The Stokes Resolvent Problem in General Unbounded Domains

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Abstract

It is well-known that the Helmholtz decomposition of L^q-spaces fails to exist for certain unbounded smooth domains unless $q = 2$. Hence also the Stokes operator is not well-defined for these domains when $q \neq 2$. In this paper, we generalize a new approach to the Stokes problem in general unbounded smooth domains from the three-dimensional case, see [5], to the n-dimensional one, $n \geq 2$, replacing the space $L^q, 1 < q < \infty$, by $\tilde{L}^q$ where $\tilde{L}^q = L^q \cap L^2$ for $q \geq 2$ and $\tilde{L}^q = L^q + L^2$ for $1 < q < 2$. In particular, we show that the Stokes operator is well-defined in $\tilde{L}^q$ for every unbounded domain of uniform $C^{1,1}$-type in $\mathbb{R}^n$, $n \geq 2$, and generates an analytic semigroup.

2000 Mathematics Subject Classification: Primary 76D05, Secondary 35Q30

Keywords: General unbounded domains; domains of uniform $C^{1,1}$-type; Stokes operator, Stokes resolvent; Stokes semigroup

1 Introduction

Throughout this paper, $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, means a general unbounded domain with uniform $C^{1,1}$-boundary $\partial\Omega \neq \emptyset$, see Definition 1.1 below. As is well-known, the standard approach to the Stokes equations in L^q-spaces, $1 < q < \infty$, cannot be extended to general unbounded domains unless $q = 2$. One reason is the fact that the Helmholtz decomposition fails to exist for certain unbounded smooth domains on L^q, $q \neq 2$, see [3], [10]. On the other hand, in L^2 the Helmholtz projection

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and the Stokes operator are well-defined for every domain, the Stokes operator is self-adjoint and generates a bounded analytic semigroup. This observation was used in [5] to consider in the three-dimensional case the Helmholtz decomposition in the space

$$\tilde{L}^q(\Omega) = \begin{cases} L^q(\Omega) \cap L^2(\Omega), & 2 \le q < \infty \\ L^q(\Omega) + L^2(\Omega), & 1 < q < 2 \end{cases},$$

and to define and to analyze the Stokes operator in the space

$$\tilde{L}^q_\sigma(\Omega) = \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \end{cases}.$$

It was proved that for every unbounded domain $\Omega \subseteq \mathbb{R}^3$ of uniform C^2-type the Stokes operator in $\tilde{L}^q_\sigma$ satisfies the usual resolvent estimate, that it generates an analytic semigroup and has maximal regularity. Moreover, the Helmholtz decomposition of $\tilde{L}^q(\Omega)$ exists for every unbounded domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, of uniform $C^{1,1}$-type, see [6].

To describe this result, we introduce the space of gradients

$$\tilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases},$$

where $G^q(\Omega) = \{\nabla p \in L^q(\Omega) : p \in L^q_{\mathrm{loc}}(\Omega)\}$ and recall the notion of domains of uniform C^k– and $C^{k,1}$-type.

Definition 1.1 *A domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is called a uniform C^k-domain of type (α, β, K), $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, $K > 0$, if for each $x_0 \in \partial\Omega$ we can choose a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, and a C^k-function $h(y')$, $|y'| \le \alpha$, with C^k-norm $\|h\|_{C^k} \le K$ such that the neighborhood*

$$U_{\alpha,\beta,h}(x_0) := \{y = (y', y_n) \in \mathbb{R}^n : \ |y_n - h(y')| < \beta, \ |y'| < \alpha\}$$

of x_0 implies $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', h(y')) : \ |y'| < \alpha\}$ and

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_n) : \ h(y') - \beta < y_n < h(y'), \ |y'| < \alpha\} = U_{\alpha,\beta,h}(x_0) \cap \Omega.$$

By analogy, a domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is called a uniform $C^{k,1}$-domain of type (α, β, K), $k \in \mathbb{N} \cup \{0\}$, if the functions h mentioned above may be chosen in $C^{k,1}$ such that the $C^{k,1}$-norm satisfies $\|h\|_{C^{k,1}} \le K$.

Theorem 1.2 [6] *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a uniform C^1-domain of type (α, β, K) and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition*

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}^q_\sigma(\Omega), \ \nabla p \in \tilde{G}^q(\Omega),$$

satisfying the estimate

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le c\|u\|_{\tilde{L}^q}, \tag{1.1}$$

where $c = c(\alpha, \beta, K, q) > 0$. In particular, the Helmholtz projection $\tilde{P}_q$ defined by $\tilde{P}_q u = u_0$ is a bounded linear projection on $\tilde{L}^q(\Omega)$ with range $\tilde{L}^q_\sigma(\Omega)$ and kernel $\tilde{G}^q(\Omega)$. Moreover, $\tilde{L}^q_\sigma(\Omega)$ is the closure in $\tilde{L}^q(\Omega)$ of the space $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$, $\big(\tilde{L}^q_\sigma(\Omega)\big)' = \tilde{L}^{q'}_\sigma(\Omega)$ and $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$, $q' = \frac{q}{q-1}$.

