On the Essential Spectrum of a Stokes–Type Operator Arising from Flow around a Rotating Body in the L^q–Framework

Reinhard Farwig[1], Šárka Nečasová[2,3], Jiří Neustupa[2,4]

Abstract

We present the description of the essential spectrum of a linear Stokes–type operator arising from the mathematical model of fluid flow around rotating bodies. The operator is considered in the space $L^q_\sigma(\Omega)$ where $1 < q < \infty$ and $\Omega \subset \mathbb{R}^3$ is an exterior domain.

MSC 2000: Primary 76D05; Secondary 35Q30, 42B25, 47A10

Keywords: Stokes–type operator, essential spectrum, rotating obstacles

1 Introduction

The problem of motion of a rigid body in a liquid has attracted the attention of scientists for more than a century. The first systematic study of this subject was initiated by the pioneering works by Kirchhoff [21] and Kelvin [28], regarding the motion of one or more bodies in an inviscid liquid. After that many mathematicians have furnished significant contributions to this field under different assumptions on the body and on the fluid. We would like to quote the work of Brenner [3] concerning the steady motion of one or more bodies in a linear viscous liquid in the Stokes approximation as well as Weinberger [29], [30], Serre [27] regarding the fall of a body in an incompressible Navier–Stokes fluid under the influence of gravity and Borchers [1] for the existence of a weak solution. Among more recent articles we refer to Hishida [17], [18], Farwig, Hishida and Müller [7], Farwig [5], [6], Farwig, Neustupa [9], Farwig, Krbec, Nečasová [8], Galdi [11], [12], Galdi, Silvestre [13], [14], Geissert, Heck, Hieber [15], Dintelmann, Geissert, Hieber [4], Hishida, Shibata [19], Gunther, Hudspeth, Thomann [16], Martin, Starovoitov, Tucsnak [24], Kračmar, Nečasová, Penel [22], [23] and references in these papers.

In this paper we consider a three–dimensional rigid body rotating with the constant angular velocity $\boldsymbol{\omega} = (0, 0, 1)$ and we assume that its complement in $\mathbb{R}^3$ (denoted by $\Omega(t)$ at time t) is filled with a viscous incompressible fluid. The velocity $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t)$ of the moving

[1]Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany, e-mail: farwig@mathematik.tu-darmstadt.de

[2]Mathematical Institute of Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

[3]e-mail: matus@math.cas.cz

[4]e-mail: neustupa@math.cas.cz

fluid and the pressure $q = q(\boldsymbol{y}, t)$ solve the nonlinear problem

$$\begin{aligned} \partial_t \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q &= \widetilde{\boldsymbol{f}} && \text{for } \boldsymbol{y} \in \Omega(t), \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{for } \boldsymbol{y} \in \Omega(t), \\ \boldsymbol{v}(\boldsymbol{y}, t) &= \boldsymbol{\omega} \times \boldsymbol{y} && \text{for } \boldsymbol{y} \in \partial\Omega(t), \\ \boldsymbol{v}(\boldsymbol{y}, t) &\to \boldsymbol{0} && \text{as } |\boldsymbol{y}| \to \infty \end{aligned} \tag{1.1}$$

where $t > 0$. The coefficient of viscosity is denoted by ν and $\widetilde{\boldsymbol{f}} = \widetilde{\boldsymbol{f}}(\boldsymbol{y}, t)$ is an external body force. The time–dependent exterior domain $\Omega(t)$ is defined, due to the rotation of the body with the angular velocity $\boldsymbol{\omega}$, by the formula

$$\Omega(t) = O(t)\Omega$$

where $\Omega \subset \mathbb{R}^3$ is the exterior of the body at time $t = 0$ and $O(t)$ denotes the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t, & -\sin t, & 0 \\ \sin t, & \cos t, & 0 \\ 0, & 0, & 1 \end{pmatrix}. \tag{1.2}$$

After the change of variables

$$\boldsymbol{x} = O(t)^T \boldsymbol{y}$$

and passing to the new functions

$$\boldsymbol{u}(\boldsymbol{x}, t) = O(t)^T \boldsymbol{v}(\boldsymbol{y}, t), \quad p(\boldsymbol{x}, t) = q(\boldsymbol{y}, t)$$

as well as to the new force term $\boldsymbol{f}(\boldsymbol{x}, t) = O(t)^T \widetilde{\boldsymbol{f}}(\boldsymbol{y}, t)$, we arrive at the modified Navier–Stokes problem

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{for } \boldsymbol{x} \in \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{for } \boldsymbol{x} \in \Omega, \\ \boldsymbol{u}(\boldsymbol{x}, t) &= \boldsymbol{\omega} \times \boldsymbol{x} && \text{for } \boldsymbol{x} \in \partial\Omega, \\ \boldsymbol{u}(\boldsymbol{x}, t) &\to \boldsymbol{0} && \text{as } |\boldsymbol{x}| \to \infty \end{aligned} \tag{1.3}$$

where $t > 0$. The first equation in (1.3) contains two new linear terms, the classical Coriolis force term $\boldsymbol{\omega} \times \boldsymbol{u}$ (up to a multiplicative constant) and the term $(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}$ which is not subordinate to the Laplace operator in the unbounded domain Ω. The associated linearized steady problem represents the modified Stokes system

$$\begin{aligned} -\nu \Delta \boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{x}) &= \boldsymbol{0} && \text{for } \boldsymbol{x} \in \partial\Omega, \\ \boldsymbol{u} &\to \boldsymbol{0} && \text{as } |\boldsymbol{x}| \to \infty. \end{aligned} \tag{1.4}$$

The problem (1.4) was analyzed for $\Omega = \mathbb{R}^3$ in L^q–spaces, $1 < q < \infty$, in [7], proving the estimate

$$\|\nu \nabla^2 \boldsymbol{u}\|_q + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}\|_q + \|\nabla p\|_q \le c_1 \|\boldsymbol{f}\|_q \tag{1.5}$$

where $\|\,.\,\|_q$ denotes the norm in $L^q(\Omega)^3$ or in $L^q(\Omega)^9$ and c_1 is a positive constant independent of $\boldsymbol{f}$, $\boldsymbol{u}$ and p. Similar results were obtained in [5] and [6] in the case when the last condition in (1.4) was replaced by $\boldsymbol{u} \to \boldsymbol{u}_\infty$ (for $|\boldsymbol{x}| \to \infty$) where $\boldsymbol{u}_\infty$ is a non–zero constant translational velocity, parallel to $\boldsymbol{\omega}$. For related L^q–results on weak solutions to (1.3) we refer to [17], for the investigation of several auxiliary linear problems to [25], [26], and for weak solutions to the Oseen system corresponding to (1.4) with anisotropic weights see [22] ($q = 2$) or [23] ($1 < q < +\infty$).

