by

$$\overset{\nabla}{\mathbf{S}} = \mathbf{u} \cdot \nabla \mathbf{S} - \mathbf{S} \nabla \mathbf{u} - (\nabla \mathbf{u})^t \mathbf{S} \quad (2.2)$$

The Cauchy stress tensor is given by  $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$ , where  $p$  represents the pressure. The equations of conservation of momentum and mass hold in the domain  $\Omega$ ,

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbf{S} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad (2.3)$$

where  $\rho > 0$  is the (constant) density of the fluid and  $\mathbf{f}$  is an external force. Decomposing the extra-stress tensor  $\mathbf{S}$  into the sum of its Newtonian part  $\boldsymbol{\tau}_s = 2\mu \frac{\lambda_2}{\lambda_1} D\mathbf{u}$  and its viscoelastic part  $\boldsymbol{\tau}$ , we rewrite (2.1)-(2.3) as

$$\begin{cases} -\frac{\lambda_2}{\lambda_1} \Delta \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \nabla \cdot \boldsymbol{\tau}, \\ \nabla \cdot \mathbf{u} = 0, \\ \boldsymbol{\tau} + \lambda_1 \overset{\nabla}{\boldsymbol{\tau}} = 2\mu(1 - \frac{\lambda_2}{\lambda_1}) D\mathbf{u}. \end{cases} \quad (2.4)$$

We consider the dimensionless form of this system by introducing the following quantities  $x = \frac{\tilde{x}}{L}$ ,  $u = \frac{\tilde{u}}{U}$ ,  $p = \frac{\tilde{p}L}{\mu U}$ ,  $\mathbf{f} = \frac{\tilde{\mathbf{f}}L^2}{\mu U}$ ,  $\boldsymbol{\tau} = \frac{\tilde{\boldsymbol{\tau}}L}{\mu U}$ , where the symbol  $\tilde{\cdot}$  is attached to dimensional parameters ( $L$  represents a reference length and  $U$  a characteristic velocity of the flow). We also set  $\varepsilon = 1 - \frac{\lambda_2}{\lambda_1}$ , and introduce the Reynolds and Weissenberg numbers  $\mathcal{Re} = \frac{\rho UL}{\mu}$ ,  $\mathcal{We} = \frac{\lambda_1 U}{L}$ , respectively. The dimensionless system takes the form

$$\begin{cases} -(1 - \varepsilon) \Delta \mathbf{u} + \mathcal{Re} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \nabla \cdot \boldsymbol{\tau}, \\ \nabla \cdot \mathbf{u} = 0, \\ \boldsymbol{\tau} + \mathcal{We} (\mathbf{u} \cdot \nabla \boldsymbol{\tau} - g(\nabla \mathbf{u}, \boldsymbol{\tau})) = 2\varepsilon D\mathbf{u}, \end{cases} \quad (2.5)$$

with  $g(\nabla \mathbf{u}, \boldsymbol{\tau}) = \boldsymbol{\tau} \nabla \mathbf{u} + (\nabla \mathbf{u})^t \boldsymbol{\tau}$ . This system is closed with a Dirichlet homogeneous boundary condition  $\mathbf{u} = 0$ .

### 3. Formulation in Rectangular Toroidal Coordinates

Due to the geometric characteristics of the curved pipe (see Fig. 1) it is convenient to write system (2.5) in the rectangular toroidal coordinates  $(\tilde{x}_i)$  defined with respect to the rectangular Cartesian coordinates  $(\tilde{y}_i)$  through the relations

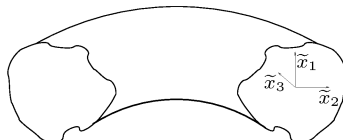


Figure 1. A segment of the curved pipe with centerline radius  $R$  and cross-sectional radius  $r_0$

$$\tilde{x}_1 = \tilde{y}_3, \quad \tilde{x}_2 = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2} - R, \quad \tilde{x}_3 = R \arctan \frac{\tilde{y}_2}{\tilde{y}_1}. \quad (3.1)$$

The orthonormal basis corresponding to the  $(\tilde{x}_i)$  basis will be denoted by  $(\mathbf{e}_i)$ . In the rectangular toroidal coordinates system, planes of constant  $\tilde{x}_3$  are perpendicular to the pipe centerline. We restrict our attention to curved pipes of constant but arbitrary cross-section  $\Sigma$  (with boundary  $\partial\Sigma$ ), namely the projection of the pipe surface on planes of constant  $\tilde{x}_3$  are independent of  $\tilde{x}_3$ . Introducing

$$r_0 = \sup_{\tilde{x} \in \Sigma} |\tilde{x}|, \quad x_i = \frac{\tilde{x}_i}{r_0}, \quad \delta = \frac{r_0}{R},$$

( $\delta \in [0, 1[$  is the pipe curvature ratio) we see that the corresponding non-dimensional coordinates system is given by

$$x_1 = y_3, \quad x_2 = \sqrt{y_1^2 + y_2^2} - \frac{1}{\delta}, \quad x_3 = \frac{1}{\delta} \arctan \frac{y_2}{y_1}.$$

By using standard arguments we can rewrite system (2.5) in the rectangular toroidal coordinates, with the differential operators defined below (see Section 6: Appendix). We consider the simpler case of fully developed flows, i.e. the components of the velocity vector and of the stress tensor with respect to the new basis are independent of the axial variable  $x_3$  ( $\frac{\partial u_i}{\partial x_3} = \frac{\partial \tau_{ij}}{\partial x_3} \equiv 0 \quad i, j = 1, 2, 3$ ) and the axial component of the pressure gradient  $\frac{\partial p}{\partial x_3} = p^*$  is a constant. System (2.5) takes the form

$$\begin{cases} -(1 - \varepsilon)\Delta \mathbf{u} + \mathcal{R}e \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \nabla \cdot \boldsymbol{\tau}, & \text{in } \Sigma \\ \nabla' \cdot (\beta \mathbf{u}) = 0, & \text{in } \Sigma \\ \boldsymbol{\tau} + \mathcal{W}e \mathbf{u} \cdot \nabla' \boldsymbol{\tau} = \mathcal{W}e \mathcal{G}(\mathbf{u}, \boldsymbol{\tau}) + 2\varepsilon D\mathbf{u}, & \text{in } \Sigma \\ \mathbf{u} = 0 & \text{on } \partial\Sigma \end{cases} \quad (3.2)$$

where

$$\Delta \mathbf{u} = \Delta' \mathbf{u} + \frac{\delta}{\beta} \frac{\partial \mathbf{u}}{\partial x_2} - \frac{\delta^2}{\beta^2} (u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3) \quad \beta \equiv \beta(x_2) = 1 + \delta x_2,$$

$$\begin{aligned}
 \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{u} \cdot \nabla' \mathbf{u} + \frac{\delta}{\beta} (u_3 u_2 \mathbf{e}_3 - u_3^2 \mathbf{e}_2), & \nabla p &= \nabla' p + \frac{v^*}{\beta} \mathbf{e}_3, \\
 \nabla \cdot \boldsymbol{\tau} &= \nabla' \boldsymbol{\tau} + \frac{\delta}{\beta} (\tau_{12} \mathbf{e}_1 + (\tau_{22} - \tau_{33}) \mathbf{e}_2 + 2\tau_{23} \mathbf{e}_3), \\
 \mathcal{G}(\mathbf{u}, \boldsymbol{\tau}) &= g(\nabla' \mathbf{u}, \boldsymbol{\tau}) + \frac{\delta}{\beta} \sum_{i=1}^2 (u_2 \tau_{i3} - u_3 \tau_{i2}) \mathbf{e}_i \mathbf{e}_3 + \frac{2\delta}{\beta} (u_2 \tau_{33} - u_3 \tau_{32}) \mathbf{e}_3 \mathbf{e}_3, \\
 D\mathbf{u} &= D' \mathbf{u} + \frac{\delta}{2\beta} (2u_2 \mathbf{e}_3 \mathbf{e}_3 - u_3 (\mathbf{e}_3 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_3)).
 \end{aligned}$$

## 4. Numerical Approximation of the Problem

This section is devoted to the numerical approximation of the nonlinear problem (3.2) using finite element methods. In order to simplify the estimates and without loss of generality we consider  $\mathcal{R}e = 0$ , corresponding to creeping flows. This system is of mixed elliptic-hyperbolic type and will be decoupled into a Stokes-like problem and a tensorial transport equation that will be studied separately. A fixed-point argument will be used to prove the existence and uniqueness of the approximated solutions as well as a priori error estimates.

### 4.1. Discrete Stokes-like system

We first consider the numerical approximation of the following Stokes-like system

$$\begin{cases}
 -\Delta' \mathbf{u} - \frac{\delta}{\beta} \frac{\partial \mathbf{u}}{\partial x_2} + \left(\frac{\delta}{\beta}\right)^2 (u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3) + \nabla' p = \mathbf{f} & \text{in } \Sigma, \\
 \nabla' \cdot (\beta \mathbf{u}) = 0 & \text{in } \Sigma, \\
 \mathbf{u} = 0 & \text{on } \partial \Sigma,
 \end{cases} \quad (4.1)$$

and derive sharp estimates with respect to the curvature ratio  $\delta$ . Notice that in the particular case of  $\delta \equiv 0$ , we recover the classical Stokes system.

Let us give a more suitable equivalent weak formulation of this problem. For fixed  $\delta$  we set  $\mathbf{X}^\beta = \{\phi \in \mathbf{H}_0^1(\Sigma) \mid \nabla' \cdot (\beta \phi) = 0\}$ . The space  $\mathbf{X}_\beta$  is a Hilbert space and its norm will be usually denoted by  $\|\cdot\|_1$ . Its dual space is denoted by  $\mathbf{X}'_\beta$  the associated norm by  $\|\cdot\|_{-1}$  and the duality pairing between  $\mathbf{X}_\beta$  and  $\mathbf{X}'_\beta$  by  $\langle \cdot, \cdot \rangle$ . Observing that, for all  $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Sigma)$ ,

$$(\beta \nabla' \mathbf{u}, \nabla' \mathbf{v}) = 2(\beta D' \mathbf{u}, D' \mathbf{v}) - \left(\frac{1}{\beta} \nabla' \cdot (\beta \mathbf{u}), \nabla' \cdot (\beta \mathbf{v})\right) + \left(\frac{\delta^2}{\beta} u_2, v_2\right),$$

we consider the bilinear form  $\mathcal{A}_\delta$  defined by

$$\mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) \equiv 2(\beta D' \mathbf{u}, D' \mathbf{v}) + 2\left(\frac{\delta^2}{\beta} u_2, v_2\right) + \left(\frac{\delta^2}{\beta} u_3, v_3\right). \quad (4.2)$$

A weak solution of problem (4.1) is defined as follows

**Definition 4.1.** Let  $\mathbf{f} \in \mathbf{X}'_\beta$ . A pair  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Sigma) \times L^2(\Sigma)$  is a weak solution of (3.2) if,

$$\begin{cases}
 \mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) - (p, \nabla' \cdot (\beta \mathbf{v})) = \langle \mathbf{f}, \beta \mathbf{v} \rangle, & \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Sigma), \\
 (q, \nabla' \cdot (\beta \mathbf{u})) = 0, & \text{for all } q \in L_0^2(\Sigma).
 \end{cases} \quad (4.3)$$

Using Poincaré and Korn inequalities we can easily prove

**Proposition 4.1.** *For every  $\delta \in [0, 1[$ , the bilinear form  $\mathcal{A}_\delta$  is continuous and coercive in  $(\mathbf{H}_0^1(\Sigma))^2$ .*

Here we suppose that the vector field  $\mathbf{f}$  has the form  $\mathbf{f} = \nabla \cdot \tau + \mathbf{F}$ , where  $\tau$  is a given tensor and  $\mathbf{F}$  a given vector and rewrite the weak formulation (4.3) as

$$\begin{cases} \mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) - (p, \nabla' \cdot (\beta \mathbf{v})) + (\tau, \beta D\mathbf{v}) - (\mathbf{F}, \beta \mathbf{v}) = 0 & \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Sigma) \\ (q, \nabla' \cdot (\beta \mathbf{u})) = 0 & \text{for all } q \in L_0^2(\Sigma). \end{cases} \quad (4.4)$$

The natural Galerkin approximation of problem (4.4) is a mixed method associated to the approximation of saddle point problems, in which we consider two bilinear forms and two approximation spaces satisfying a compatibility condition. Let  $(\mathcal{T}_h)_h$  be a family of regular triangulations with non-degenerate triangles and define the following finite element spaces

$$\begin{aligned} \mathbf{X}_h &= \{ \mathbf{v}_h \in \mathbf{C}(\bar{\Sigma}) \cap \mathbf{H}_0^1(\Sigma) \mid \mathbf{v}_h|_K \in \mathbb{P}_2(K), \forall K \in \mathcal{T}_h \}, \\ Q_h &= \{ q_h \in C(\bar{\Sigma}) \cap L_0^2(\Sigma) \mid q_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h \}, \\ \mathbf{X}_{\beta, h} &= \{ \mathbf{v}_h \in \mathbf{X}_h \mid \nabla' \cdot (\beta \mathbf{v}_h) = 0 \}. \end{aligned}$$

This pair of spaces  $(\mathbf{X}_h, Q_h)$  corresponds to the so-called *Hood-Taylor* finite element method, and verifies a compatibility condition known as the *discrete LBB* (or *inf-sup*) condition (see e.g. [17]), which reads as follows:

$$\exists \gamma^* (\text{independent of } h) \text{ s.t. } \inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{|(q_h, \nabla' \cdot \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathbf{X}_h} \|q_h\|_{Q_h}} \geq \gamma^*.$$

Due to the particular form of our problem, and assuming that  $\delta < \gamma^*$ , we have

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{|(q_h, \nabla' \cdot (\beta \mathbf{v}_h))|}{\|\mathbf{v}_h\|_{\mathbf{X}_h} \|q_h\|_{Q_h}} \geq \gamma^* - \delta. \quad (4.5)$$

To problem (4.4), we associate the following approximated problem

Find  $(\mathbf{u}_h, p_h)$  in  $\mathbf{X}_h \times Q_h$  such that:

$$\begin{cases} \mathcal{A}_\delta(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla' \cdot (\beta \mathbf{v}_h)) + (\tau, \beta D\mathbf{v}_h) - (\mathbf{F}, \beta \mathbf{v}_h) = 0 & \text{for all } \mathbf{v}_h \in \mathbf{X}_h \\ (q_h, \nabla' \cdot (\beta \mathbf{u}_h)) = 0 & \text{for all } q_h \in Q_h. \end{cases} \quad (4.6)$$

The next theorem deals with the existence and uniqueness of an approximated solution to problem (4.6), as well as the corresponding estimates.

**Theorem 4.1.** *Let  $(\tau, \mathbf{F}) \in (\mathbf{L}^2(\Sigma))^2$ . If  $\delta < \gamma^*$ , then system (4.6) admits a unique solution  $(\mathbf{u}_h, p_h)$ , and there exists a positive constant  $C \equiv C(\Sigma)$  (independent of  $\delta$  and of  $h$ ) such that following inequality*

$$(\mathcal{A}_\delta(\mathbf{u}_h, \mathbf{u}_h))^{1/2} \leq C(\|\tau\|_{\beta^{1/2}} + \frac{1}{(1-\delta)^{1/2}} \|\mathbf{F}\|_0)$$

is satisfied. Moreover, if  $(\mathbf{u}, p)$  is the solution of system (4.4), then the following estimate holds

$$\begin{aligned} & (\mathcal{A}_\delta(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2} \\ & \leq C_1 \left(1 + \frac{1}{(\gamma^* - \delta)(1 - \delta)^{1/2}}\right) \inf_{\mathbf{v}_h \in \mathbf{X}_h} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2} + C \inf_{q_h \in Q_h} \|p - q_h\|_0, \end{aligned} \quad (4.7)$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $\delta$  and of  $h$ .