Using the Helmholtz projection $\tilde{P}_q$ we define the Stokes operator $\tilde{A}_q$ as an operator with domain

$$\mathcal{D}(\tilde{A}^q) = \begin{cases} D^q(\Omega) \cap D^2(\Omega), & 2 \le q < \infty \\ D^q(\Omega) + D^2(\Omega), & 1 < q < 2 \end{cases},$$

where $D^q(\Omega) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, by setting

$$\tilde{A}^q u = -\tilde{P}_q \Delta u, \quad u \in \mathcal{D}(\tilde{A}^q).$$

Let I be the identity and $\mathcal{S}_\varepsilon = \{0 \neq \lambda \in \mathbb{C}; |\arg \lambda| < \frac{\pi}{2} + \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$. Then our main result reads as follows:

Theorem 1.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform $C^{1,1}$-domain of type (α, β, K) and let $1 < q < \infty$, $\delta > 0$. Then*

$$\tilde{A}_q = -\tilde{P}_q \Delta : \mathcal{D}(\tilde{A}_q) \subset \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$$

is a densely defined closed operator. For any $0 < \varepsilon < \frac{\pi}{2}$ and for all $\lambda \in \mathcal{S}_\varepsilon$, its resolvent $(\lambda I + \tilde{A}_q)^{-1} : \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ is well-defined and $u = (\lambda I + \tilde{A}_q)^{-1} f$, $f \in \tilde{L}^q_\sigma(\Omega)$, satisfies the resolvent estimate

$$\|\lambda u\|_{\tilde{L}^q_\sigma} + \|\nabla^2 u\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q_\sigma}, \quad |\lambda| \ge \delta, \tag{1.2}$$

where $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$.

Corollary 1.4 *Under the assumptions of Theorem 1.3 the Stokes operator $\tilde{A}_q$ satisfies the duality relation*

$$\langle \tilde{A}_q u, v\rangle = \langle u, \tilde{A}_{q'} v\rangle \quad \text{for all} \quad u \in \mathcal{D}(\tilde{A}_q),\ v \in \mathcal{D}(\tilde{A}_{q'}). \tag{1.3}$$

and generates an analytic semigroup $e^{-t\tilde{A}_q}$ with bound

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q_\sigma} \le M e^{\delta t} \|f\|_{\tilde{L}^q_\sigma}, \quad f \in \tilde{L}^q_\sigma,\ t \ge 0, \tag{1.4}$$

where $M = M(q, \delta, \alpha, \beta, K) > 0$.

Moreover, let $f \in \tilde{L}^q(\Omega)$. Then the Stokes resolvent equation

$$\lambda u - \Delta u + \nabla p = f, \ \operatorname{div} u = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

has a unique solution $(u, \nabla p) \in \mathcal{D}(\tilde{A}_q) \times \tilde{G}^q(\Omega)$ defined by $u = (\lambda I + \tilde{A}_q)^{-1}\tilde{P}_q f$ and $\nabla p = (I - \tilde{P}_q)(f + \Delta u)$ satisfying

$$\|\lambda u\|_{\tilde{L}^q} + \|\nabla^2 u\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \leq C\|f\|_{\tilde{L}^q}, \tag{1.5}$$

with a constant $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$.

Note that the bound $\delta > 0$ in Theorem 1.3 and Corollary 1.4 may be chosen arbitrarily small, but that it is not clear whether $\delta = 0$ is allowed for a general unbounded domain and whether the semigroup $e^{-t\tilde{A}_q}$ is uniformly bounded in $\tilde{L}^q_\sigma$ for $0 \leq t < \infty$.

2 Preliminaries

Let us recall some properties of sum and intersection spaces known from interpolation theory, cf. [2], [13].

Consider two (complex) Banach spaces X_1, X_2 with norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$, respectively, and assume that both X_1 and X_2 are subspaces of a topological vector space V with continuous embeddings. Further, we assume that $X_1 \cap X_2$ is a dense subspace of both X_1 and X_2. Then the sum space

$$X_1 + X_2 := \{u_1 + u_2; \ u_1 \in X_1, \ u_2 \in X_2\} \subseteq V$$

is a well-defined Banach space with the norm

$$\|u\|_{X_1+X_2} := \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2}; \ u = u_1 + u_2, \ u_1 \in X_1, \ u_2 \in X_2\}.$$

The intersection space $X_1 \cap X_2$ is a Banach space with norm

$$\|u\|_{X_1 \cap X_2} = \max(\|u\|_{X_1}, \ \|u\|_{X_2}).$$

Suppose that X_1 and X_2 are reflexive Banach spaces. Then an argument using weakly convergent subsequences yields the following property: Given $u \in X_1 + X_2$ there exist $u_1 \in X_1$, $u_2 \in X_2$ with $u = u_1 + u_2$ such that

$$\|u\|_{X_1+X_2} = \|u_1\|_{X_1} + \|u_2\|_{X_2}.$$

The dual space $(X_1 + X_2)'$ of $X_1 + X_2$ is given by $X_1' \cap X_2'$, and we get

$$(X_1 + X_2)' = X_1' \cap X_2'$$

with the natural pairing $\langle u, f\rangle = \langle u_1, f\rangle + \langle u_2, f\rangle$ for all $u = u_1 + u_2 \in X_1 + X_2$, $f \in X_1' \cap X_2'$. Thus it holds

$$\|u\|_{X_1+X_2} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|f\|_{X_1'\cap X_2'}}; 0 \neq f \in X_1' \cap X_2'\right\}$$

and

$$\|f\|_{X_1'\cap X_2'} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|u\|_{X_1+X_2}}; 0 \neq u = u_1 + u_2 \in X_1 + X_2\right\};$$

see [2], [13]. By analogy,

$$(X_1 \cap X_2)' = X_1' + X_2'$$

with the natural pairing $\langle u, f_1 + f_2\rangle = \langle u, f_1\rangle + \langle u, f_2\rangle$ for $u \in X_1 \cap X_2$ and $f = f_1 + f_2 \in X_1' + X_2'$.