In this paper, assuming $1 < q < \infty$, we describe the essential spectrum $\sigma_{ess}(L_q)$ of the Stokes–type operator L_q, defined by the equation

$$L_q\boldsymbol{u} \;=\; \nu\Pi_\sigma\Delta\boldsymbol{u} + \Pi_\sigma\left[(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}\right] - \Pi_\sigma\,\boldsymbol{\omega}\times\boldsymbol{u}. \tag{1.6}$$

The symbol Π_σ denotes the Helmholtz projection of $L^q(\Omega)^3$ onto the subspace of $\boldsymbol{v} \in L^q(\Omega)^3$ such that $\operatorname{div}\boldsymbol{v} = 0$ in the sense of distributions and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces. This subspace is usually denoted by $L^q_\sigma(\Omega)$. Thus L_q is a linear operator in $L^q_\sigma(\Omega)$ with domain

$$D(L_q) \;=\; \{\boldsymbol{v}\in W^{2,q}(\Omega)^3\cap W^{1,q}_0(\Omega)^3\cap L^q_\sigma(\Omega);\;\; (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\in L^q(\Omega)^3\}.$$

We shall further treat $D(L_q)$ as a Banach space with the norm

$$\|\boldsymbol{v}\|_{D(L_q)} \;:=\; \|\boldsymbol{v}\|_{2,q} + \|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\|_q$$

where $\|\,.\,\|_{2,q}$ denotes the norm in $W^{2,q}(\Omega)^3$. The essential spectrum is already known from [9] for $q = 2$:

$$\sigma_{ess}(L_2) \;=\; \{\lambda = \alpha + \mathrm{i}k \in \mathbb{C};\;\; \alpha \le 0,\;\; k\in\mathbb{Z}\}. \tag{1.7}$$

We are going to show that the same identity holds in the case of a general $q \in (1,\infty)$.

2 The case $\Omega = \mathbb{R}^3$

If $\Omega = \mathbb{R}^3$ and $\boldsymbol{u} \in D(L_q)$, then both terms $\nu\Delta\boldsymbol{u}$ and $[(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}] - \boldsymbol{\omega}\times\boldsymbol{u}$ belong to $L^q_\sigma(\mathbb{R}^3)$. Therefore, the projection Π_σ in (1.6) can be omitted and the operator L_q can be simplified to

$$L_q\boldsymbol{u} \;=\; \nu\Delta\boldsymbol{u} + [(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}] - \boldsymbol{\omega}\times\boldsymbol{u}.$$

The next theorem provides information on solutions of the resolvent equation

$$L_q\boldsymbol{u} - \lambda\boldsymbol{u} \;=\; \boldsymbol{f} \tag{2.1}$$

for $\boldsymbol{f} \in L^q_\sigma(\mathbb{R}^3)$.

Theorem 2.1 *Suppose that $1 < q < \infty$, $\boldsymbol{f} \in L^q_\sigma(\mathbb{R}^3)$ and $\lambda = \alpha + \mathrm{i}\beta$ (α, $\beta \in \mathbb{R}$) where either $\alpha > 0$ or $\beta \ne k$ for all $k \in \mathbb{Z}$. Then the equation (2.1) has a unique solution $\boldsymbol{u} \in D(L_q)$. There exists a real constant c_2 depending only on λ and q such that $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_q \;\le\; c_2\,\|\boldsymbol{f}\|_q\,. \tag{2.2}$$

Consequently, λ belongs to the resolvent set of the operator L_q.

Proof. Equation (2.1) can be written in the form

$$\nu\Delta\boldsymbol{u} + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u} - \lambda\boldsymbol{u} = \boldsymbol{f}. \tag{2.3}$$

Of course, $\boldsymbol{u}$ is also required to satisfy the condition $\operatorname{div}\boldsymbol{u} = 0$. Due to the geometry of the problem it is reasonable to use cylindrical coordinates (r,θ,x_3) in $\mathbb{R}^3$. Then the term $(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}$, which equals $-x_2\,\partial_1\boldsymbol{u} + x_1\,\partial_2\boldsymbol{u}$, can be expressed as

$$(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} \;=\; \partial_\theta\boldsymbol{u}.$$

At first suppose that $\boldsymbol{f}$ belongs to Schwartz' space $\mathcal{S}(\mathbb{R}^3)^3$ of so-called rapidly decreasing functions. We shall denote by $\mathcal{F}$ the Fourier transform, by $\mathcal{F}_{-1}$ its inverse, by $\widehat{\ }$ Fourier images of functions, by $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$ their Cartesian variables, and we put $s = |\boldsymbol{\xi}|$. Then we look for a solution $\boldsymbol{u}$ of (2.1) in the form

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \mathcal{F}_{-1}[\,\widehat{\boldsymbol{u}}\,](\boldsymbol{x}) \;=\; \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}x\cdot\xi}\,\widehat{\boldsymbol{u}}(\boldsymbol{\xi})\;\mathrm{d}\boldsymbol{\xi}. \tag{2.4}$$

Substituting (2.4) to (2.3), we obtain

$$(\lambda+\nu s^2)\widehat{\boldsymbol{u}} - \partial_\phi\widehat{\boldsymbol{u}} + \boldsymbol{\omega}\times\widehat{\boldsymbol{u}} = -\widehat{\boldsymbol{f}}. \tag{2.5}$$

Here $\partial_\phi\widehat{\boldsymbol{u}}$ denotes the angular derivative

$$\partial_\phi\widehat{\boldsymbol{u}} \;=\; (\boldsymbol{\omega}\times\boldsymbol{\xi})\cdot\nabla\widehat{\boldsymbol{u}} \;\equiv\; -\xi_2\,\frac{\partial\widehat{\boldsymbol{u}}}{\partial\xi_1} + \xi_1\,\frac{\partial\widehat{\boldsymbol{u}}}{\partial\xi_2}$$

when using cylindrical coordinates (ρ,ϕ,ξ_3) in the space of Fourier variables. The equation $\operatorname{div}\boldsymbol{u} = 0$ leads to the condition $\mathrm{i}\boldsymbol{\xi}\cdot\widehat{\boldsymbol{u}} = 0$. Now $\widehat{\boldsymbol{u}}$ can be considered to be a solution of the first order ordinary differential equation (2.5) with respect to the angular variable ϕ. Writing $\widehat{\boldsymbol{u}}$ in the form

$$\widehat{\boldsymbol{u}}(\rho,\phi,\xi_3) \;=\; O(\phi)\,\widehat{\boldsymbol{v}}(\rho,\phi,\xi_3),$$

one verifies that $\partial_\phi\widehat{\boldsymbol{u}} \;=\; O(\phi)\,\partial_\phi\widehat{\boldsymbol{v}} + \boldsymbol{\omega}\times[O(\phi)\widehat{\boldsymbol{v}}]$ and that (2.5) is equivalent to the equation

$$-\partial_\phi\widehat{\boldsymbol{v}} + (\lambda+\nu s^2)\widehat{\boldsymbol{v}} \;=\; -O(\phi)^T\widehat{\boldsymbol{f}}.$$

Its solution $\widehat{\boldsymbol{v}}$ satisfies

$$\begin{aligned}
\widehat{\boldsymbol{v}}(\rho,\phi+2\pi,\xi_3) \;&=\; \mathrm{e}^{2\pi(\lambda+\nu s^2)}\,\widehat{\boldsymbol{v}}(\rho,\phi,\xi_3) + \int_\phi^{\phi+2\pi} \mathrm{e}^{(\lambda+\nu s^2)(\phi+2\pi-t)}\;O(t)^T\widehat{\boldsymbol{f}}(\rho,t,\xi_3)\;\mathrm{d}t\\
&=\; \mathrm{e}^{2\pi(\lambda+\nu s^2)}\,\widehat{\boldsymbol{v}}(\rho,\phi,\xi_3) + \int_0^{2\pi} \mathrm{e}^{(\lambda+\nu s^2)(2\pi-t)}\;O(t+\phi)^T\widehat{\boldsymbol{f}}(\rho,t+\phi,\xi_3)\;\mathrm{d}t.
\end{aligned}$$

