On singular limits for the full Navier-Stokes-Fourier system

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1 Introduction

Scale analysis, in combination with rigorous estimates of observed data, has become an important tool in the study of complex systems of equations arising in mathematical fluid mechanics. Many textbooks as well as research monographs explain how scaling arguments lead to simplified systems of equations that capture the essential piece of information on a particular fluid flow suppressing unimportant phenomena. These systems arise because of a singularity in the governing equations related to the flow regime in question. As a result of huge scale differences in atmospheric flows, such an approach has become of particular relevance both on the theoretical level and in numerical simulations of models arising in meteorology (see the survey paper by Klein et al. [26]).

Further discussion in the present paper is based on the full **Navier-Stokes-Fourier system** of equations governing the time evolution of the *density* $\varrho = \varrho(t,x)$, the *velocity* $\mathbf{u} = \mathbf{u}(t,x)$, and the *temperature* $\vartheta = \vartheta(t,x)$ of a compressible, viscous, and heat conducting fluid:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho \nabla_x F, \tag{1.2}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \sigma, \tag{1.3}$$

$$\frac{d}{\mathrm{d}t}\int_\Omega \Big(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho F\Big)\ \mathrm{d}x = 0, \tag{1.4}$$

where the *pressure* p, the *specific entropy* s, and the *specific internal energy* e are interrelated through **Gibbs' equation**

$$\vartheta Ds = De + pD\Big(\frac{1}{\varrho}\Big). \tag{1.5}$$

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The *viscous stress tensor* $\mathbb{S}$ is given by classical **Newton's rheological law**

$$\mathbb{S} = \mu(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\ \mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) + \eta(\vartheta)\ \mathrm{div}_x \mathbf{u}\ \mathbb{I} \tag{1.6}$$

while the *heat flux* $\mathbf{q}$ is determined through **Fourier's law**

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta. \tag{1.7}$$

The *entropy production rate* σ satisfies

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathrm{Ma}^2 \mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x \vartheta|^2\Big), \tag{1.8}$$

and the symbol $\nabla_x F = \nabla_x F(x)$ denotes a given potential driving force.

Problem (1.1 - 1.8) will be supplemented with *conservative* boundary conditions specified below, compatible with the total energy balance expressed through (1.4). One can check that (1.8) is *equivalent* to that standard relation

$$\sigma = \frac{1}{\vartheta}\Big(\mathrm{Ma}^2 \mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x \vartheta|^2\Big) \tag{1.9}$$

provided all quantities in (1.1 - 1.4) are sufficiently smooth (see [16]). On the other hand, it is well-known that the physically admissible *weak solutions* of the *inviscid system* do dissipate mechanical energy even though no viscosity is explicitly present in the equations. Although this is probably less likely to happen in the viscous case, it is still an outstanding open problem whether or not (1.9) holds even if the fluid is incompressible (see the classical work of Leray [28], the relevant comments by Galdi [22], or Nagasawa for the most recent results [36]).

The symbols Ma and Fr stand for dimensionless parameters called *Mach number* and *Froude number*, respectively. The main objective of the present paper is to review some recent results on the asymptotic behaviour of solutions to system (1.1 - 1.8) in the regime when

$$\mathrm{Ma} \to 0,\ \ \mathrm{Fr} \to 0.$$

The "incompressible limit" $\mathrm{Ma} \to 0$ for various systems arising in mathematical fluid dynamics was studied in the seminal work by Klainerman and Majda [25] (see also Ebin [12]). One can distinguish two kinds of qualitatively different results based on different techniques. The first approach applies to *strong solutions* defined on possibly short time intervals, the length of which, however, is independent of the value of the parameter $\mathrm{Ma} \to 0$. In this framework, the most recent achievements for system (1.1 - 1.8) can be found in the papers by Alazard [2], [1] (for earlier results see the survey papers by Danchin [8], Métivier and Schochet [34], Schochet [38], and the references cited therein). The second group of results is based on a global-in-time existence theory for the *weak solutions* of the underlying primitive system of equations, asserting convergence towards solutions of the target system on an arbitrary time interval. Results of this type for the isentropic Navier-Stokes system have been obtained by Lions and Masmoudi [30], [31], and later extended by Desjardins et al. [9], Bresch et al. [5]. Our main aim here is to present a similar theory that applies to solutions of the complete Navier-Stokes-Fourier system.