*Proof.* Due to Proposition 4.1, the bilinear form  $\mathcal{A}_\delta$  is continuous and coercive. Moreover, the inf-sup condition (4.5) is satisfied. Classical arguments ensure the existence and uniqueness of a solution  $(\mathbf{u}_h, p_h)$  to problem (4.6). Setting  $\mathbf{v}_h = \mathbf{u}_h$  in (4.6) and using the Poincaré and the Korn inequalities, we obtain

$$\begin{aligned} \mathcal{A}_\delta(\mathbf{u}_h, \mathbf{u}_h) & \leq \|\tau\|_{\beta^{1/2}} \|D\mathbf{u}_h\|_{\beta^{1/2}} + \|\mathbf{F}\|_0 \|\beta\mathbf{u}_h\|_0 \\ & \leq C \|\tau\|_{\beta^{1/2}} (\mathcal{A}_\delta(\mathbf{u}_h, \mathbf{u}_h))^{1/2} + C \|\mathbf{F}\|_0 \left(\frac{1+\delta}{\sqrt{1-\delta}}\|D'\mathbf{u}_h\|_{\beta^{1/2}} + \right. \\ & \quad \left. (1+\delta)^{1/2} \sum_{i=2}^3 \|(\mathbf{u}_h)_i\|_{\delta\beta^{-1/2}}\right) \\ & \leq C \left(\|\tau\|_{\beta^{1/2}} + \frac{1}{(1-\delta)^{1/2}}\|\mathbf{F}\|_0\right) (\mathcal{A}_\delta(\mathbf{u}_h, \mathbf{u}_h))^{1/2}. \end{aligned}$$

This gives the first estimate. The proof of estimate (4.7) is split into two steps.

*Step 1.* It is obvious that if  $(\mathbf{u}_h, p_h)$  is a solution to problem (4.6) and  $(\mathbf{u}, p)$  is a solution to problem (4.4) we have

$$\mathcal{A}_\delta(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) = -(p, \nabla'(\beta\mathbf{v}_h)) \quad \text{for all } \mathbf{v}_h \in \mathbf{X}_{\beta,h}.$$

Let  $\mathbf{w}_h \in \mathbf{X}_{\beta,h}$  and set  $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h \in \mathbf{X}_{\beta,h}$ . The last identity together with the continuity of the bilinear form  $\mathcal{A}_\delta$ , gives

$$\begin{aligned} \mathcal{A}_\delta(\mathbf{v}_h, \mathbf{v}_h) & = \mathcal{A}_\delta(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + \mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) \\ & \leq |(\beta^{1/2}(p - q_h), \beta^{1/2} \nabla' \cdot \mathbf{v}_h + \delta\beta^{-1/2} \mathbf{v}_{h2})| + |\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h)| \\ & \leq 2\|p - q_h\|_0 (\|D'\mathbf{v}_h\|_{\beta^{1/2}} + \|\mathbf{v}_{h2}\|_{\delta\beta^{-1/2}}) + \\ & \quad C(\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} (\mathcal{A}_\delta(\mathbf{v}_h, \mathbf{v}_h))^{1/2} \\ & \leq C(\|p - q_h\|_0 + (\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2}) (\mathcal{A}_\delta(\mathbf{v}_h, \mathbf{v}_h))^{1/2}, \end{aligned}$$

for all  $q_h \in Q_h$ , where  $C \equiv C(\Sigma)$ . Therefore

$$(\mathcal{A}_\delta(\mathbf{v}_h, \mathbf{v}_h))^{1/2} \leq C((\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} + \|p - q_h\|_0),$$

and for all  $(\mathbf{w}_h, q_h) \in \mathbf{X}_{\beta,h} \times Q_h$ , we have

$$(\mathcal{A}_\delta(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2} \leq C((\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} + \|p - q_h\|_0).$$

This yields the following estimate

$$(\mathcal{A}_\delta(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2} \leq C \left( \inf_{\mathbf{w}_h \in \mathbf{X}_{\beta,h}} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} + \inf_{q_h \in Q_h} \|p - q_h\|_0 \right), \quad (4.8)$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $\delta$  and  $h$ .

*Step 2.* To derive the error bound, we estimate the right-hand side term in (4.8). Let  $\mathbf{v}_h$  be an element of  $\mathbf{X}_h$ . The inf-sup condition (4.5) together with classical arguments ensure existence of a unique  $\mathbf{z}_h \in (\mathbf{X}_{\beta,h})^\perp$  such that

$$(\nabla' \cdot (\beta \mathbf{z}_h), q_h) = (\nabla' \cdot (\beta(\mathbf{u} - \mathbf{v}_h)), q_h),$$

and

$$\begin{aligned} \|\mathbf{z}_h\|_1 &\leq \frac{1}{\gamma^* - \delta} \|\nabla' \cdot (\beta(\mathbf{u} - \mathbf{v}_h))\|_0 \leq \frac{1}{\gamma^* - \delta} \|\beta D(\mathbf{u} - \mathbf{v}_h)\|_0 \\ &\leq \frac{1}{\gamma^* - \delta} \|\beta^{1/2}\|_\infty (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2} \leq \frac{(1+\delta)^{1/2}}{\gamma^* - \delta} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2} \end{aligned}$$

Therefore, we have

$$(\mathcal{A}_\delta(\mathbf{z}_h, \mathbf{z}_h))^{1/2} \leq \frac{C}{(\gamma^* - \delta)(1 - \delta)^{1/2}} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2}. \quad (4.9)$$

On the other hand, if we set  $\mathbf{w}_h = \mathbf{z}_h + \mathbf{v}_h$ , then for all  $q_h \in Q_h$

$$(\nabla' \cdot (\beta \mathbf{w}_h), q_h) = (\nabla' \cdot (\beta(\mathbf{u} - \mathbf{v}_h)), q_h) + (\nabla' \cdot (\beta \mathbf{v}_h), q_h) = 0,$$

and thus  $\mathbf{w}_h \in \mathbf{X}_{\beta,h}$ . Furthermore, from (4.9) we get

$$(\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} \leq \left(1 + \frac{C}{(\gamma^* - \delta)(1 - \delta)^{1/2}}\right) (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2}.$$

As  $\mathbf{v}_h$  is arbitrary, this implies

$$\inf_{\mathbf{w}_h \in \mathbf{X}_{\beta,h}} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{w}_h, \mathbf{u} - \mathbf{w}_h))^{1/2} \leq \left(1 + \frac{C}{(\gamma^* - \delta)(1 - \delta)^{1/2}}\right) \inf_{\mathbf{v}_h \in \mathbf{X}_h} (\mathcal{A}_\delta(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h))^{1/2}.$$

Estimate (4.7) is then a direct consequence of the combination of the last inequality with (4.8).  $\square$

Using standard arguments it is easy to prove the following estimate

**Lemma 4.1.** *Let  $\gamma \in [0, 1[$  and let  $\mathbf{v} \in \mathbf{H}^{k+1}(\Sigma)$ , for  $k = 1, 2$ . There exists  $\widehat{\mathbf{v}}_\gamma \in \mathbf{X}_h$  satisfying*

$$\mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \psi) = 0, \quad \text{for all } \psi \in \mathbf{X}_h. \quad (4.10)$$

Moreover, there exists a constant  $C$ , independent of  $h$  and  $\gamma$  such that

$$(\mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \mathbf{v} - \widehat{\mathbf{v}}_\gamma))^{1/2} \leq C \left(1 + \frac{h}{(1-\gamma)^{1/2}}\right) h^k \|\mathbf{v}\|_{k+1}. \quad (4.11)$$

*Proof.* Let  $\widehat{\mathbf{v}}_\gamma$  be the solution of the following problem

$$\mathcal{A}_\gamma(\widehat{\mathbf{v}}_\gamma, \psi) = (-\gamma \Delta' \mathbf{v} - \gamma \frac{\partial \mathbf{v}}{\partial x_2} + \frac{\gamma}{\beta_\gamma} (v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3), \psi) \quad \forall \psi \in \mathbf{X}_h$$



with  $\beta_\gamma = 1 + \gamma x_2$ . A straightforward integration by parts shows that (4.10) is satisfied. Let  $\mathbf{w} \in \mathbf{X}_h$  and set  $\psi = \widehat{\mathbf{v}}_\gamma - \mathbf{w}$ . Taking into account (4.10), we obtain

$$\begin{aligned} \mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \mathbf{v} - \widehat{\mathbf{v}}_\gamma) &= \mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \mathbf{v} - \widehat{\mathbf{v}}_\gamma + \psi) = \mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \mathbf{v} - \mathbf{w}) \\ &\leq 2(1 + \gamma)^{1/2} \|D'(\mathbf{v} - \widehat{\mathbf{v}}_\gamma)\|_{\beta_\gamma^{1/2}} \|D'(\mathbf{v} - \mathbf{w})\|_0 + \\ &\quad \frac{2}{(1-\gamma)^{1/2}} \sum_{i=2}^3 \|(\mathbf{v} - \widehat{\mathbf{v}}_\gamma)_i\|_{\gamma\beta_\gamma^{-1/2}} \|(\mathbf{v} - \mathbf{w})_i\|_0 \\ &\leq C(\mathcal{A}_\gamma(\mathbf{v} - \widehat{\mathbf{v}}_\gamma, \mathbf{v} - \widehat{\mathbf{v}}_\gamma))^{1/2} (\|D'(\mathbf{v} - \mathbf{w})\|_0 + \frac{1}{(1-\gamma)^{1/2}} \|\mathbf{v} - \mathbf{w}\|_0). \end{aligned}$$

Choosing  $\mathbf{w} = \Pi_h \mathbf{v} \in \mathbf{X}_h$ , the interpolant of  $\mathbf{v}$ , and using standard interpolation estimates, we deduce (4.11).  $\square$

If  $(\mathbf{u}, p)$  is a regular solution of problem (4.6) we prove the following estimate.

**Corollary 4.1.** *Assume that the hypotheses of Theorem 4.1 are satisfied. If  $(\mathbf{u}, p) \in \mathbf{H}^{k+1}(\Sigma) \times \mathbf{H}^k(\Sigma)$  ( $k = 1, 2$ ), then the following estimate holds*

$$(\mathcal{A}_\delta(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2} \leq Ch^k \left( \left(1 + \frac{1}{(\gamma^* - \delta)(1-\delta)^{1/2}}\right) \left(1 + \frac{h}{(1-\delta)^{1/2}}\right) \|\mathbf{u}\|_{k+1} + \|p\|_k \right),$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $\delta$  and of  $h$ .

*Proof.* This result is a direct consequence of Theorem 4.1, Lemma 4.1 and of classical interpolation error estimates.  $\square$

## 4.2. Discrete transport equation

In this section, we consider the tensorial steady transport equation

$$\tau + We \phi \cdot \nabla' \tau = \mathbf{G} \quad \text{in } \Sigma, \quad (4.12)$$

where  $\phi$  is a vector field in  $\mathbf{X}_{\beta,h}$ , and where  $\mathbf{G}$  is a tensor in  $\mathbf{L}^2(\Sigma)$ .

We use a discontinuous Galerkin finite element method to approximate this transport equation. Let  $T_h$  and  $\mathbf{T}_h$  be the following discretization spaces

$$T_h = \{\tau_h \in L^2(\Sigma) \mid \tau_h|_K \in \mathcal{P}_1, \forall K \in \mathcal{T}_h\}, \quad \mathbf{T}_h = (T_h)^{3 \times 3}.$$

For an element  $K \in \mathcal{T}_h$ , let  $\partial K$  denote its boundary. The inflow edge corresponding to the element  $K$  is defined by

$$\partial K^-(\phi) = \{\mathbf{x} \in \partial K \mid \phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\},$$

where  $\mathbf{n}$  is the unit outward normal vector. For  $\mathbf{x} \in \partial K$  such that  $\phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \neq 0$ , we define the left hand and right hand limits  $\sigma^-$  and  $\sigma^+$  as

$$\sigma^-(\mathbf{x}) = \lim_{\epsilon \rightarrow 0^-} \sigma(\mathbf{x} + \epsilon \phi(\mathbf{x})), \quad \sigma^+(\mathbf{x}) = \lim_{\epsilon \rightarrow 0^+} \sigma(\mathbf{x} + \epsilon \phi(\mathbf{x})).$$

Let  $\mathcal{B}_h$  be the trilinear form defined by

$$\mathcal{B}_h(\phi, \tau, \sigma) = (\phi \cdot \nabla' \tau, \sigma)_h + \frac{1}{2}(\nabla' \cdot \phi) \tau, \sigma - \langle \tau^+ - \tau^-, \sigma^+ \rangle_{h, \phi}, \quad (4.13)$$

where  $(\sigma, \tau)_h = \sum_{K \in \mathcal{T}_h} (\sigma, \tau)_K$ ,  $\langle \sigma, \tau \rangle_{h, \phi} = \sum_{K \in \mathcal{T}_h} \int_{\partial K^-(\phi)} \sigma : \tau \phi \cdot \mathbf{n} ds$  and  $\langle \langle \sigma \rangle \rangle_{h, \phi} = \langle \sigma \rangle_{h, \phi}^{1/2}$ .

The discontinuous Galerkin method applied to problem (4.12) can be stated in the following way:

Determine  $\tau_h \equiv \tau \in \mathbf{T}_h$  such that

$$(\tau, \beta \sigma) + We \mathcal{B}_h(\beta \phi, \tau, \sigma) = (\mathbf{G}, \beta \sigma), \quad \text{for all } \sigma \in \mathbf{T}_h. \quad (4.14)$$

**Proposition 4.2.** *Let  $(\phi, \tau) \in \mathbf{X}_h \times \mathbf{T}_h$ ,  $\epsilon = (\epsilon_{ij})$  is a  $3 \times 3$  matrix such that  $\epsilon_{ij} \in \{0, 1\}$ , and set  $\tau_\epsilon \equiv (\tau_{ij} \epsilon_{ij})$ . Then,*

$$(\tau, \beta \tau_\epsilon) + We \mathcal{B}_h(\beta \phi, \tau, \tau_\epsilon) = \|\tau_\epsilon\|_{\beta^{1/2}}^2 + \frac{We}{2} \langle \langle \tau_\epsilon^- - \tau_\epsilon^+ \rangle \rangle_{h, \beta \phi}^2. \quad (4.15)$$

*Proof.* Integration by parts shows that

$$(\beta \phi \cdot \nabla' \tau_\epsilon, \tau_\epsilon)_h + (\beta \phi \cdot \nabla' \tau_\epsilon, \tau_\epsilon)_h + \frac{1}{2}((\nabla' \cdot (\beta \phi)) \tau_\epsilon, \tau_\epsilon) = \langle \tau_\epsilon^-, \tau_\epsilon^- \rangle_{h, \beta \phi} - \langle \tau_\epsilon^+, \tau_\epsilon^+ \rangle_{h, \beta \phi},$$

and thus,

$$\mathcal{B}_h(\beta \phi, \tau, \tau_\epsilon) = -(\beta \phi \cdot \nabla' \tau_\epsilon, \tau_\epsilon)_h - \frac{1}{2}((\nabla' \cdot (\beta \phi)) \tau_\epsilon, \tau_\epsilon) + \langle \tau_\epsilon^-, \tau_\epsilon^- - \tau_\epsilon^+ \rangle_{h, \beta \phi}.$$

Therefore,

$$\mathcal{B}_h(\beta \phi, \tau_\epsilon, \tau_\epsilon) = \frac{1}{2} \langle \tau_\epsilon^+ - \tau_\epsilon^-, \tau_\epsilon^+ - \tau_\epsilon^- \rangle_{h, \beta \phi} = \frac{1}{2} \langle \langle \tau_\epsilon^+ - \tau_\epsilon^- \rangle \rangle_{h, \beta \phi}^2.$$

The conclusion follows by observing that  $(\beta \tau, \tau_\epsilon) = \|\tau_\epsilon\|_{\beta^{1/2}}^2$ .  $\square$

The following result states existence and uniqueness of solutions for the approximated problem (4.14).