Consider closed subspaces $L_1 \subseteq X_1$, $L_2 \subseteq X$ with norms $\|\cdot\|_{L_1} = \|\cdot\|_{X_1}$, $\|\cdot\|_{L_2} = \|\cdot\|_{X_2}$ and assume that $L_1 \cap L_2$ is dense in both L_1 and L_2. Then $\|u\|_{L_1\cap L_2} = \|u\|_{X_1\cap X_2}$, $u \in L_1 \cap L_2$, and an elementary argument using the Hahn-Banach theorem shows that also

$$\|u\|_{L_1+L_2} = \|u\|_{X_1+X_2}, \quad u \in L_1 + L_2. \tag{2.1}$$

In particular, we need the following special case. Let $B_1 : \mathcal{D}(B_1) \to X_1$, $B_2 : \mathcal{D}(B_2) \to X_2$ be closed linear operators with dense domains $\mathcal{D}(B_1) \subseteq X_1$, $\mathcal{D}(B_2) \subseteq X_2$ equipped with graph norms

$$\|u\|_{\mathcal{D}(B_1)} = \|u\|_{X_1} + \|B_1u\|_{X_1}, \quad \|u\|_{\mathcal{D}(B_2)} = \|u\|_{X_2} + \|B_2u\|_{X_2}.$$

We assume that $\mathcal{D}(B_1) \cap \mathcal{D}(B_2)$ is dense in both $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$ in the corresponding graph norms. Each functional $F \in \mathcal{D}(B_i)'$, $i = 1, 2$, is given by some pair $f, g \in X_i'$ in the form $\langle u, F\rangle = \langle u, f\rangle + \langle B_iu, g\rangle$. Using (2.1) with $L_i = \{(u, B_iu); u \in \mathcal{D}(B_i)\} \subseteq X_i \times X_i$, $i = 1, 2$, and the equality of norms $\|\cdot\|_{(X_1\times X_1)+(X_2\times X_2)}$ and $\|\cdot\|_{(X_1+X_2)\times(X_1+X_2)}$ on $(X_1 \times X_1) + (X_2 \times X_2)$, we conclude that for each $u \in \mathcal{D}(B_1) + \mathcal{D}(B_2)$ with decomposition $u = u_1 + u_2$, $u_1 \in \mathcal{D}(B_1)$, $u_2 \in \mathcal{D}(B_2)$,

$$\|u\|_{\mathcal{D}(B_1)+\mathcal{D}(B_2)} = \|u_1 + u_2\|_{X_1+X_2} + \|B_1u_1 + B_2u_2\|_{X_1+X_2}. \tag{2.2}$$

Concerning Definition 1.1 for domains of uniform $C^{1,1}$-type we introduce further notations and discuss some properties. Obviously, the axes e_i, $i = 1, \dots, n$, of the new coordinate system (y', y_n) may be chosen in such a way that $e_1, \dots, e_{n-1}$ are tangential to $\partial\Omega$ at x_0. Hence at $y' = 0$ we have $h(y') = 0$ and $\nabla' h(y') = (\partial h/\partial y_1, \dots, \partial h/\partial y_{n-1})(y') = 0$. Since $h \in C^{1,1}$, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $\|h\|_{C^1} \leq M_0$ is satisfied.

It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius $r > 0$ with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^{1,1}$ of type (α, β, K)

$$\overline{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \text{ if } x_j \in \partial\Omega, \quad \overline{B}_j \subset \Omega \text{ if } x_j \in \Omega. \tag{2.3}$$

Here j runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if Ω is bounded, and $j \in \mathbb{N}$ if Ω is unbounded. Moreover, as an important consequence, the covering $\{B_j\}$ of Ω may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these balls have a nonempty intersection. Related to this covering, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, such that

$$0 \le \varphi_j \le 1, \ \operatorname{supp} \varphi_j \subset B_j, \quad \text{and} \quad \sum_{\mathrm{j}=1}^{\mathrm{N}} \varphi_{\mathrm{j}} = 1 \ \text{ or } \ \sum_{\mathrm{j}=1}^{\infty} \varphi_{\mathrm{j}} = 1 \ \text{ on } \ \Omega. \tag{2.4}$$

The functions φ_j may be chosen so that $|\nabla\varphi_j(x)| + |\nabla^2\varphi_j(x)| \le C$ uniformly in j and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If Ω is unbounded, then Ω can be represented as the union of an increasing sequence of bounded uniform $C^{1,1}$-domains $\Omega_k \subset \Omega$, $k \in \mathbb{N}$,

$$\Omega_1 \subset \ldots \subset \Omega_k \subset \Omega_{k+1} \subset \ldots, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \tag{2.5}$$

where each Ω_k is of the same type (α', β', K'). Without loss of generality we assume that $\alpha = \alpha'$, $\beta = \beta'$, $K = K'$.