Since $\widehat{\boldsymbol{v}}$ is a 2π–periodic function in the variable ϕ, we have

$$\begin{aligned}
\widehat{\boldsymbol{v}}(\rho,\phi,\xi_3) \;&=\; \frac{1}{1-\mathrm{e}^{2\pi(\lambda+\nu s^2)}}\int_0^{2\pi} \mathrm{e}^{(\lambda+\nu s^2)(2\pi-t)}\;O(t+\phi)^T\widehat{\boldsymbol{f}}(\rho,t+\phi,\xi_3)\;\mathrm{d}t,\\
\widehat{\boldsymbol{u}}(\rho,\phi,\xi_3) \;&=\; \int_0^{2\pi} \frac{\mathrm{e}^{(\lambda+\nu s^2)(2\pi-t)}}{1-\mathrm{e}^{2\pi(\lambda+\nu s^2)}}\;O(t)^T\widehat{\boldsymbol{f}}(\rho,t+\phi,\xi_3)\;\mathrm{d}t.
\end{aligned} \tag{2.6}$$

Note that $s^2 = |\boldsymbol{\xi}|^2 = \rho^2 + \xi_3^2$. Returning to the Cartesian variables $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ in (2.6), we obtain

$$\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) = \int_0^{2\pi} \Psi(\lambda, \boldsymbol{\xi}, t)\; O(t)^T \widehat{\boldsymbol{f}}\big(O(t)\boldsymbol{\xi}\big)\;\mathrm{d}t = \int_0^{2\pi} \Psi(\lambda, \boldsymbol{\xi}, t)\; O(t)^T \mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\;\mathrm{d}t \tag{2.7}$$

where

$$\Psi(\lambda, \boldsymbol{\xi}, t) = \frac{\mathrm{e}^{(\lambda+\nu s^2)(2\pi - t)}}{1 - \mathrm{e}^{2\pi(\lambda+\nu s^2)}}.$$

As is easily seen, the function Ψ satisfies the estimate

$$\big|\Psi(\lambda, \boldsymbol{\xi}, t)\big| = \mathrm{e}^{-\alpha t} \left| \frac{\mathrm{e}^{-\nu s^2 t}}{\mathrm{e}^{-2\pi(\alpha+\nu s^2)} - \mathrm{e}^{2\pi \mathrm{i}\beta}} \right| \le \begin{cases} \dfrac{\mathrm{e}^{-2\pi\alpha}}{\mathrm{dist}(\mathrm{e}^{2\pi\mathrm{i}\beta}; \mathbb{R}_+)} & \text{if} \quad \alpha \le 0,\; \beta \notin \mathbb{Z}, \\[2ex] \dfrac{1}{1 - \mathrm{e}^{-2\pi\alpha}} & \text{if} \quad \alpha > 0\,; \end{cases}$$

here $\mathbb{R}_+$ denotes the positive part of the real axis. Thus, for fixed $\lambda = \alpha + \mathrm{i}\beta$ satisfying the assumptions of the theorem, $|\Psi(\lambda, \,\cdot\,, t)|$ is bounded in $\mathbb{R}^3$ with an upper bound independent of t and $\boldsymbol{\xi}$. Further, if $i \in \{1; 2; 3\}$, then

$$\begin{aligned} \frac{\partial \Psi}{\partial \xi_i} &= 2\nu\xi_i\, \frac{(2\pi - t)\,\big[\mathrm{e}^{(\lambda+\nu s^2)(4\pi - t)} - \mathrm{e}^{(\lambda+\nu s^2)(2\pi - t)}\big] - 2\pi\, \mathrm{e}^{(\lambda+\nu s^2)(4\pi - t)}}{\big[1 - \mathrm{e}^{2\pi(\lambda+\nu s^2)}\big]^2} \\ &= -2\nu\xi_i\, \frac{(2\pi - t)\,\mathrm{e}^{-(\lambda+\nu s^2)(2\pi + t)} + t\,\mathrm{e}^{-(\lambda+\nu s^2)t}}{\big[\mathrm{e}^{-2\pi(\lambda+\nu s^2)} - 1\big]^2}, \end{aligned}$$

and $|\xi_i|\,\big|\partial\Psi/\partial\xi_i\big|$ can be estimated as follows:

$$\begin{aligned} |\xi_i| \left| \frac{\partial \Psi}{\partial \xi_i} \right| &\le 2\nu s^2\, \frac{(2\pi - t)\,\mathrm{e}^{-(\alpha+\nu s^2)(2\pi + t)} + t\,\mathrm{e}^{-(\alpha+\nu s^2)t}}{\big|\mathrm{e}^{-2\pi(\lambda+\nu s^2)} - 1\big|^2} \\ &\le 2\nu\, \mathrm{e}^{-\alpha t}\, \frac{(2\pi - t)\, s^2\, \mathrm{e}^{-2\pi\nu s^2}\, \mathrm{e}^{-2\pi\alpha} + s^2 t\, \mathrm{e}^{-\nu s^2 t}}{|\mathrm{e}^{-2\pi\mathrm{i}\beta}|^2\, \big|\mathrm{e}^{-2\pi(\alpha+\nu s^2)} - \mathrm{e}^{2\pi\mathrm{i}\beta}\big|^2} \\ &\le \begin{cases} 2\nu\, \mathrm{e}^{-2\pi\alpha}\, \dfrac{2\pi\, c_3\, \mathrm{e}^{-2\pi\alpha} + c_4}{[\mathrm{dist}(\mathrm{e}^{2\pi\mathrm{i}\beta}; \mathbb{R}_+)]^2} & \text{if} \quad \alpha \le 0,\; \beta \notin \mathbb{Z}, \\[2ex] 2\nu\, \dfrac{2\pi\, c_3 + c_4}{[1 - \mathrm{e}^{-2\pi\alpha}]^2} & \text{if} \quad \alpha > 0. \end{cases} \end{aligned}$$

The upper bound of $|\xi_i|\,\big|\partial\Psi/\partial\xi_i\big|$ does not depend on t and on $\boldsymbol{\xi}$. We can similarly estimate all other terms of the form

$$|\xi_1|^{\kappa_1}\, |\xi_2|^{\kappa_2}\, |\xi_3|^{\kappa_3} \left| \frac{\partial^{\kappa_1+\kappa_2+\kappa_3}\Psi}{\partial\xi_1^{\kappa_1}\, \partial\xi_2^{\kappa_2}\, \partial\xi_3^{\kappa_3}} \right|$$

where κ_1, κ_2, κ_3 are equal to zero or one.

Let us now define the function $\boldsymbol{u}$ by

$$\boldsymbol{u}(\boldsymbol{x}) = \int_0^{2\pi} O(t)^T\; \mathcal{F}_{-1}\Big[\Psi(\lambda, \boldsymbol{\xi}, t)\; \mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\Big](\boldsymbol{x})\;\mathrm{d}t. \tag{2.8}$$