**Theorem 4.2.** *Problem (4.14) admits a unique solution  $\tau_h$ . Moreover, the following estimate holds*

$$\|\tau_h\|_{\beta^{1/2}} \leq \|\mathbf{G}\|_{\beta^{1/2}}.$$

*Proof.* Existence of a unique solution to problem (4.14) is a consequence of the Lax-Milgram Theorem and of Proposition 4.2. Setting  $\sigma = \tau_h$  in (4.14), we obtain

$$\|\tau_h\|_{\beta^{1/2}}^2 \leq \|\tau_h\|_{\beta^{1/2}}^2 + \frac{We}{2} \langle \langle \tau_h^+ - \tau_h^- \rangle \rangle_{h, \beta \phi}^2 = (\mathbf{G}, \beta \tau_h) \leq \|\mathbf{G}\|_{\beta^{1/2}} \|\tau_h\|_{\beta^{1/2}}. \quad \square$$

### 4.3. Setting of the approximated problem

Using notation already introduced, the approximate problem is defined as follows

Find  $(\mathbf{u}_h, p_h, \tau_h) \equiv (\mathbf{u}, p, \tau) \in \mathbf{X}_h \times Q_h \times \mathbf{T}_h$  solution of

$$\begin{cases} (1 - \varepsilon)\mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) + (p, \nabla' \cdot (\beta \mathbf{v})) = -(p^*, \mathbf{v}_3) - (\tau, \beta D\mathbf{v}), & \text{for all } \mathbf{v} \in \mathbf{X}_h \\ (q, \nabla' \cdot (\beta \mathbf{u})) = 0, & \text{for all } q \in Q_h \\ (\tau, \beta \sigma) + We \mathcal{B}_h(\beta \mathbf{u}, \tau, \sigma) = (We \mathcal{G}(\mathbf{u}, \tau) + 2\varepsilon D\mathbf{u}, \beta \sigma), & \text{for all } \sigma \in \mathbf{T}_h \end{cases} \quad (4.16)$$

Assuming that the discrete inf-sup condition (4.5) is satisfied, then system (4.16) is equivalent to the following problem

Find  $(\mathbf{u}_h, \tau_h) \equiv (\mathbf{u}, \tau) \in \mathbf{X}_{\beta, h} \times \mathbf{T}_h$  solution of

$$\begin{cases} (1 - \varepsilon)\mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) = -(p^*, \mathbf{v}_3) - (\tau, \beta D\mathbf{v}), & \text{for all } \mathbf{v} \in \mathbf{X}_{\beta, h} \\ (\tau, \beta \sigma) + We \mathcal{B}_h(\beta \mathbf{u}, \tau, \sigma) = (We \mathcal{G}(\mathbf{u}, \tau) + 2\varepsilon D\mathbf{u}, \beta \sigma), & \text{for all } \sigma \in \mathbf{T}_h \end{cases} \quad (4.17)$$

Next theorem establishes the main results of this paper: existence of a unique solution to problem (4.17) and the corresponding error estimates.

**Theorem 4.3.** *Assume that the curvature ratio  $\delta$  satisfies  $\delta < \gamma^*$ . Let  $(\bar{\mathbf{u}}, \bar{p}, \bar{\tau}) \in \mathbf{H}^3(\Sigma) \times H^2(\Sigma) \times \mathbf{H}^2(\Sigma)$  be a strong solution of system (3.2) satisfying  $\|\bar{\mathbf{u}}\|_3 + \|\bar{p}\|_2 + \|\bar{\tau}\|_2 \leq \kappa$ , let  $\alpha$  and  $\theta$  be two positive constants satisfying  $2\varepsilon < 2\alpha\varepsilon < 1 + \varepsilon$ ,  $\theta = 1 - (2\alpha - 1)\varepsilon$ . There exist positive constants  $C^*$ ,  $C^{**}$ ,  $C$  and  $h_0$  independent of  $h$ ,  $\kappa$ ,  $We$ ,  $\delta$ ,  $\varepsilon$ , such that if*

$$\kappa \leq \frac{\theta \min(C^*, \varepsilon\theta C^{**})(1 - \delta)}{\alpha We(1 + (\alpha - 1)^{-1/2})},$$

then for every  $h$  satisfying  $h \leq \max(\theta^2, \sqrt{1 - \delta}, h_0)$ , problem (4.16)<sub>1,2</sub> admits a unique solution  $(\bar{\mathbf{u}}_h, \bar{p}_h, \bar{\tau}_h) \in \mathbf{X}_{\beta, h} \times Q_h \times \mathbf{T}_h$ . Moreover, the following estimate holds

$$(\mathcal{A}_\delta(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h, \bar{\mathbf{u}} - \bar{\mathbf{u}}_h))^{1/2} + \|\bar{\tau} - \bar{\tau}_h\|_{\beta^{1/2}} \leq C\lambda h^{3/2},$$

where  $\lambda = \frac{8\kappa}{\theta} \max\left(\left(1 + \frac{h_{\max}}{(1 - \delta)^{1/2}}\right)h_{\max}^{1/2}, \frac{\alpha We \kappa}{(1 - \delta)(\alpha - 1)^{1/2}} + \frac{(\alpha We \kappa)^{1/2}}{(1 - \delta)^{1/2}} + 1, \frac{8\theta\alpha We \kappa}{1 - \delta}\right)$ .

To prove existence and uniqueness of solutions to problem (4.16), we consider the following composite mapping

$$\begin{aligned} \Phi_\omega : \mathbf{X}_{\beta, h} \times \mathbf{T}_h &\longrightarrow \mathbf{X}_{\beta, h} \times \mathbf{T}_h \\ (\phi, \varphi) &\longmapsto (\mathbf{u}, \tau) \end{aligned}$$

defined through the coupled system

$$\begin{cases} \mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) = \varepsilon\mathcal{A}_\delta(\phi, \mathbf{v}) - (p^*, \mathbf{v}_3) - (\tau, \beta D\mathbf{v}), \\ (\tau, \beta \sigma) + We \mathcal{B}_h(\beta \phi, \tau, \sigma) = (We \mathcal{G}(\phi, \varphi) + 2\varepsilon D\phi, \beta \sigma). \end{cases} \quad (4.18)$$

Following [4] and [22] where a model of Oldroyd's type has been studied in bounded domains, we decompose the proof into three parts. We first prove that  $\Phi_\omega$  is well defined and bounded on bounded sets. Next, we prove that there exists a ball  $B_h$  with center  $(\bar{\mathbf{u}}, \bar{\tau})$ , such that  $B_h$  is nonempty and  $\Phi_\omega(B_h) \subset B_h$ . Finally, we conclude that  $\Phi_\omega$  is a contraction. These assertions will be respectively proved in Lemma 5.1, Lemma 5.2 and Lemma 5.3.

#### 4.4. Useful estimates

The aim of this section is to prove some useful auxiliary results. We begin by recalling some classical interpolation estimates. Let  $\tau \in \mathbf{H}^2(\Sigma)$ , and let  $\hat{\tau}$  be its orthogonal projection in  $\mathbf{L}^2(\Sigma)$ , i.e.  $(\tau - \hat{\tau}, \sigma) = 0$ , for all  $\sigma \in \mathbf{T}_h$ . Then the following estimates hold

$$\|\tau - \hat{\tau}\|_0 + h\|\tau - \hat{\tau}\|_{1,h} \leq C_0 h^2 \|\tau\|_2. \quad (4.19)$$

$$\|\tau - \hat{\tau}\|_{0,K} + h_K \|\tau - \hat{\tau}\|_{1,K} \leq C_0 h_K^2 \|\tau\|_{2,K} \quad \text{for all } K \in \mathcal{T}_h \quad (4.20)$$

$$\|\tau - \hat{\tau}\|_{0,\Gamma_h} \leq C_0 h^{3/2} \|\tau\|_2, \quad (4.21)$$

where  $C_0$  is a positive constant independent of  $h$ . In the next lemma, we consider the nonlinear terms appearing in the transport equation, and prove some corresponding estimates.

**Lemma 4.2.** *Let  $(\phi, \phi_0) \in (\mathbf{H}_0^1(\Sigma))^2$ ,  $(\varphi, \varphi_0) \in (\mathbf{L}^2(\Sigma))^2$ , and let  $\sigma \in \mathbf{T}_h$ . Then, for all  $(\mathbf{v}, \tau) \in (\mathbf{W}^{1,\infty}(\Sigma) \cap \mathbf{H}_0^1(\Sigma)) \times \mathbf{L}^\infty(\Sigma)$ , we have*

$$\begin{aligned} & |(\mathcal{G}(\phi, \varphi) - \mathcal{G}(\phi_0, \varphi_0), \beta\sigma)| \\ & \leq C \|D'(\phi - \phi_0)\|_{\beta^{1/2}} \left( \frac{1}{h(1-\delta)} \|\varphi - \tau\|_{\beta^{1/2}} + \frac{1}{(1-\delta)^{1/2}} \|\tau\|_\infty \right) \|\sigma\|_{\beta^{1/2}} + \\ & \quad \frac{C}{h(1-\delta)} \|D'(\phi_0 - \mathbf{v})\|_{\beta^{1/2}} \|\varphi - \varphi_0\|_{\beta^{1/2}} \|\sigma\|_{\beta^{1/2}} + \\ & \quad C \sum_{i=2}^3 \|(\phi - \phi_0)_i\|_{\delta\beta^{-1/2}} \left( \frac{1}{h(1-\delta)^{1/2}} \|\varphi - \tau\|_{\beta^{1/2}} + \|\tau\|_\infty \right) \sum_{i=1}^3 \|\sigma_{i3}\|_{\beta^{1/2}} + \\ & \quad \frac{C}{h(1-\delta)^{1/2}} \sum_{i=2}^3 \|(\phi_0 - \mathbf{v})_i\|_{\delta\beta^{-1/2}} \|\varphi - \varphi_0\|_{\beta^{1/2}} \sum_{i=1}^3 \|\sigma_{i3}\|_{\beta^{1/2}} + \\ & \quad C \|\varphi - \varphi_0\|_{\beta^{1/2}} (\|\nabla' \mathbf{v}\|_\infty \|\sigma\|_{\beta^{1/2}} + \frac{\delta}{1-\delta} \sum_{i=2}^3 \|\mathbf{v}_i\|_\infty \sum_{i=1}^3 \|\sigma_{i3}\|_{\beta^{1/2}}), \end{aligned}$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $h$  and  $\delta$ .

*Proof.* By using standard arguments, we obtain

$$\begin{aligned} & |(\mathcal{G}(\phi, \varphi) - \mathcal{G}(\phi_0, \varphi_0), \beta\sigma)| \leq |(\mathcal{G}(\phi - \phi_0, \tau) + \mathcal{G}(\mathbf{v}, \varphi - \varphi_0), \beta\sigma)| + \\ & \quad |(\mathcal{G}(\phi - \phi_0, \varphi - \tau) + \mathcal{G}(\phi_0 - \mathbf{v}, \varphi - \varphi_0), \beta\sigma)| \\ & \leq C (\|\nabla'(\phi - \phi_0)\|_0 \|\varphi - \tau\|_{\beta^{1/2}} + \|\nabla'(\phi_0 - \mathbf{v})\|_0 \|\varphi - \varphi_0\|_{\beta^{1/2}}) \|\sigma\|_\infty + \\ & \quad C (\|\nabla'(\phi - \phi_0)\|_0 \|\tau\|_\infty + \|\nabla' \mathbf{v}\|_\infty \|\varphi - \varphi_0\|_{\beta^{1/2}}) \|\sigma\|_{\beta^{1/2}} + \\ & \quad C \sum_{i=2}^3 \|(\phi - \phi_0)_i\|_{\delta\beta^{-1/2}} \sum_{i=1}^3 (\|\varphi - \tau\|_{\beta^{1/2}} \|\sigma_{i3}\|_\infty + \|\tau\|_\infty \|\sigma_{i3}\|_{\beta^{1/2}}) + \\ & \quad C \sum_{i=2}^3 \|(\phi_0 - \mathbf{v})_i\|_{\delta\beta^{-1/2}} \|\varphi - \varphi_0\|_{\beta^{1/2}} \sum_{i=1}^3 \|\sigma_{i3}\|_\infty + \\ & \quad C \sum_{i=2}^3 \|\delta\beta^{-1} \mathbf{v}_i\|_\infty \|\varphi - \varphi_0\|_{\beta^{1/2}} \sum_{i=1}^3 \|\sigma_{i3}\|_{\beta^{1/2}}. \end{aligned} \quad (4.22)$$

The result is proved combining the following inverse inequality

$$\|\sigma\|_\infty \leq \frac{C}{h} \|\sigma\|_0 \leq \frac{C}{h(1-\delta)^{1/2}} \|\sigma\|_{\beta^{1/2}}, \quad (4.23)$$

together with (4.22), the Poincaré and the Korn inequalities.  $\square$

In the next propositions, we study useful properties of the trilinear form  $\mathcal{B}_h$  defined in (4.13).

**Proposition 4.3.** *Let  $(\mathbf{v}, \phi) \in (\mathbf{H}_0^1(\Sigma))^2$  and let  $\tau \in \mathbf{H}^2(\Sigma)$ . Then, for all  $\sigma \in \mathbf{L}^2(\Sigma)$ , the following estimate holds*

$$\begin{aligned} & |\mathcal{B}_h(\beta\mathbf{v}, \tau, \sigma) - \mathcal{B}_h(\beta\phi, \tau, \sigma)| \\ & \leq C \left( \frac{1}{(1-\delta)^{1/2}} \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}} + \|(\mathbf{v} - \phi)_2\|_{\delta\beta^{-1/2}} \right) \|\tau\|_2 \|\sigma\|_{\beta^{1/2}}, \end{aligned}$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $\delta$ .