Using the partition of unity $\{\varphi_j\}$ we will perform the analysis of the Stokes operator by starting from well-known results for certain bounded and unbounded domains. For this reason, given $h \in C^{1,1}(\mathbb{R}^{n-1})$ satisfying $h(0) = 0$, $\nabla' h(0) = 0$ and with compact support contained in the $(n-1)$-dimensional ball of radius r, $0 < r = r(\alpha, \beta, K) < \alpha$, and center 0, we introduce the bounded domain

$$H = H_{\alpha,\beta,h;r} = \{y \in \mathbb{R}^n : \ h(y') - \beta < y_n < h(y'), \ |y'| < \alpha\} \cap B_r(0);$$

here we assume that $\overline{B_r(0)} \subset \{y \in \mathbb{R}^n : \ |y_n - h(y')| < \beta, |y'| < \alpha\}$.

On H we consider the classical Sobolev spaces $W^{k,q}(H)$ and $W_0^{k,q}(H)$, $k \in \mathbb{N}$, the dual space $W^{-1,q}(H) = \big(W_0^{1,q'}(H)\big)'$ and the space

$$L_0^q(H) = \Big\{u \in L^q(H) : \ \int_H u \, dx = 0\Big\}$$

of L^q-functions with vanishing mean on H.

Lemma 2.1 *Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h;r}$.*

(i) There exists a bounded linear operator

$$R : L^q_0(H) \to W^{1,q}_0(H)^n$$

such that $\operatorname{div} \circ R = I$ *on* $L^q_0(H)$ *and* $R\big(L^q_0(H) \cap W^{1,q}_0(H)\big) \subset W^{2,q}_0(H)$*. Moreover, there exists a constant* $C = C(\alpha, \beta, K, q) > 0$ *such that*

$$\begin{array}{lll} \|Rf\|_{W^{1,q}} & \leq & C\|f\|_{L^q(H)} \quad \textit{for all} \quad f \in L^q_0(H) \\ \|Rf\|_{W^{2,q}} & \leq & C\|f\|_{W^{1,q}(H)} \quad \textit{for all} \quad f \in L^q_0(H) \cap W^{1,q}_0(H)\,. \end{array} \tag{2.6}$$

(ii) There exists $C = C(\alpha, \beta, K, q) > 0$ *such that for every* $p \in L^q_0(H)$

$$\|p\|_q \leq C\|\nabla p\|_{W^{-1,q}} = C \sup\Big\{\frac{|\langle p, \operatorname{div} v\rangle|}{\|\nabla v\|_{q'}} : \; 0 \neq v \in W^{1,q'}_0(H)\Big\}. \tag{2.7}$$

(iii) For given $f \in L^q(H)$ *let* $u \in L^q_\sigma(H) \cap W^{1,q}_0(H) \cap W^{2,q}(H)$*,* $p \in W^{1,q}(H)$ *satisfy the Stokes resolvent equation* $\lambda u - \Delta u + \nabla p = f$ *with* $\lambda \in \mathcal{S}_\varepsilon$*. Moreover, assume that* $\operatorname{supp} u \cup \operatorname{supp} p \subset B_r(0)$*. Then there are constants* $\lambda_0 = \lambda_0(q, \alpha, \beta, K) > 0$*,* $C = C(q, \alpha, \beta, K) > 0$ *such that*

$$\|\lambda u\|_{L^q(H)} + \|u\|_{W^{2,q}(H)} + \|\nabla p\|_{L^q(H)} \leq C\|f\|_{L^q(H)} \tag{2.8}$$

if $|\lambda| \geq \lambda_0$*.*

Proof: (i) It is well known that there exists a bounded linear operator $R : L^q_0(H) \to W^{1,q}_0(H)^n$ such that $u = Rf$ solves the divergence problem $\operatorname{div} u = f$. Moreover, the estimate $(2.6)_1$ holds with $C = C(\alpha, \beta, K, q) > 0$, see [8], III, Theorem 3.1. The second part follows from [8], III, Theorem 3.2.

(ii) A duality argument and (i) yield (ii), see [6], [11], II.2.1.

(iii) We extend u, p by zero so that $(u, \nabla p)$ may be considered as a solution of the Stokes resolvent system in a *bent half space*; then we refer to [4], Theorem 3.1, (i). ■

Now let $\Omega \subseteq \mathbb{R}^n$ be a *bounded* $C^{1,1}$-domain. Obviously, such a domain is of type (α, β, K). We collect several results on Sobolev embedding estimates and on the Stokes operator A_q, $1 < q < \infty$.