Applying Lizorkin's multiplier theorem (see e.g. [10], p. 375) and using the estimates of Ψ and its derivatives discussed above, we derive the inequality

$$\left\| \mathcal{F}_{-1}\Big[\Psi(\lambda,\boldsymbol{\xi},t)\ \mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\Big] \right\|_q \le c_5\,\|\boldsymbol{f}\|_q \tag{2.9}$$

where $c_5 = c_5(\lambda, q)$. Then (2.8) and (2.9) imply that $\boldsymbol{u} \in L^q_\sigma(\mathbb{R}^3)$ and that it satisfies (2.2). Moreover, by construction, $\boldsymbol{u}$ is a solution of (2.3) in the sense of distributions. In order to show that $\boldsymbol{u}$ is a strong solution, we still need to verify that it belongs to the domain of the operator L_q. Consider the auxiliary problem

$$\begin{aligned} \nu\Delta\boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \nabla p \;&=\; \boldsymbol{f} + \lambda\boldsymbol{u} \qquad \text{in } \mathbb{R}^3, \\ \operatorname{div}\boldsymbol{v} \;&=\; 0 \qquad\qquad\quad \text{in } \mathbb{R}^3 \end{aligned} \tag{2.10}$$

for the unknown function $\boldsymbol{v}$. Theorem 1.1 in [7] says that the problem (2.10) has a locally integrable solution $(\boldsymbol{v}, p)$ which satisfies

$$\|\nu\nabla^2\boldsymbol{v}\|_q + \|\partial_\theta\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v}\|_q + \|\nabla p\|_q \le c_6\left(\|\boldsymbol{f}\|_q + |\lambda|\,\|\boldsymbol{u}\|_q\right) \tag{2.11}$$

where $c_6 = c_6(q)$. Moreover, the set of all solutions of (2.10) has the structure $(\boldsymbol{v}+\boldsymbol{w}, p)$ where

$$\boldsymbol{w} \;=\; a\boldsymbol{\omega} + b\boldsymbol{\omega}\times\boldsymbol{x} + c(x_1, x_2, -2x_3); \qquad a,\ b,\ c \in \mathbb{R}.$$

Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ differ at most in the additive linear vector field $a\boldsymbol{\omega}+b\boldsymbol{\omega}\times\boldsymbol{x}+c(x_1,x_2,-2x_3)$ which, however, does not affect the validity of (2.11) because $\nabla^2\boldsymbol{w} = \boldsymbol{0}$ and $\partial_\theta\boldsymbol{w} - \boldsymbol{\omega}\times\boldsymbol{w} = \boldsymbol{0}$. Hence $\boldsymbol{u}$ satisfies (2.11), too, and this inequality and (2.2) yield

$$\|\nu\nabla^2\boldsymbol{u}\|_q + \|\partial_\theta\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u}\|_q + \|\nabla p\|_q \le (c_6 + |\lambda|\,c_2)\,\|\boldsymbol{f}\|_q. \tag{2.12}$$

In particular, $\boldsymbol{u}$ defined by (2.8) lies in $D(L_q)$ and is a strong solution of (2.1).

The extension of this result from $\boldsymbol{f} \in \mathcal{S}(\mathbb{R}^3)^3$ to all $\boldsymbol{f} \in L^q(\mathbb{R}^3)^3$ follows from the density of $\mathcal{S}(\mathbb{R}^3)^3$ in $L^q(\mathbb{R}^3)^3$. $\square$

Theorem 2.2 *Suppose that $1 < q < \infty$ and $\lambda = \alpha + \mathrm{i}\beta$ where $\alpha \le 0$ and $\beta = k \in \mathbb{Z}$. Then λ belongs to the essential spectrum of the operator L_q.*

Proof. Transforming the problem (2.3) to cylindrical coordinates (r, θ, x_3) and denoting by $\boldsymbol{u}$ (respectively $\boldsymbol{f}$) the triplet (u_r, u_θ, u_3) (respectively (f_r, f_θ, f_3)), we obtain that

$$\nu\Delta\boldsymbol{u} + \partial_\theta\boldsymbol{u} - \lambda\boldsymbol{u} = \boldsymbol{f}, \qquad \partial_r(ru_r) + \partial_\theta u_\theta + \partial_3(ru_3) = 0. \tag{2.13}$$

Suppose that $\lambda = \alpha + \mathrm{i}k$ where $\alpha < 0$ and $k \in \mathbb{Z}$. Let $\eta \in C^\infty([0,1])$ be equal to zero on $[0, \frac14]$, increasing on $[\frac14, \frac34]$ and equal to one on $[\frac34, 1]$. We put

$$v_r^n(r) \;=\; \begin{cases} 0 & \text{for } \ 0 \le r \le n, \\ \eta\Big(\dfrac{r-n}{n}\Big) & \text{for } \ n \le r \le 2n, \\ \cos\big[\zeta(r-2n)\big] & \text{for } \ 2n \le r \le 2n + n^2T, \\ \eta\Big(\dfrac{3n+n^2T-r}{n}\Big) & \text{for } \ 2n+n^2T \le r \le 3n+n^2T, \\ 0 & \text{for } \ 3n+n^2T \le r, \end{cases}$$

where $\zeta = \sqrt{-\alpha/\nu}$ and $T = 2\pi/\zeta$ is the period of the function $\cos\bigl[\zeta(r-2n)\bigr]$ (as a function of the variable r). Moreover, let $\varphi \in C_0^\infty(-\infty,+\infty)$ be equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$ and to 0 on $(-\infty,-1]\cup[1,+\infty)$. Then we define $\boldsymbol{u}^n = \bigl(u_r^n,\, u_\theta^n,\, u_3^n\bigr)$ by

$$\begin{aligned}
u_r^n(r,\theta,x_3) &= \gamma_n\, v_r^n(r)\, \varphi\Bigl(\frac{x_3}{n}\Bigr)\, \mathrm{e}^{\mathrm{i}k\theta}, \\
u_\theta^n(r,\theta,x_3) &= -\frac{1}{\mathrm{i}k}\, \partial_r\bigl[r\, u_r^n(r,\theta,x_3)\bigr] \;=\; -\frac{\gamma_n}{\mathrm{i}k}\, \Bigl[v_r^n(r) + r\, \frac{\mathrm{d}v_r^n(r)}{\mathrm{d}r}\Bigr]\, \varphi\Bigl(\frac{x_3}{n}\Bigr)\, \mathrm{e}^{\mathrm{i}k\theta}, \\
u_3^n(r,\theta,x_3) &= 0,
\end{aligned}$$

where the constant γ_n is chosen so that $\|\boldsymbol{u}^n\|_q = 1$. Note that the function $\boldsymbol{u}^n$ satisfies the second equation in (2.13), the equation of continuity. If we calculate γ_n, then the decisive term for large n comes from $u_\theta^n(r,\theta,x_3)$ for $2n \le r \le 2n + n^2T$, to be more precise, it comes from $-(\gamma_n/\mathrm{i}k)\, r\, (\mathrm{d}v_r^n/\mathrm{d}r)\, \varphi\, \mathrm{e}^{\mathrm{i}k\theta}$. All other terms are of lower order in powers of n. Now the L^q–norm of the mentioned term can be estimated from below as follows:

$$\begin{aligned}
&\Bigl\| \frac{\gamma_n}{\mathrm{i}k}\, r\, \frac{\mathrm{d}v_r^n(r)}{\mathrm{d}r}\, \varphi\Bigl(\frac{x_3}{n}\Bigr)\, \mathrm{e}^{\mathrm{i}k\theta} \Bigr\|_q \\
&= \frac{\gamma_n}{|k|} \biggl(\int_{2n}^{2n+n^2T} r^q \Bigl| \frac{\mathrm{d}}{\mathrm{d}r} \cos\bigl[\zeta(r-2n)\bigr] \Bigr|^q r\, \mathrm{d}r \int_0^{2\pi} |\mathrm{e}^{\mathrm{i}k\theta}|^q\, \mathrm{d}\theta \int_{-\infty}^{+\infty} \Bigl|\varphi\Bigl(\frac{x_3}{n}\Bigr)\Bigr|^q \mathrm{d}x_3 \biggr)^{1/q} \\
&= c_7\, \gamma_n \biggl(\int_{2n}^{2n+n^2T} r^{q+1} \Bigl|\sin\bigl[\zeta(r-2n)\bigr]\Bigr|^q \mathrm{d}r \biggr)^{1/q} \biggl(n \int_{-1}^{1} |\varphi(s)|^q\, \mathrm{d}s \biggr)^{1/q} \\
&= c_8\, \gamma_n \biggl(\sum_{i=1}^{n^2} \int_{2n+(i-1)T}^{2n+iT} r^{q+1} \Bigl|\sin\bigl[\zeta(r-2n)\bigr]\Bigr|^q \mathrm{d}r \biggr)^{1/q} n^{1/q} \\
&\ge c_8\, \gamma_n \biggl(\sum_{i=1}^{n^2} [2n+(i-1)T]^{q+1} \int_{2n+(i-1)T}^{2n+iT} \Bigl|\sin\bigl[\zeta(r-2n)\bigr]\Bigr|^q \mathrm{d}r \biggr)^{1/q} n^{1/q} \\
&\ge c_9\, \gamma_n \biggl(\sum_{i=1}^{n^2} (i-1)^{q+1} \biggr)^{1/q} n^{1/q} \\
&\ge c_{10}\, \gamma_n\, n^{2+5/q}.
\end{aligned}$$