*Proof.* Since  $\tau$  is regular, we have  $\langle \tau^+ - \tau^-, \sigma \rangle_{h, \beta\phi} = \langle \tau^+ - \tau^-, \sigma \rangle_{h, \beta\mathbf{v}} = 0$ , and we easily prove that

$$\begin{aligned} |\mathcal{B}_h(\beta\mathbf{v}, \tau, \sigma) - \mathcal{B}_h(\beta\phi, \tau, \sigma)| & \leq C (\|\mathbf{v} - \phi\|_{L^4} \|\tau\|_{W^{1,4}} + \\ & (\|\nabla' \cdot (\mathbf{v} - \phi)\|_0 + \|\mathbf{v}_2 - \phi_2\|_{\delta\beta^{-1/2}}) \|\tau\|_\infty) \|\sigma\|_{\beta^{1/2}} \\ & \leq C \left( \frac{1}{(1-\delta)^{1/2}} \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}} + \|(\mathbf{v} - \phi)_2\|_{\delta\beta^{-1/2}} \right) \|\tau\|_2 \|\sigma\|_{\beta^{1/2}}. \end{aligned} \quad \square$$

**Proposition 4.4.** *Let  $\tau \in \mathbf{H}^2(\Sigma)$ ,  $\hat{\tau}$  be its orthogonal projection in  $\mathbf{L}^2(\Sigma)$  and let  $(\phi, \sigma) \in \mathbf{X}_h \times \mathbf{T}_h$ . Then for all  $\mathbf{v} \in \mathbf{H}^3(\Sigma)$ , the following estimate holds*

$$\begin{aligned} |\mathcal{B}_h(\beta\phi, \tau - \hat{\tau}, \sigma)| & \leq C \left( \frac{1}{(1-\delta)^{1/2}} \|D'(\phi - \mathbf{v})\|_{\beta^{1/2}} + h \|\mathbf{v}\|_3 \right) h^{1/2} \|\tau\|_2 \|\sigma\|_{\beta^{1/2}} + \\ & C \left( \frac{1}{(1-\delta)^{1/2}} \|(\phi - \mathbf{v})_2\|_{\delta\beta^{-1/2}} + h \|\mathbf{v}\|_3 \right) h \|\tau\|_2 \|\sigma\|_{\beta^{1/2}} + \\ & \frac{C}{(h(1-\delta))^{1/4}} \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}}^{1/2} h^{3/2} \|\tau\|_2 \langle \sigma^- - \sigma^+ \rangle_{h, \beta\phi} + \\ & C \left( 1 + \frac{h^{3/2}}{(1-\delta)^{1/2}} \right)^{1/2} \|\mathbf{v}\|_3^{1/2} h^{3/2} \|\tau\|_2 \langle \sigma^- - \sigma^+ \rangle_{h, \beta\phi}, \end{aligned}$$

where  $C \equiv C(\Sigma)$  is a positive constant independent of  $h$  and  $\delta$ .

*Proof.* An integration by parts gives

$$\mathcal{B}_h(\beta\phi, \tau - \hat{\tau}, \sigma) = -(\beta\phi \cdot \nabla' \sigma, \tau - \hat{\tau})_h - \frac{1}{2} (\nabla' \cdot (\beta\phi) \sigma, \tau - \hat{\tau}) - \langle (\tau - \hat{\tau})^-, \sigma^- - \sigma^+ \rangle_{h, \beta\phi}. \quad (4.24)$$

Let  $\mathbf{v}_0$  be the  $P_1$  continuous interpolate of  $\mathbf{v}$  on  $\mathcal{T}_h$ . Since  $\nabla' \sigma$  is  $P_0$  on each triangle  $K$ , then  $\beta\mathbf{v}_0 \cdot \nabla' \sigma$  is  $P_2$  on each  $K$ ;  $\hat{\tau}$  being the orthogonal projection of  $\tau$  on  $\mathcal{T}_h$  in  $\mathbf{L}^2(\Sigma)$ , it follows that  $(\beta\mathbf{v}_0 \cdot \nabla' \sigma, \tau - \hat{\tau})_h = 0$ , and consequently,

$$\begin{aligned} |(\beta\phi \cdot \nabla' \sigma, \tau - \hat{\tau})_h| & = |(\beta(\phi - \mathbf{v}_0) \cdot \nabla' \sigma, \tau - \hat{\tau})_h| \leq C \|\phi - \mathbf{v}_0\|_{L^4} \|\beta\sigma\|_{W^{1,4}, h} \|\tau - \hat{\tau}\|_0 \\ & \leq C (\|D'(\phi - \mathbf{v})\|_0 + \|\mathbf{v} - \mathbf{v}_0\|_1) \|\beta\sigma\|_{W^{1,4}, h} \|\tau - \hat{\tau}\|_0. \end{aligned}$$

This inequality together with the following estimates

$$\|\mathbf{v} - \mathbf{v}_0\|_1 \leq Ch\|\mathbf{v}\|_3, \quad \|\beta\sigma\|_{W^{1,4},h} \leq Ch^{-3/2} \|\beta\sigma\|_0,$$

gives

$$\begin{aligned} |(\beta\phi \cdot \nabla' \sigma, \tau - \hat{\tau})_h| &\leq C(\|D'(\phi - \mathbf{v})\|_0 + h\|\mathbf{v}\|_3) h^{1/2} \|\tau\|_2 \|\sigma\|_{\beta^{1/2}} \\ &\leq C\left(\frac{h^{1/2}}{(1-\delta)^{1/2}} \|D'(\phi - \mathbf{v})\|_{\beta^{1/2}} + h^{3/2}\|\mathbf{v}\|_3\right) \|\tau\|_2 \|\sigma\|_{\beta^{1/2}}. \end{aligned} \quad (4.25)$$

On the other hand, taking into account (4.19) and (4.23), we have

$$\begin{aligned} |(\beta \nabla' \cdot \phi \sigma, \tau - \hat{\tau})| &\leq C(\|\nabla' \cdot (\phi - \mathbf{v})\|_0 \|\sigma\|_\infty + \|\nabla' \cdot \mathbf{v}\|_\infty \|\beta\sigma\|_0) \|\tau - \hat{\tau}\|_0 \\ &\leq C\left(\frac{1}{h(1-\delta)^{1/2}} \|D'(\phi - \mathbf{v})\|_{\beta^{1/2}} + \|\mathbf{v}\|_3\right) \|\tau - \hat{\tau}\|_0 \|\sigma\|_{\beta^{1/2}} \\ &\leq C\left(\frac{h}{(1-\delta)^{1/2}} \|D'(\phi - \mathbf{v})\|_{\beta^{1/2}} + h^2\|\mathbf{v}\|_3\right) \|\tau\|_2 \|\sigma\|_{\beta^{1/2}}. \end{aligned} \quad (4.26)$$

Similar arguments show that

$$\delta |(\phi_2(\tau - \hat{\tau}), \sigma)| \leq C\left(\frac{h}{(1-\delta)^{1/2}} \|(\phi - \mathbf{v})_2\|_{\delta\beta^{-1/2}} + h^2\|\mathbf{v}\|_3\right) \|\tau\|_2 \|\sigma\|_{\beta^{1/2}}. \quad (4.27)$$

Finally, observe that

$$|\langle (\tau - \hat{\tau})^-, \sigma^- - \sigma^+ \rangle_{h,\beta\phi}| \leq C\|\phi\|_\infty^{1/2} \|\tau - \hat{\tau}\|_{\Gamma_h} \langle \sigma^- - \sigma^+ \rangle_{h,\beta\phi}. \quad (4.28)$$

From (4.11), we have

$$\begin{aligned} \|\phi\|_\infty &\leq \|\mathbf{v}\|_\infty + \|\mathbf{v} - \hat{\mathbf{v}}_{\gamma=0}\|_\infty + \|\hat{\mathbf{v}}_{\gamma=0} - \phi\|_\infty \\ &\leq \|\mathbf{v}\|_\infty + \|\mathbf{v} - \hat{\mathbf{v}}_{\gamma=0}\|_{W^{1,4}} + Ch^{-1/2} \|\hat{\mathbf{v}}_{\gamma=0} - \phi\|_1 \\ &\leq C(\|\mathbf{v}\|_3 + C_0h \|\mathbf{v}\|_{W^{2,4}}) + h^{-1/2} (\|D'(\hat{\mathbf{v}}_{\gamma=0} - \mathbf{v})\|_0 + \|D'(\mathbf{v} - \phi)\|_0) \\ &\leq C(\|\mathbf{v}\|_3 + \frac{1}{(1-\delta)^{1/2}h^{1/2}} (\|D'(\hat{\mathbf{v}}_{\gamma=0} - \mathbf{v})\|_{\beta^{1/2}} + \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}})) \\ &\leq C\left((1 + \frac{h^{3/2}}{(1-\delta)^{1/2}})\|\mathbf{v}\|_3 + \frac{1}{(1-\delta)^{1/2}h^{1/2}} \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}}\right). \end{aligned} \quad (4.29)$$

Due to (4.21), (4.28) and (4.29), it follows that

$$\begin{aligned} |\langle (\tau - \hat{\tau})^-, \sigma^- - \sigma^+ \rangle_{h,\beta\phi}| &\leq C\left(1 + \frac{h^{3/2}}{(1-\delta)^{1/2}}\right)^{1/2} \|\mathbf{v}\|_3^{1/2} h^{3/2} \|\tau\|_2 \langle \sigma^- - \sigma^+ \rangle_{h,\beta\phi} + \\ &\quad \frac{C}{(h(1-\delta))^{1/4}} \|D'(\mathbf{v} - \phi)\|_{\beta^{1/2}}^{1/2} h^{3/2} \|\tau\|_2 \langle \sigma^- - \sigma^+ \rangle_{h,\beta\phi}. \end{aligned} \quad (4.30)$$

The conclusion follows from (4.24), (4.25), (4.26), (4.27) and (4.30).  $\square$

## 5. Existence and Uniqueness of the Approximated Solutions

Let us prove the first assertion.

**Lemma 5.1.** *The mapping  $\Phi_w$  is well defined and is bounded on bounded sets.*

*Proof.* Due to Theorem 4.1, system (4.18)<sub>1</sub> admits a unique solution  $\mathbf{u}$ , and

$$(\mathcal{A}_\delta(\mathbf{u}, \mathbf{u}))^{1/2} \leq C(\|\tau\|_{\beta^{1/2}} + \varepsilon(\mathcal{A}_\delta(\phi, \phi))^{1/2} + \frac{|p^*|}{(1-\delta)^{1/2}}). \quad (5.1)$$

On the other hand, due to Theorem 4.2, problem (4.18)<sub>2</sub> admits a unique solution satisfying

$$\|\tau\|_{\beta^{1/2}} \leq We\|\mathcal{G}(\phi, \varphi)\|_{\beta^{1/2}} + 2\varepsilon\|D\phi\|_{\beta^{1/2}} \leq C\left(\frac{We}{(1-\delta)^{1/2}}\|\varphi\|_\infty + \varepsilon\right)(\mathcal{A}_\delta(\phi, \phi))^{1/2}. \quad (5.2)$$

Taking into account (5.1) and (5.2), we finally get

$$(\mathcal{A}_\delta(\mathbf{u}, \mathbf{u}))^{1/2} \leq C\left(\varepsilon + \frac{We}{(1-\delta)^{1/2}}\|\varphi\|_\infty\right)(\mathcal{A}_\delta(\phi, \phi))^{1/2} + \frac{|p^*|}{(1-\delta)^{1/2}}. \quad \square$$

**Lemma 5.2.** *Assume that the hypotheses of Theorem (4.3) are fulfilled and  $\delta < \gamma^*$ . There exist positive constants  $C^*$ ,  $C_0$ ,  $C$  and  $h_0$  independent of  $h$ ,  $\kappa$ ,  $We$ ,  $\delta$ ,  $\varepsilon$ , such that if the following condition holds*

$$\kappa \leq \frac{\theta C^*(1-\delta)}{8\alpha We((\alpha-1)^{-1/2} + 1)}, \quad (5.3)$$

then for every  $h$  satisfying  $h \leq \max(\theta^2, \sqrt{1-\delta}, h_0)$ , the ball

$$B_h = \{(\mathbf{u}, \sigma) \equiv (\mathbf{u}', \mathbf{u}_3, \sigma) \in \mathbf{X}_h^\beta \times \mathbf{T}_h \mid \mathcal{A}_\delta(\bar{\mathbf{u}}' - \mathbf{u}', \bar{\mathbf{u}}' - \mathbf{u}') \leq \lambda^2 h^3, \\ \mathcal{A}_\delta(\bar{\mathbf{u}}_3 - \mathbf{u}_3, \bar{\mathbf{u}}_3 - \mathbf{u}_3) \leq \lambda^2 h^3, \quad \|\bar{\tau} - \sigma\|_{\beta^{1/2}} \leq 4\lambda h^{3/2}\}$$

is nonempty. Moreover,  $\Phi_w(B_h) \subset B_h$ .

*Proof.* From (4.11) and (4.19), we see that in order to ensure that  $(\hat{\mathbf{u}}_\delta, \hat{\tau})$  belongs to  $B_h$ , it is sufficient that

$$C_0\left(1 + \frac{1}{(1-\delta)^{1/2}}\right)\kappa h^2 \leq \lambda h^{3/2}, \quad (5.4)$$

which is clearly satisfied if  $h \leq h_0 = \frac{\lambda^2(1-\delta)}{(C_0\kappa(1+(1-\delta)^{1/2}))^2}$ . Our aim now is to prove that  $\Phi_w$  maps  $B_h$  into itself. Let  $(\phi, \sigma) \in B_h$  and let  $(\mathbf{u}, \tau)$  be its image by  $\Phi_w$ . The strong solution  $(\bar{\mathbf{u}}, \bar{p}, \bar{\tau})$  of problem (3.2) satisfies the consistency relation:

$$\begin{cases} (1-\varepsilon)\mathcal{A}_\delta(\bar{\mathbf{u}}, \mathbf{v}) + (\bar{p}, \nabla' \cdot (\beta\mathbf{v})) + (p^*, \mathbf{v}_3) + (\bar{\tau}, \beta D\mathbf{v}) = 0, \\ (\bar{\tau}, \sigma) + We\mathcal{B}_h(\bar{\mathbf{u}}, \bar{\tau}, \sigma) = (We\mathcal{G}(\bar{\mathbf{u}}, \bar{\tau}) + 2\varepsilon\beta D\bar{\mathbf{u}}, \sigma), \end{cases} \quad (5.5)$$

for all  $(\mathbf{v}, \sigma) \in \mathbf{X}_{\beta,h} \times \mathbf{T}_h$ . Therefore, by subtracting (5.5) from (4.18), we obtain

$$\begin{cases} \mathcal{A}_\delta(\tilde{\mathbf{u}} - \bar{\mathbf{u}}, \mathbf{v}) = \varepsilon\mathcal{A}_\delta(\phi - \bar{\mathbf{u}}, \mathbf{v}) + (\bar{\tau} - \tilde{\tau}, \beta D\mathbf{v}) - (\bar{p}, \nabla' \cdot (\beta\mathbf{v})) \\ (\tilde{\tau} - \bar{\tau}, \sigma) + We(\mathcal{B}_h(\phi, \tilde{\tau}, \sigma) - \mathcal{B}_h(\bar{\mathbf{u}}, \bar{\tau}, \sigma)) = (We(\mathcal{G}(\phi, \varphi) - \mathcal{G}(\bar{\mathbf{u}}, \bar{\tau})) + \\ 2\varepsilon D(\phi - \bar{\mathbf{u}}), \sigma), \end{cases} \quad (5.6)$$

for all  $(\mathbf{v}, \sigma) \in \mathbf{X}_{\beta, h} \times \mathbf{T}_h$ . Let us begin with general considerations. Throughout the sequel, we set  $\mathbf{u} \equiv \mathbf{u}_\delta = \tilde{\mathbf{u}} - \hat{\mathbf{u}}_\delta$  and  $\tau = \tilde{\tau} - \hat{\tau}$ . Taking into account (5.6)<sub>1</sub>, the fact of  $\hat{\tau}$  be orthogonal projection of  $\tilde{\tau}$  in  $L^2$  and (4.10), for every  $\mathbf{v}_h \in \mathbf{X}_{\beta, h}$ , we have

$$\mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) = \mathcal{A}_\delta(\tilde{\mathbf{u}} - \hat{\mathbf{u}}, \mathbf{v}) = \varepsilon \mathcal{A}_\delta(\phi - \bar{\mathbf{u}}, \mathbf{v}) - (\bar{p} - \hat{p}, \nabla' \cdot (\beta \mathbf{v})) - (\tau, \beta D \mathbf{v}). \quad (5.7)$$

Similarly, (5.6)<sub>2</sub> implies that for every  $\sigma \in \mathbf{T}_h$ , we have

$$\begin{aligned} (\tau, \sigma) + We \mathcal{B}_h(\phi, \tau, \sigma) &= We(\mathcal{B}_h(\bar{\mathbf{u}}, \bar{\tau}, \sigma) - \mathcal{B}_h(\phi, \bar{\tau}, \sigma) + \mathcal{B}_h(\phi, \bar{\tau} - \hat{\tau}, \sigma)) + \\ &We(\mathcal{G}(\phi, \varphi) - \mathcal{G}(\bar{\mathbf{u}}, \bar{\tau}), \sigma) + 2\varepsilon(D(\phi - \bar{\mathbf{u}}), \sigma). \end{aligned} \quad (5.8)$$

Since the curvature ratio  $\delta$  is non zero, some crossed-terms appear in the formulation and the standard approach consisting of a global management of the variables should be adapted. The idea is to decouple the problem by considering the first two components of the velocity field (and the corresponding stress tensor components), and next deal with the third component. In this order, the rest of the proof is split into three steps.