Lemma 2.2 *(i) Let* $1 < q < \infty$*,* $0 < M \leq 1$*. Then there exists some* $C = C(q, M, \alpha, \beta, K) > 0$ *such that*

$$\|\nabla u\|_{L^q} \leq M\|\nabla^2 u\|_{L^q} + C\|u\|_{L^q} \tag{2.9}$$

for all $u \in W^{2,q}(\Omega)$*.*

(ii) If $2 \leq q < \infty$*,* $0 < M \leq 1$*, then there exists a constant* $C = C(q, M, \alpha, \beta, K) > 0$ *such that*

$$\|u\|_{L^q} \leq M\|\nabla^2 u\|_{L^q} + C\big(\|\nabla^2 u\|_{L^2} + \|u\|_{L^2}\big) \tag{2.10}$$

for all $u \in W^{2,q}(\Omega)$*.*

Proof: The proofs of (i), (ii) are easily reduced to the case $u \in W_0^{2,q}(\Omega')$, $\overline{\Omega} \subset \Omega'$, Ω' a bounded $C^{1,1}$-domain, using an extension operator on Sobolev spaces the norm of which is shown to depend only on q and (α, β, K). In (ii) we choose an $r \in [2, q)$ such that $\|u\|_{L^q} \le M\|\nabla^2 u\|_{L^r} + C\|u\|_{L^r}$ and use the interpolation inequality

$$\|v\|_{L^r} \le \gamma\Big(\frac{1}{\varepsilon}\Big)^{1/\gamma}\|v\|_{L^2} + (1-\gamma)\varepsilon^{1/(1-\gamma)}\|v\|_{L^q}, \tag{2.11}$$

with $\gamma \in (0,1)$, $\frac{1}{r} = \frac{\gamma}{2} + \frac{1-\gamma}{q}$, for $v = u$ and $v = \nabla^2 u$ for suitable $\varepsilon > 0$ to get (2.10). For basic details see [1], IV, Theorem 4.28, [7] and [11], II.1.3. ■

Lemma 2.3 *Let $1 < q < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain.*

(i) The Stokes operator $A_q = -P_q\Delta : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$, where $\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$, satisfies the resolvent estimate

$$\|\lambda u\|_{L^q} + \|A_q u\|_{L^q} \le C\|f\|_{L^q}, \quad C = C(\varepsilon, q, \Omega) > 0, \tag{2.12}$$

where $u \in \mathcal{D}(A_q)$, $\lambda u + A_q u = f \in L^q_\sigma(\Omega)$ and $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$. In particular, it holds the estimate

$$\|u\|_{W^{2,q}} \le C\|A_q u\|_{L^q}, \quad C = C(q, \Omega).$$

Moreover,

$$\langle A_q u, v\rangle = \langle u, A_{q'} v\rangle \quad \text{for all} \quad u \in \mathcal{D}(A_q),\ v \in \mathcal{D}(A_{q'})$$

and $A_q' = A_{q'}$.

(ii) If $q = 2$, then the resolvent problem $\lambda u + A_2 u = f \in L^2_\sigma(\Omega)$, $\lambda \in \mathcal{S}_\varepsilon$, has a unique solution $u \in \mathcal{D}(A_2)$ satisfying the estimate

$$\|\lambda u\|_{L^2} + \|A_2 u\|_{L^2} \le C\|f\|_{L^2} \tag{2.13}$$

with the constant $C = 1 + 2/\cos\varepsilon$ independent of Ω. Moreover, A_2 is selfadjoint and

$$\langle A_2 u, u\rangle = \|A_2^{\frac{1}{2}} u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2, \quad u \in \mathcal{D}(A_2)\,. \tag{2.14}$$

Proof: For (i) see [4], [9], [12]. For (ii) – including even general unbounded domains – we refer to [11]. ■

Note that in the resolvent estimate (2.12) it is not yet clear how the constant C will depend on the the underlying bounded domain Ω.

3 Proofs

3.1 A preliminary result for bounded Ω

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain of type (α, β, K). Then the graph norm $\|u\|_{\mathcal{D}(A_q)} = \|u\|_{L^q} + \|A_q u\|_{L^q}$ is equivalent to the norm $\|u\|_{W^{2,q}}$ on $\mathcal{D}(A_q)$ with constants only depending on q, α, β, K. More precisely,*

$$C_1 \|u\|_{W^{2,q}} \leq \|u\|_{\mathcal{D}(A_q)} \leq C_2 \|u\|_{W^{2,q}}, \quad u \in \mathcal{D}(A_q), \tag{3.1}$$

with $C_1 = C_1(q, \alpha, \beta, K) > 0$, $C_2 = C_2(q, \alpha, \beta, K) > 0$.

Proof: We use the system of functions $\{h_j\}$, $1 \leq j \leq N$, the covering of Ω by balls $\{B_j\}$, and the partition of unity $\{\varphi_j\}$ as described in Section 2. Let

$$U_j = U^-_{\alpha,\beta,h_j}(x_j) \cap B_j \ \text{ if } \ x_j \in \partial\Omega \ \text{ and } U_j = B_j \ \text{ if } \ x_j \in \Omega, \ 1 \leq j \leq N.$$

Given $f \in L^q_\sigma(\Omega)$ and $u \in \mathcal{D}(A_q)$ satisfying $A_q u = f$, i.e. $-\Delta u + \nabla p = f$, $\operatorname{div} u = 0$ in Ω, let $w_j = R\big((\nabla\varphi_j) \cdot u\big) \in W^{2,q}_0(U_j)$ be the solution of the divergence equation $\operatorname{div} w_j = \operatorname{div}(\varphi_j u) = (\nabla\varphi_j) \cdot u$ in U_j, $1 \leq j \leq N$. Moreover, let $M_j = M_j(p)$ be the constant such that $p - M_j \in L^q_0(U_j)$. By Lemma 2.1 and the equation $\nabla p = f + \Delta u$ we conclude that $\|w_j\|_{W^{1,q}(U_j)} \leq C\|u\|_{L^q(U_j)}$, $\|w_j\|_{W^{2,q}(U_j)} \leq C\|u\|_{W^{1,q}(U_j)}$ as well as