The constant c_{10} is independent of n. It is even easier to derive the upper estimate

$$\Bigl\| \frac{\gamma_n}{\mathrm{i}k}\, r\, \frac{\mathrm{d}v_r^n(r)}{\mathrm{d}r}\, \varphi\Bigl(\frac{x_3}{n}\Bigr)\, \mathrm{e}^{\mathrm{i}k\theta} \Bigr\|_q \;\le\; c_{11}\, \gamma_n\, n^{2+5/q}.$$

From these estimates and from the condition $\|\boldsymbol{u}^n\|_q = 1$, we obtain the inequalities

$$\frac{c_{12}}{n^{2+5/q}} \;\le\; \gamma_n \;\le\; \frac{c_{13}}{n^{2+5/q}} \qquad \text{for all } n \in \mathbb{N} \tag{2.14}$$

where $c_{12} \equiv c_{12}(\alpha,k,q)$ and $c_{13} \equiv c_{13}(\alpha,k,q)$ are appropriate positive constants.

Substituting $\boldsymbol{u}^n$ to the left hand side in the first equation of (2.13), we calculate that the corresponding right hand side is $\boldsymbol{f}^n = \boldsymbol{g}^n\, \mathrm{e}^{\mathrm{i}k\theta}$ where $\boldsymbol{g}^n \equiv (g_r^n, g_\theta^n, g_3^n)$ depends only on the variables r, x_3. To be more precise,

$$
\begin{aligned}
g_r^n(r,x_3) &= 0 \qquad \text{for } 0 \le r \le n, && (2.15)\\
g_r^n(r,x_3) &= \frac{\gamma_n\nu}{n^2}\,\eta''\Big(\frac{r-n}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) + \frac{\gamma_n\nu}{rn}\,\eta'\Big(\frac{r-n}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big)\\
&\quad - \frac{\gamma_n\nu k^2}{r^2}\,\eta\Big(\frac{r-n}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) + \frac{\gamma_n\nu}{n^2}\,\eta\Big(\frac{r-n}{n}\Big)\,\varphi''\Big(\frac{x_3}{n}\Big)\\
&\quad - \gamma_n\,\alpha\,\eta\Big(\frac{r-n}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) \qquad \text{for } n \le r \le 2n, && (2.16)\\
g_r^n(r,x_3) &= \frac{\gamma_n\nu}{n^2}\,\cos\big[\zeta(r-2n)\big]\,\varphi''\Big(\frac{x_3}{n}\Big) - \frac{\gamma_n\nu}{r}\,\zeta\,\sin\big[\zeta(r-2n)\big]\,\varphi\Big(\frac{x_3}{n}\Big)\\
&\quad - \frac{\gamma_n k^2\nu}{r^2}\,\cos\big[\zeta(r-2n)\big]\,\varphi\Big(\frac{x_3}{n}\Big) \qquad \text{for } 2n \le r \le 2n+n^2T, && (2.17)
\end{aligned}
$$

and

$$
\begin{aligned}
g_r^n(r,x_3) &= \frac{\gamma_n\nu}{n^2}\,\eta''\Big(\frac{3n+n^2T-r}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) - \frac{\gamma_n\nu}{rn}\,\eta'\Big(\frac{3n+n^2T-r}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big)\\
&\quad - \frac{\gamma_n\nu k^2}{r^2}\,\eta\Big(\frac{3n+n^2T-r}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) + \frac{\gamma_n\nu}{n^2}\,\eta\Big(\frac{3n+n^2T-r}{n}\Big)\,\varphi''\Big(\frac{x_3}{n}\Big)\\
&\quad - \gamma_n\,\alpha\,\eta\Big(\frac{3n+n^2T-r}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big) \qquad \text{for } 2n+n^2T \le r \le 3n+n^2T, && (2.18)\\
g_r^n(r,x_3) &= 0 \qquad \text{for } 3n+n^2T \le r. && (2.19)
\end{aligned}
$$

By analogy with the relation between u_r^n and u_θ^n, the function g_θ^n equals $-\partial_r\big[r\,g_r^n(r,x_3)\big]/\mathrm{i}k$. Furthermore, $g_3^n(r,x_3) = 0$ for all $r \ge 0$ and $x_3 \in (-\infty,+\infty)$.

Now we need to estimate the norm of $\boldsymbol{f}^n$ in $L^q(\mathbb{R}^3)^3$ and to show that

$$
\|\boldsymbol{f}^n\|_q \longrightarrow 0 \qquad \text{for } n \to \infty. \tag{2.20}
$$

Let us begin with the first term on the right hand side of (2.16), multiplied by $\mathrm{e}^{\mathrm{i}k\theta}$. This term is defined for $n \le r \le 2n$, $0 \le \theta \le 2\pi$ and $-\infty < x_3 < +\infty$. Thus its contribution to the upper bound of $\|\boldsymbol{f}^n\|_q$ is

$$
\begin{aligned}
&\bigg(\int_n^{2n}\int_0^{2\pi}\int_{-\infty}^{+\infty}\Big|\frac{\gamma_n\nu}{n^2}\,\eta''\Big(\frac{r-n}{n}\Big)\,\varphi\Big(\frac{x_3}{n}\Big)\,\mathrm{e}^{\mathrm{i}k\theta}\Big|^q\,r\,\mathrm{d}x_3\,\mathrm{d}\theta\,\mathrm{d}r\bigg)^{1/q}\\
&\le\; C\,\frac{\gamma_n}{n^2}\,\bigg(\int_n^{2n}\Big|\eta''\Big(\frac{r-n}{n}\Big)\Big|^q\,r\,\mathrm{d}r\int_{-\infty}^{+\infty}\Big|\varphi\Big(\frac{x_3}{n}\Big)\Big|^q\,\mathrm{d}x_3\bigg)^{1/q}\\
&\le\; C\,\frac{\gamma_n}{n^2}\,n^{3/q}\,;
\end{aligned}
$$

here C is a generic constant which is independent of $n \in \mathbb{N}$. The right hand side tends to zero as $n \to \infty$ due to (2.14). We arrive at the same conclusion when dealing with other terms on the right hand sides of (2.15)–(2.19) and also with all corresponding terms in the expression of g_θ^n. (Naturally, all terms must be multiplied by $\mathrm{e}^{\mathrm{i}k\theta}$.) Let us consider one more example: the first term on the right hand of (2.17) defined for $2n \le r \le 2n+n^2T$, $0 \le \theta \le 2\pi$ and