*Step 1.* Setting  $\mathbf{v} = \mathbf{u}' \equiv (\mathbf{u}_1, \mathbf{u}_2, 0) \in \mathbf{X}_h^\beta$  in (5.7), we obtain

$$\begin{aligned} \mathcal{A}_\delta(\mathbf{u}', \mathbf{u}') &= 2(\|D' \mathbf{u}'\|_{\beta^{1/2}}^2 + \|\mathbf{u}_2\|_{\delta\beta^{-1/2}}^2) = -(\tau' - 2\varepsilon D'(\phi - \bar{\mathbf{u}})', \beta D' \mathbf{u}') - \\ &\delta(\tau_{33} - 2\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2, \mathbf{u}_2) - (\bar{p} - \hat{p}, \nabla' \cdot (\beta \mathbf{u}')) \\ &\leq \|\tau' - 2\varepsilon D'(\phi - \bar{\mathbf{u}})'\|_{\beta^{1/2}} \|D' \mathbf{u}'\|_{\beta^{1/2}} + \\ &\|\tau_{33} - 2\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2\|_{\beta^{1/2}} \|\mathbf{u}_2\|_{\delta\beta^{-1/2}} + |(\bar{p} - \hat{p}, \nabla' \cdot (\beta \mathbf{u}'))|, \end{aligned} \quad (5.9)$$

where  $\tau'$  is such that  $\tau'_{ij} = \tau_{ij}$  for  $i, j = 1, 2$ , and  $\tau'_{ij} = 0$  elsewhere. Standard calculations show that

$$\begin{aligned} |(\bar{p} - \hat{p}, \nabla' \cdot (\beta \mathbf{u}'))| &\leq 2\|\bar{p} - \hat{p}\|_0 (\|D' \mathbf{u}'\|_{\beta^{1/2}} + \|\mathbf{u}_2\|_{\delta\beta^{-1/2}}) \\ &\leq C\|\bar{p} - \hat{p}\|_0 (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} \leq Ch^2 \|\bar{p}\|_2 (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2}. \end{aligned} \quad (5.10)$$

Combining (5.9) and (5.10), we obtain

$$(2\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} \leq (\|\tau' - 2\varepsilon D'(\phi - \bar{\mathbf{u}})'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2\|_{\beta^{1/2}}^2)^{1/2} + C\kappa h^2. \quad (5.11)$$

On the other hand, straightforward calculations show that

$$\begin{aligned} \|\tau' - 2\varepsilon D'(\phi - \bar{\mathbf{u}})'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2\|_{\beta^{1/2}}^2 &= \|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2 + \\ &2\varepsilon^2 \mathcal{A}_\delta((\phi - \bar{\mathbf{u}})', (\phi - \bar{\mathbf{u}})') - 4\varepsilon(\beta \tau', D'(\phi - \bar{\mathbf{u}})') - 4\varepsilon\delta(\tau_{33}, \phi_2 - \bar{\mathbf{u}}_2). \end{aligned} \quad (5.12)$$

Setting  $\sigma = \beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)$  in (5.8), and using Proposition 4.2, we deduce that

$$\|\tau'\|_{\beta^{1/2}}^2 + \frac{We}{2} \langle \langle (\tau^- - \tau^+) \rangle \rangle_{h, \beta \phi}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2 + \frac{We}{2} \langle \langle (\tau^- - \tau^+)_{33} \rangle \rangle_{h, \beta \phi}^2$$



$$= \Theta + 2\varepsilon(D'(\phi - \bar{\mathbf{u}})', \beta\tau') + 2\varepsilon\delta(\tau_{33}, \phi_2 - \bar{\mathbf{u}}_2). \quad (5.13)$$

where

$$\begin{aligned} \Theta &= We(\mathcal{B}_h(\beta\bar{\mathbf{u}}, \bar{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3) - \mathcal{B}_h(\beta\phi, \bar{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3)) + \\ &\quad We\mathcal{B}_h(\beta\phi, \bar{\tau} - \hat{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3) + We(\mathcal{G}(\phi, \varphi) - \mathcal{G}(\bar{\mathbf{u}}, \bar{\tau}), \beta(\tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3)) \end{aligned}$$

From (5.12) and (5.13), it follows that

$$\begin{aligned} \|\tau - 2\varepsilon D'(\phi - \bar{\mathbf{u}})'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon\frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2\|_{\beta^{1/2}}^2 &= 2\varepsilon^2\mathcal{A}_\delta(\phi' - \bar{\mathbf{u}}', \phi' - \bar{\mathbf{u}}') - \\ &\quad \|\tau'\|_{\beta^{1/2}}^2 - We\langle\langle(\tau^- - \tau^+)'\rangle\rangle_{h,\beta\phi}^2\tau_{33}\|_{\beta^{1/2}}^2 - We\langle\langle(\tau^- - \tau^+)_{33}\rangle\rangle_{h,\beta\phi}^2 - 2\Theta. \end{aligned} \quad (5.14)$$

Let us estimate  $\Theta$ . Due to Proposition 4.3, the following estimate holds

$$|\mathcal{B}_h(\beta\bar{\mathbf{u}}, \bar{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3) - \mathcal{B}_h(\beta\phi, \bar{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3)| \leq \frac{\kappa\lambda h^{3/2}}{C(1-\delta)^{1/2}}(\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}). \quad (5.15)$$

Similarly, Proposition 4.4 leads to

$$\begin{aligned} |\mathcal{B}_h(\beta\phi, \bar{\tau} - \hat{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3)| &\leq C\kappa h^{3/2}\left(\frac{\lambda h^{1/2}}{(1-\delta)^{1/2}} + \kappa\right)(\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\ &\quad C\kappa h^{3/2}\left(\kappa + \frac{\kappa h^{3/2} + \lambda h}{(1-\delta)^{1/2}}\right)^{1/2}(\langle\langle(\tau^- - \tau^+)_{33}\rangle\rangle_{h,\beta\phi} + \langle\langle(\tau^- - \tau^+)'\rangle\rangle_{h,\beta\phi}). \end{aligned} \quad (5.16)$$

Finally, setting  $(\phi_0, \tau, \sigma) \equiv (\bar{\mathbf{u}}, \bar{\tau}, \tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3)$ , due to Lemma 4.2, we have

$$|(\mathcal{G}(\bar{\mathbf{u}}, \bar{\tau}) - \mathcal{G}(\phi, \varphi), \beta(\tau' + \tau_{33}\mathbf{e}_3\mathbf{e}_3))| \leq C\frac{C\lambda h^{3/2}}{(1-\delta)^{1/2}}\left(\frac{\kappa + \lambda h^{1/2}}{(1-\delta)^{1/2}} + \kappa\right)(\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}). \quad (5.17)$$

Taking into account (5.15)-(5.17) and the Young inequalities, we obtain

$$\begin{aligned} |\Theta| &\leq Weh^{3/2}B(\langle\langle(\tau^- - \tau^+)'\rangle\rangle_{h,\beta\phi} + \langle\langle(\tau^- - \tau^+)_{33}\rangle\rangle_{h,\beta\phi}) + \\ &\quad Weh^{3/2}A(\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \\ &\leq \frac{We}{2}(\langle\langle(\tau^- - \tau^+)'\rangle\rangle_{h,\beta\phi}^2 + \langle\langle(\tau^- - \tau^+)_{33}\rangle\rangle_{h,\beta\phi}^2) + \\ &\quad \frac{1}{2}\|\tau'\|_{\beta^{1/2}}^2 + \frac{1}{2}\|\tau_{33}\|_{\beta^{1/2}}^2 + (WeAh^{3/2})^2 + We(Bh^{3/2})^2. \end{aligned} \quad (5.18)$$

where

$$A = C_1\left(\kappa^2 + \frac{\kappa\lambda + \kappa\lambda h^{1/2}}{(1-\delta)^{1/2}} + \frac{(\kappa + \lambda h^{1/2})\lambda}{1-\delta}\right), \quad B = C_1\kappa\left(\kappa + \frac{\kappa h^{3/2} + \lambda h}{(1-\delta)^{1/2}}\right)^{1/2}, \quad (5.19)$$

and  $C_1 \equiv C_1(\Sigma) > 0$ . Combining (5.14) and (5.18), we deduce that

$$\begin{aligned} \|\tau - 2\varepsilon D'(\phi - \bar{\mathbf{u}})'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon\frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_2\|_{\beta^{1/2}}^2 \\ \leq 2\varepsilon^2\mathcal{A}_\delta(\phi' - \bar{\mathbf{u}}', \phi' - \bar{\mathbf{u}}') + 2(WeAh^{3/2} + We^{1/2}Bh^{3/2})^2. \end{aligned}$$

Consequently, by taking into account (5.11), it follows that

$$\begin{aligned} (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} &\leq \varepsilon(\mathcal{A}_\delta(\phi' - \bar{\mathbf{u}}', \phi' - \bar{\mathbf{u}}'))^{1/2} + (WeA + We^{1/2}B)h^{3/2} + C\kappa h^2 \\ &\leq (\varepsilon\lambda + WeA + We^{1/2}B + C\kappa h^{1/2})h^{3/2} \end{aligned}$$

Finally, estimate (4.11) together with the fact that

$$(\mathcal{A}_\delta(\bar{\mathbf{u}}' - \tilde{\mathbf{u}}', \bar{\mathbf{u}}' - \tilde{\mathbf{u}}'))^{1/2} \leq (\mathcal{A}_\delta(\bar{\mathbf{u}}' - \hat{\mathbf{u}}'_\delta, \bar{\mathbf{u}}' - \hat{\mathbf{u}}'_\delta))^{1/2} + (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2},$$

gives

$$(\mathcal{A}_\delta((\bar{\mathbf{u}} - \tilde{\mathbf{u}})', (\bar{\mathbf{u}} - \tilde{\mathbf{u}})'))^{1/2} \leq C(1 + \frac{h}{(1-\delta)^{1/2}})\kappa h^2 + (\varepsilon\lambda + WeA + We^{1/2}B)h^{3/2} \quad (5.20)$$

*Step2.* Let  $\alpha$  satisfying  $2\varepsilon < 2\alpha\varepsilon < 1 + \varepsilon$ . Setting  $\mathbf{v}=(0, 0, \mathbf{u}_3)$  in (5.7), we obtain

$$\begin{aligned} \mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3) &= \sum_{i=1}^2 \left\| \frac{\partial \mathbf{u}_3}{\partial x_i} \right\|_{\beta^{1/2}}^2 + \|\mathbf{u}_3\|_{\delta\beta^{-1/2}}^2 = \varepsilon \mathcal{A}_\delta(\phi - \bar{\mathbf{u}}, \mathbf{u}_3) - (\tau, \beta D\mathbf{u}_3) \\ &\leq \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial}{\partial x_i}(\phi - \bar{\mathbf{u}})_3\|_{\beta^{1/2}} \left\| \frac{\partial \mathbf{u}_3}{\partial x_i} \right\|_{\beta^{1/2}} + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_3\|_{\beta^{1/2}} \|\mathbf{u}_3\|_{\delta\beta^{-1/2}} + \\ &\quad (\alpha - 1)\varepsilon \left( \sum_{i=1}^2 \left\| \frac{\partial}{\partial x_i}(\phi - \bar{\mathbf{u}})_3 \right\|_{\beta^{1/2}} \left\| \frac{\partial \mathbf{u}_3}{\partial x_i} \right\|_{\beta^{1/2}} + \|(\phi - \bar{\mathbf{u}})_3\|_{\delta\beta^{-1/2}} \|\mathbf{u}_3\|_{\delta\beta^{-1/2}} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} (\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} &\leq \left( \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial(\phi - \bar{\mathbf{u}})_3}{\partial x_i}\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_3\|_{\beta^{1/2}}^2 \right)^{1/2} + \\ &\quad (\alpha - 1)\varepsilon (\mathcal{A}_\delta((\phi - \bar{\mathbf{u}})_3, (\phi - \bar{\mathbf{u}})_3))^{1/2} \\ &\leq \left( \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial(\phi - \bar{\mathbf{u}})_3}{\partial x_i}\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta}(\phi - \bar{\mathbf{u}})_3\|_{\beta^{1/2}}^2 \right)^{1/2} + (\alpha - 1)\varepsilon \lambda h^{3/2}. \end{aligned} \quad (5.21)$$

On the other hand, straightforward calculations show that

$$\begin{aligned} \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial}{\partial x_i}(\bar{\mathbf{u}} - \phi)_3\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta}(\bar{\mathbf{u}} - \phi)_3\|_{\beta^{1/2}}^2 \\ = (\alpha\varepsilon)^2 \mathcal{A}_\delta((\phi - \bar{\mathbf{u}})_3, (\phi - \bar{\mathbf{u}})_3) + \|\tau_{13}\|_{\beta^{1/2}}^2 + 2\|\tau_{23}\|_{\beta^{1/2}}^2 - \\ 2\alpha\varepsilon \left( \sum_{i=1}^2 (\tau_{i3}, \beta \frac{\partial}{\partial x_i}(\bar{\mathbf{u}} - \phi)_3) - \delta(\tau_{23}, (\bar{\mathbf{u}} - \phi)_3) \right). \end{aligned} \quad (5.22)$$

Setting  $\sigma = \beta \sum_{i=1}^2 \tau_{i3} \mathbf{e}_i \mathbf{e}_3$  in (5.8), using Proposition 4.2 and multiplying the obtained equation by  $2\alpha$ , leads to

$$\begin{aligned} \alpha \sum_{i=1}^2 (2\|\tau_{i3}\|_{\beta^{1/2}}^2 + We \langle \langle \tau_{i3}^- - \tau_{i3}^+ \rangle \rangle_{h, \beta\phi}^2) \\ = 2\alpha \hat{\Theta} + 2\alpha\varepsilon \left( \sum_{i=1}^2 (\tilde{\tau}_{i3}, \beta \frac{\partial}{\partial x_i}(\bar{\mathbf{u}} - \phi)_3) - \delta(\tilde{\tau}_{23}, (\bar{\mathbf{u}} - \phi)_3) \right). \end{aligned} \quad (5.23)$$

with

$$\begin{aligned} \hat{\Theta} &= We \sum_{i=1}^2 (\mathcal{B}_h(\beta\phi, \bar{\tau} - \hat{\tau}, \tau_{i3} \mathbf{e}_i \mathbf{e}_3) + (\mathbf{G}(\phi, \varphi) - \mathbf{G}(\bar{\mathbf{u}}, \bar{\tau}), \beta\tau_{i3} \mathbf{e}_i \mathbf{e}_3)) + \\ &\quad We \sum_{i=1}^2 (\mathcal{B}_h(\beta\bar{\mathbf{u}}, \bar{\tau}, \tau_{i3} \mathbf{e}_i \mathbf{e}_3) - \mathcal{B}_h(\beta\phi, \bar{\tau}, \tau_{i3} \mathbf{e}_i \mathbf{e}_3)). \end{aligned}$$