The paper is organized as follows. In Section 2, we summarize all hypotheses concerning the structural properties of the nonlinear quantities appearing in the constitutive relations, and recall the underlying *existence theory* for system (1.1 - 1.8) supplemented with a suitable set of boundary and initial conditions. The convergence results for $\mathrm{Ma} \to 0$, $\mathrm{Fr} \to 0$ are established in Section 3. In particular, we shall see that for $\mathrm{Ma} \approx \varepsilon$, $\mathrm{Fr} \approx \sqrt{\varepsilon}$ the corresponding solutions of (1.1 - 1.8) tend to a solution of the Oberbeck-Boussinesq approximation. Results in the regime $\mathrm{Ma} \approx \mathrm{Fr} \approx \varepsilon$ for a reduced (barotropic) system as well as some open problems are sketched in Section 4.

2 Existence theory

2.1 Hypotheses

Modeling of fluid flows gives rise to a rich variety of mathematical problems with applications in many fields ranging from engineering to astrophysics. Motivated by the existence theory developed in [16], [19], we shall assume that the material properties of the fluid we shall deal with can be characterized through the following list of hypotheses:

- The fluid is *linearly viscous*, that means, the vicous stress tensor $\mathbb{S}$ is given by formula (1.6), where the viscosity coefficients μ and η are continuously differentiable functions of the absolute temperature ϑ satisfying

$$0 < \underline{\mu}(1 + \vartheta^{\beta}) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^3), \ 0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^3), \ \beta > \frac{2}{5}. \tag{2.1}$$

  Note that hypothesis (2.1) includes the physically relevant value $\beta = \frac{1}{2}$ (see Becker [4]).

- The fluid is a *heat conductor*, the heat flux $\mathbf{q}$ is given by Fourier's law (1.7), where the heat conductivity coefficient κ is a continuously differentiable function of the absolute temperature ϑ such that

$$0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3). \tag{2.2}$$

  Hypothesis (2.2) takes into account the effect of radiation relevant in the high temperature regimes (see Becker[4], Buet and Després [6], Zel'dovich and Raizer [41]).

- The fluid behaves like a real *monoatomic gas*. By this we mean that the pressure p can be written in the form

$$p(\varrho, \vartheta) = p_F(\varrho, \vartheta) + p_R(\vartheta), \ p_F(\varrho, \vartheta) = p_G(\varrho, \vartheta) + p_E(\varrho, \vartheta),$$

  where p_G is the classical molecular pressure, p_E denotes the pressure of the electron gas dominating in the high density (degenerate) region, and p_R is the radiation pressure due to the gas of emitted photons. Similarly, the specific internal energy e can be decomposed as

$$e(\varrho, \vartheta) = e_F(\varrho, \vartheta) + e_R(\varrho, \vartheta),$$

  where the component e_F is related to the pressure p_F through the state equation

$$p_F(\varrho,\vartheta) = \frac{2}{3}\varrho e_F(\varrho,\vartheta) \tag{2.3}$$

while

$$p_R = \frac{a}{3}\vartheta^4,\ \varrho e_R = a\vartheta^4,\ a > 0. \tag{2.4}$$

In addition, the electrons are supposed to form a Fermi gas, in particular, the internal energy remains strictly positive in the vanishing temperature limit:

$$\liminf_{\vartheta\to 0+} e_F(\varrho,\vartheta) > 0 \text{ for any given } \varrho > 0 \tag{2.5}$$

(see Chapters 1, 15 in Eliezer et al. [14], Chapter 4 in Mueller and Ruggeri [35], Gallavoti [23], among others).

Furthermore, we assume the standard *thermodynamics stability hypothesis*:

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0,\ c_v(\varrho,\vartheta) = \frac{\partial e_F(\varrho,\vartheta)}{\partial\vartheta} > 0, \tag{2.6}$$

together with a technical but physically relevant restriction

$$\limsup_{\vartheta\to 0+} c_v(\varrho,\vartheta) < \infty \text{ for any fixed } \varrho > 0 \tag{2.7}$$

(see Bechtel et al. [3]).