$-\infty < x_3 < +\infty$; its contribution to an upper bound of $\|\boldsymbol{f}^n\|_q$ is

$$
\begin{aligned}
&\bigg(\int_{2n}^{2n+n^2T} \int_0^{2\pi} \int_{-\infty}^{+\infty} \Big| \frac{\gamma_n \nu}{n^2} \cos\big[\zeta(r-2n)\big]\, \varphi''\Big(\frac{x_3}{n}\Big)\, \mathrm{e}^{\mathrm{i}k\theta} \Big|^q \, r\, \mathrm{d}x_3\, \mathrm{d}\theta\, \mathrm{d}r \bigg)^{1/q} \\
&\quad \leq \; C\, \frac{\gamma_n}{n^2} \bigg(\int_{2n}^{2n+n^2T} r\, \mathrm{d}r \int_{-\infty}^{+\infty} \Big| \varphi''\Big(\frac{x_3}{n}\Big) \Big|^q \, \mathrm{d}x_3 \bigg)^{1/q} \\
&\quad \leq \; C\, \frac{\gamma_n}{n^2}\, n^{5/q}\, .
\end{aligned}
$$

Due to (2.14), the right hand side again tends to zero as $n \to +\infty$. In this way we successively verify (2.20).

Thus we have constructed a sequence $\{\boldsymbol{u}^n\}$ in the unit sphere in $L^q_\sigma(\mathbb{R}^3)$ such that $\|(L_q - \lambda I)\boldsymbol{u}^n\|_q \to \boldsymbol{0}$ as $n \to \infty$. The sequence $\{\boldsymbol{u}^n\}$ is not compact in $L^q_\sigma(\mathbb{R}^3)$ because the intersection of supports of any infinite family of functions chosen from this sequence is empty. This proves that the chosen number $\lambda = \alpha + \mathrm{i}k$ ($\alpha < 0$, $k \in \mathbb{Z}$) is in the essential spectrum of the operator L_q. Due to the closedness of the essential spectrum, each λ of this form belongs to the essential spectrum of L_q even if $\alpha \leq 0$. $\square$

Theorems 2.1 and 2.2 imply that

$$
\sigma(L_q) \;=\; \sigma_{ess}(L_q) \;=\; \Lambda \;:=\; \big\{ \lambda = \alpha + \mathrm{i}k;\ \alpha \in \mathbb{R},\ \alpha \leq 0,\ k \in \mathbb{Z} \big\} \tag{2.21}
$$

where $\sigma(L_q)$ denotes the full spectrum of L_q.

3 The case of an exterior domain $\Omega \subset \mathbb{R}^3$

Theorem 3.1 *Let Ω be an exterior domain in $\mathbb{R}^3$. Then the essential spectrum of the operator L_q, $1 < q < \infty$, has the same form as in the case $\Omega = \mathbb{R}^3$, i.e., $\sigma_{ess}(L_q) = \Lambda$ where the set Λ is defined in* (2.21).

Proof. At first let us show that $\Lambda \subset \sigma_{ess}(L_q)$. Let $\lambda \in \mathbb{C}$ have the form $\lambda = \alpha + \mathrm{i}k$ where $\alpha < 0$ and $k \in \mathbb{Z}$. The supports of the functions $\boldsymbol{u}^n$, constructed in the proof of Theorem 2.2, are subsets of $\{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3;\ \sqrt{x_1^2 + x_2^2} > n\}$ and consequently of Ω for $n \geq n_0$, n_0 sufficiently large. It means that the sequence $\{\boldsymbol{u}^n\}$ (starting from $n = n_0$) is a non–compact sequence in the unit sphere in $L^q_\sigma(\Omega)$ satisfying

$$
\|(L_q - \lambda I)\boldsymbol{u}^n\|_q \;\longrightarrow\; 0 \qquad \text{as } n \to \infty. \tag{3.1}
$$

Consequently, $\lambda \in \sigma_{ess}(L_q)$. The inclusion $\Lambda \subset \sigma_{ess}(L_q)$ now follows from the closedness of $\sigma_{ess}(L_q)$.

Now let us verify the opposite inclusion, i.e. $\sigma_{ess}(L_q) \subset \Lambda$, by contradiction: Suppose that there exists $\lambda \in \sigma_{ess}(L_q)$ such that $\lambda \notin \Lambda$. Then the approximate nullity of the operator $L_q - \lambda I$ equals $+\infty$. Using this information we will construct by mathematical induction a sequence $\{\boldsymbol{u}^n\}$ in $L^q_\sigma(\Omega)$ satisfying $\|\boldsymbol{u}^n\|_q = 1$, (3.1) and

$$
\mathrm{dist}\big(\boldsymbol{u}^{m+1};\, \mathcal{L}_m\big) \;=\; 1 \tag{3.2}
$$

for all $m \in \mathbb{N}$, where $\mathcal{L}_m$ denotes the linear hull of the functions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^m$. Suppose that we have already constructed $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ satisfying $\|(L_q - \lambda I)\boldsymbol{u}^j\|_q \leq 1/j$ for $j = 1, \ldots, n$ and

(3.2) for all $m = 1, \dots, n-1$. To $\epsilon_{n+1} = 1/(n+1)$ there exists an infinite dimensional linear manifold M_{n+1} in $L^q_\sigma(\Omega)$ such that $\|(L_q - \lambda I)\boldsymbol{u}\|_q \le \epsilon_{n+1}$ for all $\boldsymbol{u} \in M_{n+1}$. Then due to Lemma IV.2.3 in [20], we find $\boldsymbol{u}^{n+1} \in M_{n+1}$ such that $\|\boldsymbol{u}^{n+1}\|_q = 1$ and $\operatorname{dist}\big(\boldsymbol{u}^{n+1};\, \mathcal{L}_n\big) = 1$.

The sequence $\{\boldsymbol{u}^n\}$ obviously satisfies

$$\|(L_q - \lambda I)\boldsymbol{u}^n\|_q \;\le\; \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Denote $\boldsymbol{f}^n := (L_q - \lambda I)\boldsymbol{u}^n$. Then there exists $\nabla p^n \in L^q(\Omega)^3$ such that

$$\nu\Delta\boldsymbol{u}^n + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}^n - \boldsymbol{\omega}\times\boldsymbol{u}^n - \lambda\boldsymbol{u}^n - \nabla p^n = \boldsymbol{f}^n, \qquad \operatorname{div}\boldsymbol{u}^n = 0 \tag{3.4}$$

in Ω. Using Theorem 1.1 in [7], see also estimate (1.5) in Section 1, we deduce that

$$\|\nu\nabla^2\boldsymbol{u}^n\|_q + \|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}^n - \boldsymbol{\omega}\times\boldsymbol{u}^n\|_q + \|\nabla p^n\|_q \;\le\; c_1\big(\|\boldsymbol{f}^n\|_q + |\lambda|\big) \;\le\; c_{14} \tag{3.5}$$

where the constant $c_{14} > 0$ does not depend on n. Due to (3.5) and the identity $\|\boldsymbol{u}^n\|_q = 1$, the sequence $\{\boldsymbol{u}^n\}$ is bounded in the space $D(L_q)$. Hence there exists a subsequence which is weakly convergent in $D(L_q)$. In order not to complicate the notation, we shall denote the subsequence again by $\{\boldsymbol{u}^n\}$. The subsequence naturally preserves the property (3.2).