From (5.22) and (5.23), we deduce that

$$\begin{aligned} & \sum_{i=1}^2 \|\tau_{i3} - \alpha \varepsilon \frac{\partial}{\partial x_i} (\bar{\mathbf{u}} - \phi)_3\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha \varepsilon \frac{\delta}{\beta} (\bar{\mathbf{u}} - \phi)_3\|_{\beta^{1/2}}^2 \\ &= -2\alpha \widehat{\Theta} + (\alpha \varepsilon)^2 \mathcal{A}_\delta((\phi - \bar{\mathbf{u}})_3, (\phi - \bar{\mathbf{u}})_3) + (1 - 2\alpha) \|\tau_{13}\|_{\beta^{1/2}}^2 + \\ & \quad 2(1 - \alpha) \|\tau_{23}\|_{\beta^{1/2}}^2 - \alpha We \sum_{i=1}^2 \langle \langle \tau^- - \tau^+ \rangle \rangle_{i3}^2_{h, \beta \phi}. \end{aligned} \quad (5.24)$$

Arguments similar to those used in *Step 1*, and Young inequalities imply that

$$\begin{aligned} 2\alpha |\widehat{\Theta}| &\leq (2\alpha - 1) \|\tau_{13}\|_{\beta^{1/2}}^2 + 2(\alpha - 1) \|\tau_{23}\|_{\beta^{1/2}}^2 + \alpha We \sum_{i=1}^2 \langle \langle \tau_{i3}^- - \tau_{i3}^+ \rangle \rangle_{h, \beta \phi}^2 + \\ & \quad \left(\frac{1}{2\alpha - 1} + \frac{1}{2(\alpha - 1)}\right) (\alpha We A)^2 h^3 + 2\alpha We (B)^2 h^3. \end{aligned} \quad (5.25)$$

where  $A$  and  $B$  are given by (5.19). Combining (5.24), (5.25) and (5.21)

$$\begin{aligned} (\mathcal{A}_\delta((\bar{\mathbf{u}} - \tilde{\mathbf{u}})_3, (\bar{\mathbf{u}} - \tilde{\mathbf{u}})_3))^{1/2} &\leq (2\alpha - 1) \varepsilon \lambda h^{3/2} + \left(\frac{\alpha We}{(\alpha - 1)^{1/2}} A + (2\alpha We)^{1/2} B\right) h^{3/2} + \\ & \quad C\left(1 + \frac{h}{(1 - \delta)^{1/2}}\right) \kappa h^2. \end{aligned} \quad (5.26)$$

Step 3. From (5.20) and (5.26), we can see that the following conditions

$$(\mathcal{A}_\delta((\bar{\mathbf{u}} - \tilde{\mathbf{u}})', (\bar{\mathbf{u}} - \tilde{\mathbf{u}})'))^{1/2} \leq \lambda h^{3/2}, \quad (\mathcal{A}_\delta((\bar{\mathbf{u}} - \tilde{\mathbf{u}})_3, (\bar{\mathbf{u}} - \tilde{\mathbf{u}})_3))^{1/2} \leq \lambda h^{3/2},$$

are satisfied provided that

$$\varepsilon \lambda + We A + We^{1/2} B + C\left(1 + \frac{h}{(1 - \delta)^{1/2}}\right) \kappa h^{1/2} \leq \lambda, \quad (5.27)$$

and

$$(2\alpha - 1) \varepsilon \lambda + \frac{\alpha}{(\alpha - 1)^{1/2}} We A + (2\alpha We)^{1/2} B + C\left(1 + \frac{h}{(1 - \delta)^{1/2}}\right) \kappa h^{1/2} \leq \lambda. \quad (5.28)$$

Observing that condition (5.28) is more restrictive than condition (5.27), we will focus on it. First, notice that

$$\begin{aligned} \frac{\alpha}{(\alpha - 1)^{1/2}} We A + (2\alpha We)^{1/2} B &\leq C_2 \frac{\alpha}{(\alpha - 1)^{1/2}} \frac{We}{1 - \delta} (\kappa^2 + \kappa \lambda + \lambda^2 h^{1/2}) + \\ & \quad C_2 \frac{(\alpha We)^{1/2}}{(1 - \delta)^{1/2}} (\kappa^{3/2} + \kappa \lambda^{1/2} h^{1/2}). \end{aligned} \quad (5.29)$$

Assuming that the bound  $\kappa$  satisfies (5.3) and that  $h^2 \leq \max(\theta, 1 - \delta)$ , we easily see that

$$C_2 \frac{\alpha}{(\alpha - 1)^{1/2}} \frac{We}{1 - \delta} \kappa \lambda \leq \frac{\theta(C_2 + C)}{8} C^* \lambda. \quad (5.30)$$

Setting

$$\lambda = \frac{8\kappa}{\theta} \max\left(C_0\left(1 + \frac{h_{\max}}{(1 - \delta)^{1/2}}\right) h_{\max}^{1/2}, \frac{C_2 \alpha We \kappa}{(1 - \delta)(\alpha - 1)^{1/2}} + \frac{C_2 (\alpha We \kappa)^{1/2}}{(1 - \delta)^{1/2}} + C, \frac{8\theta C_2^2 \alpha We \kappa}{1 - \delta}\right),$$

we deduce that

$$\frac{C_2 \alpha We \kappa^2}{(1 - \delta)(\alpha - 1)^{1/2}} + \frac{C_2 (\alpha We \kappa^3)^{1/2}}{(1 - \delta)^{1/2}} \leq \frac{\theta}{8} \lambda, \quad \frac{C_2 \kappa (\alpha We)^{1/2}}{(1 - \delta)^{1/2}} \leq \frac{1}{8} \lambda^{1/2}, \quad (5.31)$$

$$C\left(1 + \frac{h}{(1 - \delta)^{1/2}}\right) \kappa h^{1/2} \leq 2C\kappa h^{1/2} \leq \frac{\theta}{4} \lambda. \quad (5.32)$$

On the other hand, there exists a positive constant  $C_3$  such that

$$\lambda \leq C_3 \frac{\kappa}{\theta} \left(1 + \frac{\alpha We}{1-\delta} \left(\frac{1}{(\alpha-1)^{1/2}} + 1\right) \kappa + \frac{(\alpha We)^{1/2}}{(1-\delta)^{1/2}} \kappa^{1/2}\right), \quad (5.33)$$

which gives

$$\begin{aligned} \frac{\alpha We}{(1-\delta)(\alpha-1)^{1/2}} \lambda^2 h^{1/2} &\leq \frac{\alpha We}{(1-\delta)(\alpha-1)^{1/2}} \lambda^2 \theta \\ &\leq \frac{C_3 \alpha We}{(1-\delta)(\alpha-1)^{1/2}} \frac{\kappa}{\theta} \left(1 + \frac{\alpha We}{1-\delta} \left(\frac{1}{(\alpha-1)^{1/2}} + 1\right) \kappa + \frac{(\alpha We)^{1/2}}{(1-\delta)^{1/2}} \kappa^{1/2}\right) \lambda \theta \\ &\leq \frac{C_3}{8} C^* \left(1 + \frac{\alpha We}{1-\delta} \left(\frac{1}{(\alpha-1)^{1/2}} + 1\right) \kappa + \frac{(\alpha We)^{1/2}}{(1-\delta)^{1/2}} \kappa^{1/2}\right) \lambda \theta \\ &\leq C_4 C^* (1 + C^* + (C^*)^{1/2}) \theta \lambda. \end{aligned} \quad (5.34)$$

Combining (5.29)-(5.34), we obtain

$$\begin{aligned} \frac{\alpha We}{(\alpha-1)^{1/2}} A + (2\alpha We)^{1/2} B + C \left(1 + \frac{h}{(1-\delta)^{1/2}}\right) \kappa h^{1/2} &\leq \\ &\leq \left(\frac{1}{2} + \frac{C_2 + C}{8} C^* + (C_2 + C) C_4 C^* (1 + C^* + (C^*)^{1/2})\right) \theta \lambda. \end{aligned}$$

Finally, by choosing  $C^*$  such that:

$$\frac{1}{2} + \frac{C_2 + C}{8} C^* + (C_2 + C) C_4 C^* (1 + C^* + (C^*)^{1/2}) \leq 1,$$

it follows that

$$C \left(1 + \frac{h}{(1-\delta)^{1/2}}\right) \kappa h^{1/2} + \frac{\alpha We}{(\alpha-1)^{1/2}} A + (2\alpha We)^{1/2} B \leq (1 - (2\alpha - 1)\varepsilon) \lambda, \quad (5.35)$$

and thus (5.28) is satisfied. Finally, to end the proof of our statement, we need to bound  $\|\bar{\tau} - \tilde{\tau}\|_0$ . Observing that

$$\|\bar{\tau} - \tilde{\tau}\|_0 \leq \|\bar{\tau} - \hat{\tau}\|_0 + \|\tau\|_0 \leq C_0 \kappa h^2 + \|\tilde{\tau}\|_0$$

Due to (5.13), (5.18) and (5.27), we have

$$\begin{aligned} \|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2 &\leq 4(We Ah^{3/2})^2 + 4(We^{1/2} Bh^{3/2})^2 + (4\varepsilon \lambda h^{3/2})^2 \\ &\leq 8\left(\frac{1}{2^{1/2}} We A + \frac{We^{1/2}}{2^{1/2}} B + \varepsilon \lambda\right)^2 h^3 \leq 8(\lambda h^{3/2})^2. \end{aligned} \quad (5.36)$$

Similarly, by combining (5.23), (5.25) and (5.28), we obtain

$$\begin{aligned} \|\tau_{13}\|_{\beta^{1/2}}^2 + 2\|\tau_{23}\|_{\beta^{1/2}}^2 &\leq \frac{2}{\alpha-1} (\alpha We Ah^{3/2})^2 + 4\alpha We (Bh^{3/2})^2 + 2(\alpha\varepsilon \lambda h^{3/2})^2 \\ &\leq 2\left(\frac{\alpha}{(\alpha-1)^{1/2}} We A + (2\alpha We)^{1/2} B + \alpha\varepsilon \lambda\right) h^3 \\ &\leq 2(1 - (2\alpha - 1)\varepsilon + \alpha\varepsilon) (\lambda h^{3/2})^2 \leq 2(\lambda h^{3/2})^2. \end{aligned} \quad (5.37)$$

From inequalities (5.36) and (5.37), we get

$$\begin{aligned} \|\bar{\tau} - \tilde{\tau}\|_{\beta^{1/2}} &\leq \|\tau\|_{\beta^{1/2}} + \|\bar{\tau} - \hat{\tau}\|_{\beta^{1/2}} \leq (10)^{1/2} \lambda h^{3/2} + C_0 \kappa h^2 \\ &\leq (10)^{1/2} \lambda h^{3/2} + C_0 \kappa h_{\max}^{1/2} h^{3/2} \leq ((10)^{1/2} + \frac{\theta}{8}) \lambda h^{3/2} \leq 4\lambda h^{3/2}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 5.3.** *Assume that assumptions of Lemma 5.2 are fulfilled. There exists a constant  $C^{**}$  independent of  $\kappa, \lambda, \varepsilon, \delta, We$ , such that if*

$$\kappa \leq \frac{C^{**} \theta^2 \varepsilon (1 - \delta)}{\alpha We (1 + (1 - \alpha)^{-1/2})}, \quad (5.38)$$

then the mapping  $\Phi_w$  is a contraction into  $B_h$ .

*Proof.* Let  $(\phi_0, \varphi_0), (\phi_1, \varphi_1)$  be in  $B_h$ ,  $(\mathbf{u}_0, \tau_0), (\mathbf{u}_1, \tau_1)$  be their respective images by  $\Phi_w$ , and set  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_0, \phi = \phi_1 - \phi_0, \tau = \tau_1 - \tau_0, \varphi = \varphi_1 - \varphi_0$ . Taking into account the definition of  $\Phi_w$ , for all  $(\mathbf{v}, \sigma) \in \mathbf{X}_{\beta, h} \times \mathbf{T}_h$  we have

$$\mathcal{A}_\delta(\mathbf{u}, \mathbf{v}) = \varepsilon \mathcal{A}_\delta(\phi, \mathbf{v}) - (\tau, \beta D\mathbf{v}),$$

$$(\tau, \beta \sigma) + We(\mathcal{B}_h(\beta \phi, \tau_1, \sigma) - \mathcal{B}_h(\beta \phi_0, \tau_0, \sigma)) = (We(\mathbf{G}(\phi_1, \varphi_1) - \mathbf{G}(\phi_0, \varphi_0)) + 2\varepsilon D\phi, \beta \sigma).$$

Step 1. Arguing as in the proof of the previous lemma, selecting  $\mathbf{v} = \mathbf{u}'$  and  $\sigma = \beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)$ , we obtain

$$(2\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} \leq (\|\tau' - 2\varepsilon D'\phi'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta} \phi_2\|_{\beta^{1/2}}^2)^{1/2}, \quad (5.39)$$

$$\begin{aligned} \|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2 + \frac{We}{2} \langle \langle (\tau^+ - \tau^-)' \rangle \rangle_{h, \beta \phi_0}^2 + \frac{We}{2} \langle \langle (\tau^+ - \tau^-)_{33} \rangle \rangle_{h, \beta \phi_0}^2 \\ = 2\varepsilon(\beta D\phi, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) + \Theta, \end{aligned} \quad (5.40)$$

where  $\Theta$  is given by

$$\begin{aligned} \Theta = We(\beta \phi \cdot \nabla' \tau_1, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) + \frac{We}{2} (\nabla' \cdot (\beta \phi) \tau_1, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) + \\ We(\langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta \phi} - \langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta \phi_0}) + \\ We(\mathbf{G}(\phi_1, \varphi_1) - \mathbf{G}(\phi_0, \varphi_0), \beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|\tau' - 2\varepsilon D'\phi'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta} \phi_2\|_{\beta^{1/2}}^2 = \|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2 + \\ 2\varepsilon^2 \mathcal{A}_\delta(\phi', \phi') - 4\varepsilon(\beta \tau', D'\phi') - 4\varepsilon \delta (\tau_{33}, \phi_2). \end{aligned} \quad (5.41)$$

From identities (5.40) and (5.41), we deduce that

$$\begin{aligned} \|\tau' - 2\varepsilon D'\phi'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta} \phi_2\|_{\beta^{1/2}}^2 = 2\varepsilon^2 \mathcal{A}_\delta(\phi', \phi') - \|\tau'\|_{\beta^{1/2}}^2 - \\ We \langle \langle (\tau^+ - \tau^-)' \rangle \rangle_{h, \beta \phi_0}^2 - \|\tau_{33}\|_{\beta^{1/2}}^2 - We \langle \langle (\tau^+ - \tau^-)_{33} \rangle \rangle_{h, \beta \phi_0}^2 - 2\Theta \\ \leq 2\varepsilon^2 \mathcal{A}_\delta(\phi', \phi') - \|\tau'\|_{\beta^{1/2}}^2 - \|\tau_{33}\|_{\beta^{1/2}}^2 - 2\Theta. \end{aligned}$$