2.2 Boundary conditions

In the geometrically simplest but still physically relevant situation, the spatial domain Ω can be taken a horizontal slab bounded above and below by two lateral surfaces Γ_T and Γ_B, respectively. All physical quantities are supposed to be periodic with respect to (x_1, x_2), and, accordingly, one can identify

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ \Phi_B(x_1, x_2) < x_3 < \Phi_B(x_1, x_2)\}, \tag{2.8}$$

where $\mathcal{T}^2 = ((-\pi,\pi)|_{\{-\pi,\pi\}})^2$ is a two-dimensional torus, and Φ_B, Φ_T are scalar functions defined on $\mathcal{T}^2$. Accordingly, we set

$$\Gamma_B = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = \Phi_B(x_1, x_2)\}, \tag{2.9}$$

and

$$\Gamma_T = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = \Phi_T(x_1, x_2)\}. \tag{2.10}$$

The problem is supplemented with suitable boundary conditions on the lateral boundaries Γ_B, Γ_T in order to meet the total energy conservation principle expressed through (1.4). Typically, we can take the **no-flux boundary conditions** for the velocity field and the heat flux

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_B \cup \Gamma_T, \tag{2.11}$$

$$\mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \Gamma_B \cup \Gamma_T, \tag{2.12}$$

where $\mathbf{n}$ denotes the outer normal vector, supplemented either with the **Navier complete slip condition**

$$\mathbb{S}\mathbf{n} \times \mathbf{n} = 0 \text{ on } \Gamma_B \cup \Gamma_T, \tag{2.13}$$

or the standard **no-slip condition**

$$\mathbf{u} = 0 \text{ on } \Gamma_B \cup \Gamma_T. \tag{2.14}$$

2.3 Variational solutions

Definition 2.1

We say that a trio $\{\varrho, \mathbf{u}, \vartheta\}$ *is a* **variational solution** *of Navier-Stokes-Fourier system (1.1 - 1.8) on* $(0,T) \times \Omega$ *satisfying the boundary conditions (2.11) , (2.12), (2.13) (or (2.14)), together with the initial conditions*

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \ \vartheta(0,\cdot) = \vartheta_0$$

provided the following conditions hold:

- *The density* ϱ *is a non-negative function,* $\varrho \in L^\infty(0,T;L^{\frac{5}{3}}(\Omega))$*, the momentum* $\varrho\mathbf{u}$ *belongs to the space* $L^\infty(0,T;L^q(\Omega;R^3))$ *for a certain* $q > 1$*, and the integral identity*

$$\int_0^T \int_\Omega \Big(\varrho B(\varrho)\partial_t \varphi + \varrho B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - b(\varrho) \ \mathrm{div}_x \mathbf{u} \ \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = \tag{2.15}$$
$$- \int_\Omega \varrho_0 B(\varrho_0) \varphi(0,\cdot) \ \mathrm{d}x$$

  is satisfied for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}))$*, and any* b *such that*

$$b \in L^\infty \cap C[0,\infty), \ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2} \ \mathrm{d}z. \tag{2.16}$$

- *The velocity field* $\mathbf{u}$ *belongs to* $L^2(0,T;W^{1,p}(\Omega;R^3))$ *for a certain* $p > 1$ *and satisfies the boundary conditions (2.11) (or (2.14)) in the sense of traces, the absolute temperature* ϑ *is positive a.a. on the set* $(0,T) \times \Omega$*,*

$$\vartheta^\nu \in L^2(0,T;W^{1,2}(\Omega)) \text{ for all } \nu \in [1, \frac{3}{2}],$$

  and the integral identity

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u} \cdot \partial_t \varphi + (\varrho\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + \frac{1}{\mathrm{Ma}^2} p(\varrho, \vartheta) \ \mathrm{div}_x \varphi \Big) \ \mathrm{d}x\mathrm{d}t = \tag{2.17}$$
$$\int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

  holds for any test function $\varphi \in \mathcal{D}([0,T) \times \Omega; R^3)$*,* $\varphi \cdot \mathbf{n} = 0$ *on* $\partial\Omega$ *(or* $\varphi = 0$ *on* $\partial\Omega$ *if we impose (2.14) instead of (2.13)), where the viscous stress tensor* $\mathbb{S}$ *is given through (1.6). Here, we also tacitly assume that all quantities in (2.17) are integrable.*