Put $\boldsymbol{v}^n := (\boldsymbol{u}^{n+1} - \boldsymbol{u}^n)/\delta_n$ where $\delta_n = \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\|_q \ge 1$. Then $\{\boldsymbol{v}^n\}$ is a sequence in the unit sphere in $L^q_\sigma(\Omega)$ which converges weakly to zero in $D(L_q)$. Hence $\{\boldsymbol{v}^n\}$ converges strongly to zero in $W^{1,q}(\Omega\cap B_R(\mathbf{0}))^3$ for each $R > 0$. Note that the function $\boldsymbol{v}^n$ satisfies the equation

$$\nu\Delta\boldsymbol{v}^n + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}^n - \boldsymbol{\omega}\times\boldsymbol{v}^n - \lambda\boldsymbol{v}^n - \frac{1}{\delta_n}\,\nabla(p^{n+1} - p^n) \;=\; \frac{1}{\delta_n}\big(\boldsymbol{f}^{n+1} - \boldsymbol{f}^n\big) \tag{3.6}$$

in Ω. This equation, together with the information on the weak convergence of $\{\boldsymbol{v}^n\}$ to zero in $D(L_q)$, implies that the sequence $\{\nabla(p^{n+1} - p^n)\}$ weakly converges to zero in $L^q(\Omega)^3$. Thus, the functions p^n, which are given uniquely up to an additive constant, can be chosen so that $p^{n+1} - p^n \to 0$ strongly in $L^q(\Omega\cap B_R(\mathbf{0}))$ for each $R > 0$.

The sequence $\{\boldsymbol{v}^n\}$ does not contain any subsequence, convergent in $L^q_\sigma(\Omega)$. Let us prove this statement by contradiction: Assume that $\{\boldsymbol{v}^{k_n}\}$ is a convergent subsequence of $\{\boldsymbol{v}^n\}$ in $L^q_\sigma(\Omega)$. This subsequence has the same weak limit as $\{\boldsymbol{v}^n\}$, hence $\boldsymbol{v}^{k_n} \rightharpoonup \mathbf{0}$ in $L^q_\sigma(\Omega)$ as $n \to \infty$. The strong limit in $L^q_\sigma(\Omega)$ cannot differ from the weak limit, hence $\boldsymbol{v}^{k_n} \to \mathbf{0}$ in $L^q_\sigma(\Omega)$ as $n \to +\infty$. However, this is impossible because $\|\boldsymbol{v}^{k_n}\|_q = \|\boldsymbol{u}^{k_n+1} - \boldsymbol{u}^{k_n}\|_q \ge 1$.

Suppose that $R > 0$ is so large that the domain $\{\boldsymbol{x}\in\mathbb{R}^3;\ |\boldsymbol{x}| > R\}$ is a subset of Ω. Let ψ be an infinitely differentiable cut–off function defined in $\mathbb{R}^3$ such that

$$\psi(\boldsymbol{x}) \;=\; \begin{cases} 0 & \text{for} \quad 0 \le |\boldsymbol{x}| \le \frac{5}{4}R, \\ 1 & \text{for} \quad \frac{7}{4}R \le |\boldsymbol{x}|. \end{cases}$$

Put $\boldsymbol{w}^n(\boldsymbol{x}) := \psi(\boldsymbol{x})\,\boldsymbol{v}^n(\boldsymbol{x}) - \boldsymbol{V}^n(\boldsymbol{x})$ where $\boldsymbol{V}^n$ is a function which corrects the divergence of $\psi\boldsymbol{v}^n$ so that $\boldsymbol{w}^n$ is divergence–free, i.e., $\operatorname{div}\boldsymbol{V}^n = \nabla\psi\cdot\boldsymbol{v}^n$. Since the support of $\nabla\psi\cdot\boldsymbol{v}^n$ is contained in $\{\boldsymbol{x}\in\mathbb{R};\ \frac{5}{4}R \le |\boldsymbol{x}| \le \frac{7}{4}R\}$, it follows e.g. from [2] that an appropriate function $\boldsymbol{V}^n$ with support in $\{\boldsymbol{x}\in\mathbb{R};\ R \le |\boldsymbol{x}| \le 2R\}$ can be constructed. Moreover, there exist positive constants c_{15} and c_{16}, independent of n, such that

$$\|\boldsymbol{V}^n\|_{2,q} \;\le\; c_{15}\,\|\nabla\psi\cdot\boldsymbol{v}^n\|_{1,q} \;\le\; c_{16} \qquad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Using equation (3.6), we derive that $\boldsymbol{w}^n$ satisfies

$$
\begin{aligned}
& \nu\Delta\boldsymbol{w}^n + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{w}^n - \boldsymbol{\omega}\times\boldsymbol{w}^n - \lambda\boldsymbol{w}^n - \frac{1}{\delta_n}\,\nabla\big[\psi(p^{n+1}-p^n)\big] \\
= \ & \frac{\psi}{\delta_n}\,(\boldsymbol{f}^{n+1}-\boldsymbol{f}^n) - \frac{1}{\delta_n}\,\nabla\psi\,(p^{n+1}-p^n) + \nu\,(\Delta\psi)\,\boldsymbol{v}^n + 2\nu\,\nabla\psi\cdot\nabla\boldsymbol{v}^n - \nu\Delta\boldsymbol{V}^n \\
& + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot(\nabla\psi\otimes\boldsymbol{v}^n) - (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{V}^n + \boldsymbol{\omega}\times\boldsymbol{V}^n + \lambda\boldsymbol{V}^n. \qquad (3.8)
\end{aligned}
$$

The right hand side converges strongly to zero in $L^q(\Omega)^3$ as $n\to\infty$. (This follows from the strong convergence of $\{\boldsymbol{v}^n\}$ to zero in $W^{1,2}(\Omega\cap B_R(\boldsymbol{0}))^3$, the strong convergence of $\{p^{n+1}-p^n\}$ to zero in $L^q(\Omega\cap B_R(\boldsymbol{0}))$, from (3.7) and from the information on the support of $\nabla\psi$ and $\Delta\psi$.) Hence

$$
\|(L_q-\lambda I)\boldsymbol{w}^n\|_q \longrightarrow 0 \qquad \text{as } n\to\infty. \tag{3.9}
$$

In order to estimate the norm of $\boldsymbol{w}_n$ for large n, let us choose $\epsilon>0$ arbitrarily small. Then for all n sufficiently large

$$
\begin{aligned}
\|\boldsymbol{w}^n\|_q \ & \leq \ \bigg(\int_{|x|<2R}|\psi\boldsymbol{v}^n-\boldsymbol{V}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q} + \bigg(\int_{2R<|x|}|\boldsymbol{v}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q} \ \leq \ \epsilon+1, \\
\|\boldsymbol{w}^n\|_q \ & \geq \ \bigg(\int_{2R<|x|}|\boldsymbol{v}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q} \\
& \geq \ \bigg(\int_{\Omega}|\boldsymbol{v}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q} - \bigg(\int_{|x|<2R}|\boldsymbol{v}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q} \\
& \geq \ 1-\epsilon\,.
\end{aligned}
$$

Let us now normalize the sequence $\{\boldsymbol{w}^n\}$ by dividing each of the functions $\boldsymbol{w}^n$ by its norm in $L^q_\sigma(\Omega)$. In order to preserve a simple notation, let us denote the normalized functions again by $\boldsymbol{w}^n$. If we finally put $\boldsymbol{w}^n(\boldsymbol{x})=\boldsymbol{0}$ for $\boldsymbol{x}\in\mathbb{R}^3-\Omega$, we obtain a sequence in the unit sphere in $L^q_\sigma(\mathbb{R}^3)$, which does not contain any subsequence converging in $L^q_\sigma(\mathbb{R}^3)$, satisfying (3.9) with $\|\cdot\|_q$ being the norm in $L^q(\mathbb{R}^3)^3$. The existence of the sequence $\{\boldsymbol{w}^n\}$ with all these properties implies that λ belongs to the essential spectrum of the operator L_q, considered as an operator in $L^q_\sigma(\mathbb{R}^3)$. However, we know from Section 3 that this essential spectrum coincides with the set Λ, see (2.21). Hence $\lambda\in\Lambda$, which is the desired contradiction.