This estimate together with the following relation

$$2|\mathbf{x}| \leq |\mathbf{x}|^2 (|\mathbf{y}_1|^2 + |\mathbf{y}_2|^2)^{-1} + |\mathbf{y}_1|^2 + |\mathbf{y}_2|^2,$$

gives

$$\begin{aligned} (\|\tau' - 2\varepsilon D'\phi'\|_{\beta^{1/2}}^2 + \|\tau_{33} - 2\varepsilon \frac{\delta}{\beta} \phi_2\|_{\beta^{1/2}}^2)^{1/2} \\ \leq \varepsilon(2\mathcal{A}_\delta(\phi', \phi'))^{1/2} + |\Theta| (\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{-1/2}. \end{aligned}$$

Combining (5.39) with the last inequality, we get

$$(\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} \leq \varepsilon(\mathcal{A}_\delta(\phi', \phi'))^{1/2} + \frac{|\Theta|}{2^{1/2}}(\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{-1/2}. \quad (5.42)$$

On the other hand, from (5.40)

$$(\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{1/2} \leq \varepsilon(2\mathcal{A}_\delta(\phi', \phi'))^{1/2} + |\Theta|(\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{-1/2}.$$

Multiplying the last inequality by  $\frac{1-\varepsilon}{2^{3/2}\varepsilon}$  and using (5.42), leads to

$$\begin{aligned} (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} + \frac{1-\varepsilon}{2^{3/2}\varepsilon}(\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{1/2} \\ \leq \frac{1+\varepsilon}{2}(\mathcal{A}_\delta(\phi', \phi'))^{1/2} + \frac{1-\varepsilon}{2^{3/2}\varepsilon}|\Theta|(\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{-1/2}. \end{aligned} \quad (5.43)$$

Let us bound  $\Theta$ . Setting  $(\mathbf{v}, \tau, \sigma) \equiv (\bar{\mathbf{u}}, \bar{\tau}, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)$  in Lemma 4.2, we obtain

$$\begin{aligned} |(\mathcal{G}(\phi_1, \varphi_1) - \mathcal{G}(\phi_0, \varphi_0), \beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3))| &\leq \frac{C}{h(1-\delta)} (\|D'\phi\|_{\beta^{1/2}} \|\varphi_1 - \bar{\tau}\|_{\beta^{1/2}} + \\ &\quad \|D'(\phi_0 - \bar{\mathbf{u}})\|_{\beta^{1/2}} \|\varphi\|_{\beta^{1/2}}) (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\ &\quad \frac{C}{(1-\delta)^{1/2}} \|D'\phi\|_{\beta^{1/2}} \|\bar{\tau}\|_\infty (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\ &\quad C \|\nabla' \bar{\mathbf{u}}\|_\infty \|\varphi\|_{\beta^{1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\ &\quad \frac{C}{h(1-\delta)^{1/2}} \sum_{i=2}^3 \|\phi_i\|_{\delta\beta^{-1/2}} \|\varphi_1 - \bar{\tau}\|_{\beta^{1/2}} \|\tau_{33}\|_{\beta^{1/2}} + \\ &\quad \frac{C}{h(1-\delta)^{1/2}} \sum_{i=2}^3 \|(\phi_0 - \bar{\mathbf{u}})_i\|_{\delta\beta^{-1/2}} \|\varphi\|_{\beta^{1/2}} \|\tau_{33}\|_{\beta^{1/2}} + \\ &\quad C (\sum_{i=2}^3 \|\phi_i\|_{\delta\beta^{-1/2}} \|\bar{\tau}\|_\infty + \frac{\delta}{1-\delta} \|\bar{\mathbf{u}}\|_\infty \|\varphi\|_{\beta^{1/2}}) \|\tau_{33}\|_{\beta^{1/2}} \\ &\leq \frac{C(\kappa + \lambda h^{1/2})}{1-\delta} (\|D'\phi\|_{\beta^{1/2}} + \sum_{i=2}^3 (\|\phi_i\|_{\beta^{1/2}}) \|\tau'\|_{\beta^{1/2}} + \\ &\quad \|\tau_{33}\|_{\beta^{1/2}}) + \frac{C(\kappa + \lambda h^{1/2})}{1-\delta} \|\varphi\|_{\beta^{1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}). \end{aligned} \quad (5.44)$$

On the other hand, due to Lemma 3 in [14], we know that there exists a constant  $C$  independent of  $h$  such that

$$\begin{aligned} |\langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta\phi} - \langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta\phi_0}| \\ \leq C \|\phi\|_{L^p, \Gamma_h} \|\beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)\|_{L^q, \Gamma_h} \|[\tau_1]\|_{L^r, \Gamma_h}, \end{aligned}$$

where  $p, q, r$  are positive numbers satisfying  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$  and where  $[\cdot]$  denotes the jump across  $\Gamma_h$ . From classical inverse inequalities, for  $q > 2$

$$\|\beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)\|_{L^q, \Gamma_h} \leq Ch^{-\frac{1}{q}} \|\beta(\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)\|_{L^q} \leq Ch^{\frac{1}{q}-1} \|\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3\|_{\beta^{1/2}}.$$

Let  $\tau_0$  be the  $P_1$  interpolate of  $\bar{\tau}$ . For  $r > 2$ , we have

$$\|[\tau_1]\|_{L^r, \Gamma_h} = \|[\tau_1 - \tau_0]\|_{L^r, \Gamma_h} \leq Ch^{\frac{1}{r}-1} \|\tau_1 - \tau_0\|_0.$$

By combining these estimates and choosing  $p = 6$ , we deduce that

$$\begin{aligned}
& \left| \langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta \phi} - \langle \tau_1^+ - \tau_1^-, (\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)^+ \rangle_{h, \beta \phi_0} \right| \\
& \leq Ch^{-\frac{4}{3}} \|\phi\|_{L^6(\Sigma)} \|\tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3\|_{\beta^{1/2}} (\|\tau_1 - \bar{\tau}\|_0 + \|\bar{\tau} - \tau_0\|_0) \\
& \leq Ch^{\frac{2}{3}} \|\phi\|_{L^6(\Sigma)} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \|\bar{\tau}\|_2 + \\
& \quad \frac{Ch^{-\frac{4}{3}}}{(1-\delta)^{1/2}} \|\phi\|_{L^6(\Sigma)} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \|\bar{\tau} - \tau_0\|_{\beta^{1/2}} \\
& \leq Ch^{\frac{2}{3}} \|D'\phi\|_0 (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \|\bar{\tau}\|_2 + \\
& \quad \frac{Ch^{-\frac{4}{3}}}{(1-\delta)^{1/2}} \|D'\phi\|_0 (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \|\bar{\tau} - \tau_0\|_{\beta^{1/2}} \\
& \leq \frac{Ch^{\frac{1}{6}}}{(1-\delta)^{1/2}} (\kappa h^{1/2} + \frac{\lambda}{(1-\delta)^{1/2}}) \|D'\phi\|_{\beta^{1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \\
& \leq \frac{C\lambda h^{\frac{1}{6}}}{1-\delta} \|D'\phi\|_{\beta^{1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}).
\end{aligned} \tag{5.45}$$

Let us estimate the last term. Classical arguments show that

$$\begin{aligned}
& \left| (\beta \phi \cdot \nabla' \tau_1, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) + \frac{1}{2} (\nabla' \cdot (\beta \phi) \tau_1, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) \right| \\
& \leq \left| (\beta \phi \cdot \nabla' (\tau_1 - \bar{\tau}), \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) \right| + \frac{1}{2} \left| (\nabla' \cdot (\beta \phi) (\tau_1 - \bar{\tau}), \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) \right| + \\
& \quad \left| (\beta \phi \cdot \nabla' \bar{\tau}, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) \right| + \frac{1}{2} \left| (\nabla' \cdot (\beta \phi) \bar{\tau}, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) \right| \\
& \leq C \|\phi\|_\infty \|\tau_1 - \bar{\tau}\|_{1,h} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\
& \quad \frac{1}{2} \|\nabla' \cdot (\beta \phi)\|_0 \|\tau_1 - \bar{\tau}\|_0 (\|\tau'\|_\infty + \|\tau_{33}\|_\infty) + \\
& \quad C \|\phi\|_{L^4(\Sigma)} \|\bar{\tau}\|_{W^{1,4}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\
& \quad C (\|\nabla' \cdot \phi\|_0 + \|\phi_2\|_{\delta\beta^{-1/2}}) \|\bar{\tau}\|_\infty (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) \\
& \leq C \|\phi\|_\infty \|\tau_1 - \bar{\tau}\|_{1,h} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\
& \quad \frac{C}{(1-\delta)^{1/2}} \|D'\phi\|_{\beta^{1/2}} \|\tau_1 - \bar{\tau}\|_{\beta^{1/2}} (\|\tau'\|_\infty + \|\tau_{33}\|_\infty) + \\
& \quad \frac{C}{(1-\delta)^{1/2}} \|\phi_2\|_{\delta\beta^{-1/2}} \|\tau_1 - \bar{\tau}\|_{\beta^{1/2}} (\|\tau'\|_\infty + \|\tau_{33}\|_\infty) + \\
& \quad \frac{C\kappa}{(1-\delta)^{1/2}} \|D'\phi\|_{\beta^{1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}) + \\
& \quad C\kappa \|\phi_2\|_{\delta\beta^{-1/2}} (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}).
\end{aligned} \tag{5.46}$$

Using the inverse inequality  $\|\tau_1 - \hat{\tau}\|_{1,h} \leq \frac{C}{h} \|\tau_1 - \hat{\tau}\|_0$ , we get

$$\begin{aligned}
\|\tau_1 - \bar{\tau}\|_{1,h,\Sigma} & \leq \frac{C}{h} \|\tau_1 - \hat{\tau}\|_0 + C\kappa h \leq \frac{C}{h} \|\tau_1 - \bar{\tau}\|_0 + C\kappa h \\
& \leq \frac{C}{h(1-\delta)^{1/2}} \|\tau_1 - \bar{\tau}\|_{\beta^{1/2}} + C\kappa h \leq Ch^{1/2} \left( \frac{\lambda}{(1-\delta)^{1/2}} + \kappa h^{1/2} \right).
\end{aligned} \tag{5.47}$$

Similarly, we have

$$\|\phi\|_\infty \leq \|\phi\|_{W^{1,3}} \leq \frac{C}{h^{1/3}} \|D'\phi\|_0 \leq \frac{C}{h^{1/3}(1-\delta)^{1/2}} \|D'\phi\|_{\beta^{1/2}}, \tag{5.48}$$

Combining (4.23) and (5.46)-(5.48), we deduce that

$$\begin{aligned} & |(\beta\phi \cdot \nabla' \tau, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3) + \frac{1}{2}(\nabla' \cdot (\beta\phi)\tau, \tau' + \tau_{33} \mathbf{e}_3 \mathbf{e}_3)| \\ & \leq \frac{C(\kappa + \lambda h^{1/6})}{1-\delta} (\|D'\phi\|_{\beta^{1/2}} + \|\phi_2\|_{\delta\beta^{-1/2}}) (\|\tau'\|_{\beta^{1/2}} + \|\tau_{33}\|_{\beta^{1/2}}). \end{aligned} \quad (5.49)$$

Due to (5.44), (5.45), (5.49), we get

$$\begin{aligned} & |\Theta| (\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{-1/2} \\ & \leq \frac{CWe}{(1-\delta)^{1/2}} (\kappa + \frac{\kappa + \lambda h^{1/6}}{(1-\delta)^{1/2}}) (\|D'\phi\|_{\beta^{1/2}} + \sum_{i=2}^3 \|\phi_i\|_{\delta\beta^{-1/2}} + \|\varphi\|_{\beta^{1/2}}) \\ & \leq \frac{CWe(\kappa + \lambda)}{1-\delta} ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \|\varphi\|_{\beta^{1/2}}). \end{aligned}$$

Hence, by taking into account (5.43), we finally obtain

$$\begin{aligned} & (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} + \frac{1-\varepsilon}{2^{3/2}\varepsilon} (\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{1/2} \leq \frac{1+\varepsilon}{2} (\mathcal{A}_\delta(\phi', \phi'))^{1/2} + \\ & \frac{C(\kappa + \lambda)(1-\varepsilon)We}{2^{3/2}(1-\delta)\varepsilon} ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \|\varphi\|_{\beta^{1/2}}). \end{aligned} \quad (5.50)$$

Step 2. Arguing as in the previous step, we select  $\mathbf{v} = \mathbf{u}_3 \mathbf{e}_3$ . Then,

$$\begin{aligned} & (\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} \leq (\alpha - 1)\varepsilon (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \\ & \left( \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial \phi_3}{\partial x_i}\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta} \phi_3\|_{\beta^{1/2}}^2 \right)^{1/2}. \end{aligned} \quad (5.51)$$

Selecting  $\sigma = \beta \sum_{i=1}^2 (\tau_1)_{i3} \mathbf{e}_i \mathbf{e}_3$  in (5.8), using Proposition 4.2 and multiplying the obtained equation by  $2\alpha$

$$\alpha \sum_{i=1}^2 (2\|\tau_{i3}\|_{\beta^{1/2}}^2 + We \langle \langle \tau^- - \tau^+ \rangle \rangle_{i3}^2)_{h, \beta \phi_0} = 2\alpha \widehat{\Theta} + 2\alpha\varepsilon \left( \sum_{i=1}^2 (\tau_{i3}, \beta \frac{\partial}{\partial x_i} \phi_3) - \delta(\tau_{23}, \phi_3) \right) \quad (5.52)$$

with

$$\begin{aligned} \widehat{\Theta} = & We \sum_{i=1}^2 (\mathcal{B}_h(\beta\phi, \tau_1, \tau_{i3} \mathbf{e}_i \mathbf{e}_3) - \mathcal{B}_h(\beta\phi_0, \tau_1, \tau_{i3} \mathbf{e}_i \mathbf{e}_3)) + \\ & We \sum_{i=1}^2 (\mathbf{G}(\phi_1, \varphi_1) - \mathbf{G}(\phi_0, \varphi_0), \beta\tau_{i3} \mathbf{e}_i \mathbf{e}_3). \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial}{\partial x_i} \phi_3\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta} \phi_3\|_{\beta^{1/2}}^2 = \|\tau_{13}\|_{\beta^{1/2}}^2 + 2\|\tau_{23}\|_{\beta^{1/2}}^2 + \\ & (\alpha\varepsilon)^2 \mathcal{A}_\delta(\phi_3, \phi_3) - 2\alpha\varepsilon \left( \sum_{i=1}^2 (\tau_{i3}, \beta \frac{\partial}{\partial x_i} \phi_3) - \delta(\tau_{23}, \phi_3) \right). \end{aligned} \quad (5.53)$$

From (5.52) and (5.53), we deduce that

$$\begin{aligned} & \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial}{\partial x_i} \phi_3\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta} \phi_3\|_{\beta^{1/2}}^2 \\ & \leq -2\alpha \widehat{\Theta}_d + (\alpha\varepsilon)^2 \mathcal{A}_\delta(\phi_3, \phi_3) + (1 - 2\alpha) \|\tau_{13}\|_{\beta^{1/2}}^2 + 2(1 - \alpha) \|\tau_{23}\|_{\beta^{1/2}}^2. \end{aligned}$$