$$\|p - M_j\|_{L^q(U_j)} \leq C(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)})$$

with $C = C(q, \alpha, \beta, K) > 0$ independent of j. Finally, let $\lambda_0 > 0$ denote the constant in Lemma 2.1 (iii). Then $\varphi_j u - w_j$ satisfies the local resolvent equation

$$\begin{aligned} &\lambda_0(\varphi_j u - w_j) - \Delta(\varphi_j u - w_j) + \nabla\big(\varphi_j(p - M_j)\big) \\ &\quad = \varphi_j f + \Delta w_j - 2\nabla\varphi_j \cdot \nabla u - (\Delta\varphi_j)u + (\nabla\varphi_j)(p - M_j) + \lambda_0(\varphi_j u - w_j). \end{aligned}$$

in U_j. By (2.8) with $\lambda = \lambda_0$ and the previous *a priori* estimates we get the local inequalities

$$\|\varphi_j \nabla^2 u\|^q_{L^q(U_j)} + \|\varphi_j \nabla(p - M_j)\|^q_{L^q(U_j)} \leq C(\|f\|^q_{L^q(U_j)} + \|u\|^q_{W^{1,q}(U_j)}), \tag{3.2}$$

$1 \leq j \leq N$. Taking the sum over $j = 1, \ldots, N$ and exploiting the crucial property of the number N_0 we are led to the estimate

$$\begin{aligned} \|\nabla^2 u\|^q_{L^q(\Omega)} + \|\nabla p\|^q_{L^q(\Omega)} &= \int_\Omega \Big(\Big(\sum_j \varphi_j |\nabla^2 u| \Big)^q + \Big(\sum_j \varphi_j |\nabla p| \Big)^q \Big)\, dx \\ &\leq \int_\Omega N_0^{\frac{q}{q'}} \Big(\sum_j |\varphi_j \nabla^2 u|^q + \sum_j |\varphi_j \nabla p|^q \Big)\, dx \qquad (3.3) \\ &\leq C N_0^{\frac{q}{q'}} \Big(\sum_j \|f\|^q_{L^q(U_j)} + \sum_j \|u\|^q_{W^{1,q}(U_j)} \Big). \end{aligned}$$

Next we use (2.9) for the term $\|u\|_{W^{1,q}(U_j)}$. Choosing $M > 0$ sufficiently small in (2.9), exploiting the absorption principle and again the property of the number N_0, (3.3) may be simplified to the estimate

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C\big(\|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\big) \tag{3.4}$$

where $C = C(q, \alpha, \beta, K) > 0$. Since $f = A_q u$ and since the norm of the Helmholtz projection P_q in $L^q(\Omega)$ is bounded by $C = C(q, \alpha, \beta, K) > 0$, the proof of the lemma is complete. ■

3.2 The Stokes resolvent in a bounded domain Ω when $q \geq 2$

We consider for $\lambda \in \mathcal{S}_\varepsilon$ the resolvent equation

$$\lambda u + A_q u = \lambda u - \Delta u + \nabla p = f \quad \text{in} \quad \Omega$$

with $f \in L^q_\sigma(\Omega)$, where $1 < q < \infty$, $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$. Our aim is to prove for its solution $u \in D(A_q)$ and $\nabla p = (I - P_q)\Delta u$, the estimate

$$\|\lambda u\|_{L^q \cap L^2} + \|\nabla^2 u\|_{L^q \cap L^2} + \|\nabla p\|_{L^q \cap L^2} \leq C\|f\|_{L^q \cap L^2} \tag{3.5}$$

with $|\lambda| \geq \delta > 0$, where $\delta > 0$ is given, and $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$. Note that this estimate is well-known for bounded domains with a constant $C = C(q, \varepsilon, \delta, \Omega) > 0$. As in Subsection 3.1 let $w_j = R\big((\nabla\varphi_j) \cdot u\big) \in W^{2,q}_0(U_j)$ and choose a constant $M_j = M_j(p)$ such that $p - M_j \in L^q_0(U_j)$. Then we obtain the local equation

$$\begin{aligned} &\lambda(\varphi_j u - w_j) - \Delta(\varphi_j u - w_j) + \nabla\big(\varphi_j(p - M_j)\big) \\ &\quad = \varphi_j f + \Delta w_j - 2\nabla\varphi_j \cdot \nabla u - (\Delta\varphi_j)u - \lambda w_j + (\nabla\varphi_j)(p - M_j) \end{aligned} \tag{3.6}$$

Concerning the term λw_j, we choose in an intermediate step $r \in [2, q)$, use the interpolation estimate (2.11) for $v = u$ and get by Lemma 2.2 (i) for $M \in (0, 1)$ that

$$\|w_j\|_{L^q(U_j)} \leq C_1\|w_j\|_{W^{1,r}(U_j)} \leq M\|u\|_{L^q(U_j)} + C_2\|u\|_{L^2(U_j)};$$

here $C_i = C_i(M, q, r, \alpha, \beta, K) > 0$. Moreover, $\|\nabla^2 w_j\|_{L^q(U_j)} \leq C\|\nabla u\|_{L^q(U_j)}$. For $p - M_j$ we use (2.7) and the equation $\nabla p = -\lambda u + \Delta u + f$ to see that

$$\|p - M_j\|_{L^q(U_j)} \leq C\Big(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \sup\Big\{\frac{|\langle\lambda u, v\rangle|}{\|\nabla v\|_{q'}} : 0 \neq v \in W^{1,q'}_0(U_j)\Big\}\Big),$$