The theorem is proved. □

Remark 3.1 Note that in the case $\Omega=\mathbb{R}^3$ the identity (2.21) describes not only the essential spectrum of the operator L_q, but also its full spectrum. However, we have not proved the same proposition in the case of a general exterior domain Ω: Theorem 3.1 describes the essential spectrum of L_q and not its whole spectrum. The reason is that we can construct a solution of the resolvent equation (2.1) in $\Omega=\mathbb{R}^3$ by means of the Fourier transform which cannot be used in the case of a general exterior domain Ω.

Remark 3.2 We remind that the body in $\mathbb{R}^3$ has so far been supposed to rotate with the angular velocity $\boldsymbol{\omega}=(0,0,1)$. If the angular velocity $\boldsymbol{\omega}$ equals the vector $(0,0,\omega)$ where ω is a non–zero scalar constant, then we can prove in the same way that

$$
\sigma_{ess}(L_q) \ = \ \big\{\lambda=\alpha+\mathrm{i}k\omega;\ \alpha\in\mathbb{R},\ \alpha\leq 0,\ k\in\mathbb{Z}\big\}\,; \tag{3.10}
$$

this identity holds both in the case of the whole space $\mathbb{R}^3$ and in the case of an exterior domain Ω.

Acknowledgement: The research was supported by the Czech Academy of Sciences (CAS), Institutional Research Plan No. AV0Z10190503 (authors 2 and 3), by the Grant Agency of CAS (author 2, grant No. IAA10019505), by the Grant Agency of the Czech Republic (author 3, grant No. 201/05/0005) and by the joint research project of DAAD (No. D/04/25763) and CAS (No. D–CZ 3/05-06).

References

[1] Borchers, W., Zur Stabilität und Faktorisierungsmethode für die Navier–Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten. Habilitation Thesis, Univ. of Paderborn 1992.

[2] Borchers, W., Sohr, H., On the equations $\mathbf{rot}\, \boldsymbol{v} = \boldsymbol{g}$ and $\operatorname{div} \boldsymbol{v} = f$ with zero boundary conditions. *Hokkaido Math. J.* 19, 1990, 67–87.

[3] Brenner, H., The Stokes resistance of an arbitrary particle II. *Chem. Eng. Sci.* 19 (1959), 599–624.

[4] Dintelmann, E., Geissert, M., Hieber, M., Strong L^p solutions to the Navier–Stokes flow past moving obstacles: the case of several obstacles and time dependent velocity. TU Darmstadt, Fachbereich Mathematik, Preprint 2006.

[5] Farwig, R., An L^q-analysis of viscous fluid flow past a rotating obstacle. *Tôhoku Math. J.* 58 (2005), 129–147.

[6] Farwig, R., Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle. *Banach Center Publications* 70, Warsaw 2005, 73–84.

[7] Farwig, R., Hishida, T., Müller, D., L^q-Theory of a singular "winding" integral operator arising from fluid dynamics. *Pacific J. Math.* 215 (2004), 297–312.

[8] Farwig, R., Krbec, M., Nečasová, Š., A weighted L^q–approach to Stokes flow around a rotating body. TU Darmstadt, Fachbereich Mathematik, Preprint No. 2422, 2005.

[9] Farwig, R., Neustupa, J., On the spectrum of a Stokes-type operator arising from flow around a rotating body. TU Darmstadt, Fachbereich Mathematik, Preprint No. 2423, 2005.

[10] Galdi, G. P., *An introduction to the mathematical theory of the Navier-Stokes equations: Linearised steady problems.* Springer Tracts in Natural Philosophy, Vol. 38, 2nd edition, Springer 1998.

[11] Galdi, G. P., On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. *Handbook of Mathematical Fluid Dynamics,* Vol. 1, Ed. by S. Friedlander, D. Serre, Elsevier 2002.

[12] Galdi, G. P., Steady flow of a Navier-Stokes fluid around a rotating obstacle. *J. Elasticity* 71 (2003), 1–31.

[13] Galdi, G. P., Silvestre, A. L., Strong solutions to the Navier–Stokes equations around a rotating obstacle, *Arch. Rational Mech. Anal.* 176 (2005), 331–350

[14] Galdi, G. P., Silvestre, A. L., On the stationary motion of a Navier–Stokes liquid around a rigid body, University of Pittsburgh, preprint, 2005.

[15] Geissert, M., Heck, H., Hieber, M., L^p–theory of the Navier–Stokes flow in the exterior of a moving or rotating obstacle. TU Darmstadt, Fachbereich Mathematik, Preprint No. 2367, 2005.

[16] Gunther, R. B., Hudspeth, R. T., Thomann, E. A., Hydrodynamic flows on submerged rigid bodies – steady flow. *J. Math. Fluid Mech.* 4 (2002), 187–202.

[17] Hishida, T., L^q estimates of weak solutions to the stationary Stokes equations around a rotating body. Hokkaido Univ. Preprint Series in Math., No. 691, 2004. To appear in *J. Math. Soc. Japan.*

[18] Hishida, T., An existence theorem for the Navier–Stokes flow in the exterior of a rotating obstacle. *Arch. Rational Mech. Anal.* 150, 307–348 (1999).

[19] Hishida, T., Shibata, Y., *$L_p - L_q$ estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle.* Preprint.

[20] Kato, T., *Perturbation Theory for Linear Operators.* Springer–Verlag, Berlin–Heidelberg–New York 1966.

[21] Kirchhoff, G., Über die Bewegung eines Rotationskörpers in einer Flüssigkeit. *Crelle J.* 71 (1869), 237–281.

[22] Kračmar, S., Nečasová, Š., Penel, P., Estimates of weak solutions in anisotropically weighted Sobolev spaces to the stationary rotating Oseen equations. *IASME Transactions* 6,2 (2005), 854–861.

[23] Kračmar, S., Nečasová, Š., Penel, P., On the weak solution to the Oseen–type problem arising from flow around a rotating rigid body in the whole space. *WSEAS Transactions of Mathematics* 3,5 (2006), 243–251.

[24] San Martín, J. A., Starovoitov, V., Tucsnak, M., Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.* 161 (2002), 113–147.

[25] Nečasová, Š., On the problem of the Stokes flow and Oseen flow in $\mathbb{R}^3$ with Coriolis force arising from fluid dynamics. *IASME Transactions* 2 (2005), 1262–1270.

[26] Nečasová, Š., Asymptotic properties of the steady fall of a body in viscous fluids. *Math. Meth. Appl. Sci.* 27 (2004), 1969–1995.

[27] Serre, D., Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Jap. J. Appl. Math.* 4 (1987), 99–110.

[28] Thomson, W. (Lord Kelvin), *Mathematical and Physical Papers,* Vol. 4. Cambridge University Press 1982.

[29] Weinberger, H. F., On the steady fall of a body in a Navier-Stokes fluid. *Proc. Symp. Pure Mathematics* 23 (1973), 421–440.

[30] Weinberger, H. F., Variational properties of steady fall in Stokes flow. *J. Fluid Mech.* 52 (1972), 321–344.