The last estimate together with the following relations

$$\begin{aligned} 2\alpha|\mathbf{x}| &\leq \alpha^2|\mathbf{x}|^2((2\alpha-1)|\mathbf{y}_1|^2 + 2(\alpha-1)|\mathbf{y}_2|^2)^{-1} + (2\alpha-1)|\mathbf{y}_1|^2 + 2(\alpha-1)|\mathbf{y}_2|^2 \\ &\leq \frac{\alpha^2}{2(\alpha-1)}|\mathbf{x}|^2(|\mathbf{y}_1|^2 + |\mathbf{y}_2|^2)^{-1} + (2\alpha-1)|\mathbf{y}_1|^2 + 2(\alpha-1)|\mathbf{y}_2|^2, \end{aligned}$$

gives

$$\begin{aligned} & \left( \sum_{i=1}^2 \|\tau_{i3} - \alpha\varepsilon \frac{\partial}{\partial x_i} \phi_3\|_{\beta^{1/2}}^2 + \|\tau_{23} + \alpha\varepsilon \frac{\delta}{\beta} \phi_3\|_{\beta^{1/2}}^2 \right)^{1/2} \\ & \leq \alpha\varepsilon (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \frac{\alpha|\widehat{\Theta}|}{(2(\alpha-1))^{1/2}} \left( \sum_{i=1}^2 \|\beta^{1/2} \tau_{i3}\|_0^2 \right)^{-1/2}. \end{aligned} \quad (5.54)$$

Combining (5.51) and (5.54)

$$(\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} \leq (1-\theta)(\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \frac{\alpha|\widehat{\Theta}|}{(2(\alpha-1))^{1/2}} \left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{-1/2}. \quad (5.55)$$

where  $\theta = 1 - (2\alpha - 1)\varepsilon$ . On the other hand, from (5.52), we have

$$\left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{1/2} \leq \varepsilon (2\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + |\widehat{\Theta}| \left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{-1/2}. \quad (5.56)$$

Arguing as in the previous step, we can prove that

$$|\widehat{\Theta}| \left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{-1/2} \leq \frac{CWe(\kappa+\lambda)}{1-\delta} ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \|\varphi\|_{\beta^{1/2}}).$$

Consequently, multiplying (5.56) by  $\frac{\theta}{2\sqrt{2}\varepsilon}$  and using (5.55), we obtain

$$\begin{aligned} & (\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} + \frac{\theta}{2^{3/2}\varepsilon} \left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{1/2} \leq (1 - \frac{\theta}{2})(\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2} + \\ & \frac{CWe(\kappa+\lambda)}{1-\delta} \left( \frac{\theta}{2^{3/2}\varepsilon} + \frac{\alpha}{(2(\alpha-1))^{1/2}} \right) ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2}) + \\ & \frac{CWe(\kappa+\lambda)}{1-\delta} \left( \frac{\theta}{2^{3/2}\varepsilon} + \frac{\alpha}{(2(\alpha-1))^{1/2}} \right) \|\varphi\|_{\beta^{1/2}}. \end{aligned} \quad (5.57)$$

*Step 3.* Straightforward calculations together with (5.50), (5.57), and the fact that  $\varepsilon < \theta < 1$ , show that

$$\begin{aligned} & (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} + (\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} + \frac{\theta}{2^{3/2}\varepsilon} (\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{1/2} + \\ & \left( \sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2 \right)^{1/2} \leq (1 - \frac{\theta}{2}) ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2}) + \\ & \frac{\widetilde{C}(\kappa+\lambda)We}{1-\delta} \left( \frac{1}{\varepsilon} + \frac{\alpha}{(\alpha-1)^{1/2}} \right) ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2}) + \\ & \frac{\widetilde{C}(\kappa+\lambda)We}{1-\delta} \left( \frac{1}{\varepsilon} + \frac{\alpha}{(\alpha-1)^{1/2}} \right) (\|\varphi'\|_{\beta^{1/2}}^2 + \|\varphi_{33}\|_{\beta^{1/2}}^2)^{1/2} + \\ & \frac{\widetilde{C}(\kappa+\lambda)We}{1-\delta} \left( \frac{1}{\varepsilon} + \frac{\alpha}{(\alpha-1)^{1/2}} \right) \left( \sum_{i=1}^2 \|\varphi_{i3}\|_{\beta^{1/2}}^2 \right)^{1/2}, \end{aligned} \quad (5.58)$$

where  $\tilde{C} \equiv C(\Sigma)$ . On the other hand, taking into account conditions (5.33) and (5.38), we can see that

$$\begin{aligned} \lambda &\leq C \frac{\kappa}{\theta} \left(1 + \frac{\alpha We}{1-\delta} \left(1 + \frac{1}{(\alpha-1)^{1/2}}\right) \kappa + \frac{(\alpha We)^{1/2}}{(1-\delta)^{1/2}} \kappa^{1/2}\right) \\ &\leq C \frac{\kappa}{\theta} (1 + C^{**} + (C^{**})^{1/2}) \leq C \frac{C^{**} \theta \varepsilon (1-\delta)}{\alpha We ((\alpha-1)^{-1/2} + 1)} (1 + C^{**} + (C^{**})^{1/2}). \end{aligned}$$

Hence,

$$\frac{\tilde{C}(\kappa+\lambda) We}{1-\delta} \left(\frac{1}{\varepsilon} + \frac{\alpha}{(\alpha-1)^{1/2}}\right) \leq C \theta C^{**} (1 + C^{**} + (C^{**})^{1/2}). \quad (5.59)$$

The constant  $C^{**}$  is then chosen in such a way that the following condition

$$C C^{**} (1 + C^{**} + (C^{**})^{1/2}) \leq \frac{1}{4}. \quad (5.60)$$

is satisfied. Therefore, by combining (5.58)-(5.60), we can easily see

$$\begin{aligned} (\mathcal{A}_\delta(\mathbf{u}', \mathbf{u}'))^{1/2} + (\mathcal{A}_\delta(\mathbf{u}_3, \mathbf{u}_3))^{1/2} + \frac{\theta}{2^{3/2}\varepsilon} ((\|\tau'\|_{\beta^{1/2}}^2 + \|\tau_{33}\|_{\beta^{1/2}}^2)^{1/2} + \\ (\sum_{i=1}^2 \|\tau_{i3}\|_{\beta^{1/2}}^2)^{1/2}) &\leq (1 - \frac{\theta}{4}) ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2}) + \\ &\quad \frac{\theta}{4} ((\|\varphi'\|_{\beta^{1/2}}^2 + \|\varphi_{33}\|_{\beta^{1/2}}^2)^{1/2} + (\sum_{i=1}^2 \|\varphi_{i3}\|_{\beta^{1/2}}^2)^{1/2}) \\ &\leq (1 - \frac{\theta}{4}) \{ ((\mathcal{A}_\delta(\phi', \phi'))^{1/2} + (\mathcal{A}_\delta(\phi_3, \phi_3))^{1/2}) + \\ &\quad \frac{\theta}{2^{3/2}\varepsilon^{1/2}} ((\|\varphi'\|_{\beta^{1/2}}^2 + \|\varphi_{33}\|_{\beta^{1/2}}^2)^{1/2} + (\sum_{i=1}^2 \|\varphi_{i3}\|_{\beta^{1/2}}^2)^{1/2}) \}, \end{aligned}$$

which ensures that  $\Phi_w$  is a contraction, and completes the proof.  $\square$

## 6. Appendix: Rectangular Toroidal Coordinates

Using the rectangular toroidal coordinates defined by (3.1) we get (omiting  $\sim$  to simplify the notation):

1. The gradient operator

- Gradient of a scalar function  $\phi$ 

$$[\nabla\phi] = \mathbf{e}_1 \frac{\partial\phi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\phi}{\partial x_2} + \mathbf{e}_3 \frac{1}{\beta} \frac{\partial\phi}{\partial x_3} = \nabla'\phi + \mathbf{e}_3 \frac{1}{\beta} \frac{\partial\phi}{\partial x_3}.$$
- Gradient of a vector function  $\phi \equiv (\phi_1, \phi_2, \phi_3)$ 

$$[\nabla\phi] = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_3}{\partial x_1} \\ \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_2}{\partial x_2} & \frac{\partial\phi_3}{\partial x_2} \\ \frac{1}{\beta} \frac{\partial\phi_1}{\partial x_3} & \frac{1}{\beta} \left(\frac{\partial\phi_2}{\partial x_3} - \frac{\phi_3}{R}\right) & \frac{1}{\beta} \left(\frac{\partial\phi_3}{\partial x_3} + \frac{\phi_2}{R}\right) \end{pmatrix}$$

$$= \nabla'\phi + \frac{1}{R\beta} (\phi_2 \mathbf{e}_3 \mathbf{e}_3 - \phi_3 \mathbf{e}_3 \mathbf{e}_2) + \frac{1}{\beta} \sum_{i=1}^3 \frac{\partial\phi_i}{\partial x_3} \mathbf{e}_3 \mathbf{e}_i.$$

• The strain tensor

$$\begin{aligned} [D\phi] &= \frac{1}{2} \left( [\nabla\phi] + [\nabla\phi]^T \right) \\ &= D'\phi + \frac{1}{2R\beta} (2\phi_2 \mathbf{e}_3 \mathbf{e}_3 - \phi_3 (\mathbf{e}_3 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_3)) + \frac{1}{2\beta} \sum_{i=1}^3 \frac{\partial\phi_i}{\partial x_3} (\mathbf{e}_3 \mathbf{e}_i + \mathbf{e}_i \mathbf{e}_3), \end{aligned}$$

## 2. The divergence operator

- The divergence of a vector  $\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3)$ 

$$[\nabla \cdot \boldsymbol{\omega}] = \frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_2}{\partial x_2} + \frac{\omega_2}{R\beta} + \frac{1}{\beta} \frac{\partial \omega_3}{\partial x_3} = \frac{1}{\beta} \nabla' \cdot (\beta \boldsymbol{\omega}) + \frac{1}{\beta} \frac{\partial \omega_3}{\partial x_3}.$$
- The divergence of a tensor  $\boldsymbol{\tau} \equiv (\tau_{i,j})_{i,j=1,2,3}$ 

$$[\nabla \cdot \boldsymbol{\tau}] = \left( \sum_{j=1}^2 \frac{\partial \tau_{ji}}{\partial x_j} + \frac{\tau_{2i}}{R\beta} \right) \mathbf{e}_i + \frac{1}{R\beta} (\tau_{32} \mathbf{e}_3 - \tau_{33} \mathbf{e}_2) + \frac{1}{\beta} \sum_{i=1}^3 \frac{\partial \tau_{3i}}{\partial x_3} \mathbf{e}_i$$

$$= \frac{1}{\beta} [\nabla' \cdot (\beta \boldsymbol{\tau})] + \frac{1}{R\beta} (\tau_{32} \mathbf{e}_3 - \tau_{33} \mathbf{e}_2) + \frac{1}{\beta} \sum_{i=1}^3 \frac{\partial \tau_{3i}}{\partial x_3} \mathbf{e}_i.$$

## 3. The Laplacian operator

- The Laplacian of a scalar function  $\phi$ 

$$[\Delta \phi] = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{1}{\beta^2} \frac{\partial^2 \phi}{\partial x_3^2} + \frac{1}{R\beta} \frac{\partial \phi}{\partial x_2} = \Delta' \phi + \frac{1}{R\beta} \frac{\partial \phi}{\partial x_2} + \frac{1}{\beta^2} \frac{\partial^2 \phi}{\partial x_3^2}.$$
- The Laplacian of a vector function  $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \phi_3)$ 

$$[\Delta \boldsymbol{\phi}] = \Delta' \boldsymbol{\phi} + \frac{1}{R\beta} \frac{\partial \boldsymbol{\phi}}{\partial x_2} - \frac{1}{(R\beta)^2} (\phi_2 \mathbf{e}_2 + \phi_3 \mathbf{e}_3) + \frac{1}{\beta^2} \frac{\partial^2 \boldsymbol{\phi}}{\partial x_3^2} +$$

$$\frac{2}{R\beta^2} \left( \frac{\partial \phi_2}{\partial x_3} \mathbf{e}_3 - \frac{\partial \phi_3}{\partial x_3} \mathbf{e}_2 \right).$$

## 4. The convective operator

- For the vector fields  $\mathbf{v} = (v_1, v_2, v_3)$  and  $\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3)$ 

$$[\mathbf{v} \cdot \nabla \boldsymbol{\omega}] = \begin{pmatrix} v_1 \frac{\partial \omega_1}{\partial x_1} + v_2 \frac{\partial \omega_1}{\partial x_2} + \frac{v_3}{\beta} \frac{\partial \omega_1}{\partial x_3} \\ v_1 \frac{\partial \omega_2}{\partial x_1} + v_2 \frac{\partial \omega_2}{\partial x_2} + \frac{v_3}{\beta} \frac{\partial \omega_2}{\partial x_3} - \frac{1}{R\beta} v_3 \omega_3 \\ v_1 \frac{\partial \omega_3}{\partial x_1} + v_2 \frac{\partial \omega_3}{\partial x_2} + \frac{v_3}{\beta} \frac{\partial \omega_3}{\partial x_3} + \frac{1}{R\beta} v_3 \omega_2 \end{pmatrix}$$

$$= \mathbf{v} \cdot \nabla' \boldsymbol{\omega} + \frac{v_3}{R\beta} (\omega_2 \mathbf{e}_3 - \omega_3 \mathbf{e}_2) + \frac{v_3}{\beta} \frac{\partial \boldsymbol{\omega}}{\partial x_3}.$$
- For the vector  $\mathbf{v} = (v_1, v_2, v_3)$  and the tensor  $\boldsymbol{\tau} \equiv (\tau_1, \tau_2, \tau_3)$ 

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{11} = \mathbf{v} \cdot \nabla' \tau_{11} + \frac{v_3}{\beta} \frac{\partial \tau_{11}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{12} = \mathbf{v} \cdot \nabla' \tau_{12} - \frac{v_3}{R\beta} \tau_{13} + \frac{v_3}{\beta} \frac{\partial \tau_{12}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{13} = \mathbf{v} \cdot \nabla' \tau_{13} + \frac{v_3}{R\beta} \tau_{12} + \frac{v_3}{\beta} \frac{\partial \tau_{13}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{21} = \mathbf{v} \cdot \nabla' \tau_{21} - \frac{v_3}{R\beta} \tau_{31} + \frac{v_3}{\beta} \frac{\partial \tau_{21}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{22} = \mathbf{v} \cdot \nabla' \tau_{22} - \frac{v_3}{R\beta} (\tau_{23} + \tau_{32}) + \frac{v_3}{\beta} \frac{\partial \tau_{22}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{23} = \mathbf{v} \cdot \nabla' \tau_{23} + \frac{v_3}{R\beta} (\tau_{22} - \tau_{33}) + \frac{v_3}{\beta} \frac{\partial \tau_{23}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{31} = \mathbf{v} \cdot \nabla' \tau_{31} + \frac{v_3}{R\beta} \tau_{21} + \frac{v_3}{\beta} \frac{\partial \tau_{31}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{32} = \mathbf{v} \cdot \nabla' \tau_{32} + \frac{v_3}{R\beta} (\tau_{22} - \tau_{33}) + \frac{v_3}{\beta} \frac{\partial \tau_{32}}{\partial x_3},$$

$$[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]_{33} = \mathbf{v} \cdot \nabla' \tau_{33} + \frac{v_3}{R\beta} (\tau_{23} + \tau_{32}) + \frac{v_3}{\beta} \frac{\partial \tau_{33}}{\partial x_3}.$$

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