$C = C(q, \alpha, \beta, K) > 0$. Again we choose $r \in [2, q)$, use (2.11) for $v = \lambda u$ and get

$$\|p - M_j\|_{L^q(U_j)} \leq C\big(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \|\lambda u\|^q_{L^2(U_j)}\big) + M\|\lambda u\|_{L^q(U_j)}.$$

Furthermore, we apply to the local resolvent equation (3.6) the estimate (2.8) with λ replaced by $\lambda + \lambda'_0$ where $\lambda'_0 \geq 0$ is sufficiently large such that $|\lambda + \lambda'_0| \geq \lambda_0$ for $|\lambda| \geq \delta$, λ_0 as in (2.8).

Now we combine these estimates and are led to the local inequality

$$\begin{aligned} &\|\lambda\varphi_j u\|_{L^q(U_j)} + \|\varphi_j u\|_{L^q(U_j)} + \|\varphi_j \nabla^2 u\|_{L^q(U_j)} + \|\varphi_j \nabla p\|_{L^q(U_j)} \qquad (3.7)\\ &\le C\big(\|f\|_{L^q(U_j)} + \|u\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \|\lambda u\|^q_{L^2(U_j)}\big) + M\|\lambda u\|^q_{L^q(U_j)} \end{aligned}$$

with $C = C(M, q, \delta, \varepsilon, \alpha, \beta, K) > 0$. Taking the sum over $j = 1, \ldots, N$ in the same way as in (3.2)–(3.4) and using the crucial property of the integer N_0 we get the inequality

$$\begin{aligned} &\|\lambda u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \qquad (3.8)\\ &\quad \le C\big(\|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\Omega)} + \|\lambda u\|_{L^2(\Omega)}\big) + M\|\lambda u\|^q_{L^q(\Omega)} \end{aligned}$$

with $C = C(M, q, \delta, \varepsilon, \alpha, \beta, K) > 0$, $|\lambda| \ge \delta$. Applying (2.9) and choosing M sufficiently small we remove the terms $\|\nabla u\|_{L^q(\Omega)}$ and $\|\lambda u\|_{L^q(\Omega)}$ in (3.8) by the absorption principle. The term $\|u\|_{L^q(\Omega)}$ is removed with the help of (2.10).

Now we combine this improved inequality (3.8) with the estimate (2.13) for $|\lambda| \ge \delta$ and we apply (3.1) with $q = 2$. This proves the desired estimate (3.5) for $2 \le q < \infty$.

3.3 The case Ω bounded, $1 < q < 2$

In this case we consider for $f \in L^2_\sigma + L^q_\sigma = L^q_\sigma$ and $\lambda \in \mathcal{S}_\varepsilon$, $|\lambda| \ge \delta$, the equation $\lambda u - \Delta u + \nabla p = f$ with unique solution $u \in \mathcal{D}(A_q) + \mathcal{D}(A_2) = \mathcal{D}(A_q)$, $\nabla p = (I - \tilde{P}_q)\Delta u$. Note that $A_q = \tilde{A}_q$, $P_q = \tilde{P}_q$ and that $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^{q'}(\Omega) \cap L^2(\Omega) = L^{q'}(\Omega)$. Using $f = \lambda u - \tilde{P}_q \Delta u$, the density of $\mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2) = \mathcal{D}(A_{q'})$ in $L^{q'}_\sigma \cap L^2_\sigma$ and (3.5) (with q replaced by $q' > 2$) we obtain that

$$\begin{aligned} \|f\|_{L^2_\sigma + L^q_\sigma} &= \sup\Big\{\frac{|\langle \lambda u + \tilde{A}_q u, v\rangle|}{\|v\|_{L^{q'}_\sigma \cap L^2_\sigma}};\ 0 \ne v \in \mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2)\Big\}\\ &= \sup\Big\{\frac{|\langle u, \lambda v + \tilde{A}_{q'} v\rangle|}{\|v\|_{L^{q'}_\sigma \cap L^2_\sigma}};\ 0 \ne v \in \mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2)\Big\}\\ &= \sup\Big\{\frac{|\langle u, g\rangle|}{\|(\lambda I - \tilde{P}_{q'}\Delta)^{-1} g\|_{L^{q'}_\sigma \cap L^2_\sigma}};\ 0 \ne g \in L^{q'}_\sigma \cap L^2_\sigma\Big\} \qquad (3.9)\\ &\ge |\lambda| C^{-1} \sup\Big\{\frac{|\langle u, g\rangle|}{\|g\|_{L^{q'}_\sigma \cap L^2_\sigma}};\ 0 \ne g \in L^{q'}_\sigma \cap L^2_\sigma\Big\}. \end{aligned}$$

By Theorem 1.2 the last term $\sup\{\ldots\}$ in (3.9) defines a norm on $L^q_\sigma + L^2_\sigma$ which is equivalent to the norm $\|\cdot\|_{L^q_\sigma + L^2_\sigma}$; the constants in this norm equivalence are related to the norm of $\tilde{P}_{q'}$ and depend only on q and (α, β, K). Hence we proved the estimate $\|\lambda u\|_{L^q_\sigma + L^2_\sigma} \le C\|f\|_{L^q_\sigma + L^2_\sigma}$ and even

$$\|\lambda u\|_{L^q_\sigma + L^2_\sigma} + \|u\|_{L^q_\sigma + L^2_\sigma} + \|A_q u\|_{L^q_\sigma + L^2_\sigma} \le C\|f\|_{L^q_\sigma + L^2_\sigma}, \quad \lambda \in \mathcal{S}_\varepsilon,\ |\lambda| \ge \delta. \qquad (3.10)$$

From the equivalence of norms $\|\cdot\|_{D(A_q)}$ and $\|\cdot\|_{W^{2,q}}$, cf. (3.1), and from (2.2) with $B_1 = A_q, B_2 = A_2$, we conclude that also the norms $\|u\|_{W^{2,q}+W^{2,2}}$ and $\|u\|_{L^q_\sigma+L^2_\sigma} + \|A_q u\|_{L^q_\sigma+L^2_\sigma}$ are equivalent with constants depending only on q and (α, β, K). Then (3.10) and the identity $\nabla p = f - \lambda u + \Delta u$ lead to the estimate

$$\|\lambda u\|_{L^q_\sigma+L^2_\sigma} + \|u\|_{W^{2,q}+W^{2,2}} + \|\nabla p\|_{L^q+L^2} \le C\|f\|_{L^q_\sigma+L^2_\sigma}$$

with $C = C(q, \delta, \varepsilon, \alpha, \beta, K) > 0$. Hence we proved the inequality

$$\|\lambda u\|_{\tilde{L}^q_\sigma} + \|u\|_{\tilde{W}^{2,q}} + \|\nabla p\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q_\sigma}, \quad u \in \mathcal{D}(\tilde{A}_q), \tag{3.11}$$

with $C = C(q, \delta, \varepsilon, \alpha, \beta, K) > 0$ when $|\lambda| \ge \delta > 0$. Now the proof of Theorem 1.3 is complete for bounded domains.

3.4 The case Ω unbounded

Consider the sequence of bounded subdomains $\Omega_j \subseteq \Omega$, $j \in \mathbb{N}$, of uniform $C^{1,1}$-type as in (2.5), let $f \in \tilde{L}^q_\sigma(\Omega)$ and $f_j := \tilde{P}_q f|_{\Omega_j}$. Then consider the solution $(u_j, \nabla p_j)$ of the Stokes resolvent equation

$$\lambda u_j - \tilde{P}_q \Delta u_j = \lambda u_j - \Delta u_j + \nabla p_j = f_j, \quad \nabla p_j = (I - \tilde{P}_q)\Delta u_j \quad \text{in } \Omega_j.$$

From (3.11) we obtain the uniform estimate

$$\|\lambda u_j\|_{\tilde{L}^q_\sigma(\Omega_j)} + \|u_j\|_{\tilde{W}^{2,q}(\Omega_j)} + \|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} \le C\|f\|_{\tilde{L}^q_\sigma(\Omega)} \tag{3.12}$$

with $|\lambda| \ge \delta > 0$, $C = C(q, \delta, \varepsilon, \alpha, \beta, K) > 0$. Extending u_j and ∇p_j by 0 to vector fields on Ω we find, suppressing subsequences, weak limits

$$u = w\text{–}\lim_{j\to\infty} u_j \quad \text{in } \tilde{L}^q_\sigma(\Omega), \quad \nabla p = w\text{–}\lim_{j\to\infty} \nabla p_j \quad \text{in } \tilde{L}^q(\Omega)$$

satisfying $u \in \mathcal{D}(\tilde{A}_q)$, $\lambda u - \Delta u + \nabla p = \lambda u - \tilde{P}_q \Delta u = f$ in Ω and the *a priori* estimate (1.2). Note that each ∇p_j when extended by 0 need not be a gradient field on Ω; however, by de Rham's argument, the weak limit of the sequence $\{\nabla p_j\}$ is a gradient field on Ω. Hence we solved the Stokes resolvent problem $\lambda u + \tilde{A}_q u = \lambda u - \Delta u + \nabla p = f$ in Ω and proved (1.2).

Finally, to prove uniqueness of u we assume that there is some $v \in \mathcal{D}(\tilde{A}_q)$ and $\lambda \in \mathcal{S}_\varepsilon$ satisfying $\lambda v - \tilde{P}_q \Delta v = 0$. Given $f' \in \tilde{L}^{q'}(\Omega)$ let $u \in \mathcal{D}(\tilde{A}_{q'})$ be a solution of $\lambda u - \tilde{P}_{q'} \Delta u = \tilde{P}_{q'} f'$. Then

$$0 = \langle \lambda v - \tilde{P}_q \Delta v, u\rangle = \langle v, (\lambda - \tilde{P}_{q'}\Delta)u\rangle = \langle v, \tilde{P}_{q'} f'\rangle = \langle v, f'\rangle$$

for all $f' \in \tilde{L}^{q'}(\Omega)$; hence, $v = 0$.

Now Theorem 1.3 is completely proved. ■

Proof of Corollary 1.4: The assertions of this Corollary are proved by standard duality arguments and semigroup theory. ■

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