- *The entropy* $\varrho s(\varrho,\vartheta)$ *belongs to the space* $L^\infty(0,T;L^1(\Omega))$*, the terms* $\varrho s(\varrho,\vartheta)\mathbf{u}$*,* $\frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta$ *are integrable on the set* $(0,T)\times\Omega$*,* $\log(\vartheta)\in L^2(0,T;W^{1,2}(\Omega))$*, and the integral inequality*

$$\int_0^T\int_\Omega\Big(\varrho s(\varrho,\vartheta)\partial_t\varphi+\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi-\frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t+ \tag{2.18}$$

$$\int_0^T\int_\Omega\frac{1}{\vartheta}\Big(\mathrm{Ma}^2\ \mathbb{S}:\nabla_x\mathbf{u}+\frac{\kappa(\vartheta)}{\vartheta}|\nabla_x\vartheta|^2\Big)\varphi\ \mathrm{d}x\ \mathrm{d}t\le-\int_\Omega\varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x$$

holds for any non-negative test function $\varphi\in\mathcal{D}([0,T)\times\overline{\Omega})$*.*

- *The total energy balance*

$$E(\tau)=\int_\Omega\Big(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)+\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho F\Big)\ \mathrm{d}x= \tag{2.19}$$

$$\int_\Omega\Big(\frac{\mathrm{Ma}^2}{2}\varrho_0|\mathbf{u}_0|^2+\varrho_0 e(\varrho_0,\vartheta_0)+\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho_0 F\Big)\ \mathrm{d}x=E_0$$

holds for a.a. $\tau\in(0,T)$*.*

Relation (2.15) says that ϱ, $\mathbf{u}$ satisfy equation of continuity (1.1) in the sense of renormalized solutions introduced by DiPerna and Lions [10]. As already pointed out in Section 1, an essential ingredience of this concept of weak solutions is replacing the entropy balance by inequality (2.18) equivalent to (1.8) anticipating possible singularities concentrated on sets of zero Lebesgue measure (for relevant dicussion see [16], [19]). However, as shown in [19], both formulations (classical and variational) are entirely equivalent provided the variational solutions are sufficiently smooth.

A relevant existence theory of variational solutions of system (1.1 - 1.8) when $\Omega\subset R^3$ is a bounded regular domain was developed in [19, Theorem 2.4] (see also [16] for the necessary modifications in order to accommodate the growth conditions (2.1), and [15] for general framework). The changes to handle the case of spatial domains given through (2.8) are straightforward. Thus we report the following existence result.

Theorem 2.1 *Assume that* Ω *is given by (2.8), where* Φ_B*,* $\Phi_T\in C^{2+\nu}(\mathcal{T}^2)$*,* $\Phi_B<\Phi_T$*. Let the initial data satisfy*

$$\varrho_0,\ \vartheta_0\in L^\infty(\Omega),\ \mathbf{u}_0\in L^\infty(\Omega;R^3),\ \mathrm{ess}\inf_{x\in\mathcal{T}^2}\varrho_0>0,\ \mathrm{ess}\inf_{x\in\mathcal{T}^2}\vartheta_0>0. \tag{2.20}$$

Furthermore, suppose that p*,* s*, and* e *are interrelated through Gibbs' equation (1.5), and that hypotheses (2.3 - 2.7) hold. Let* $\mathbb{S}$*,* $\mathbf{q}$ *be given by (1.6), (1.7), respectively, where the transport coefficients satisfy (2.1), (2.2).*

Then problem (1.1 - 1.8) with the boundary conditions (2.11), (2.12), (2.13) (or (2.14)) admits a variational solution ϱ*,* ϑ*,* $\mathbf{u}$ *on the set* $(0,T)\times\Omega$ *in the sense of Definition 2.1.*

Hypothesis (2.20) is not optimal. As a matter of fact, the same result can be proved for any initial data with finite total energy and a non-negative density and temperature distribution (see [15, Chapter 7]).

3 Singular limits

3.1 The Oberbeck-Boussinesq system

We start our discussion considering a very simple geometry of the underlying physical space, namely we assume that Γ_B, Γ_T are flat. Accordingly, we can set

$$\Phi_B \equiv 0, \ \Phi_T \equiv \pi. \tag{3.1}$$

As already pointed out in the introductory part, simplified asymptotic limits derived trough scale analysis yield often a useful insight into the behaviour of more complex systems arising in mathematical fluid dynamics. A typical example is the flow of a heat conducting Newtonian fluid that can be described in the framework of the **Oberbeck-Boussinesq approximation**:

$$\left.\begin{aligned} \mathrm{div}_x \mathbf{U} = 0, \\ \overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\otimes\mathbf{U})\Big) + \nabla_x P = \mathrm{div}_x\, \mathbb{S} - r\mathbf{j}, \\ \overline{\varrho}\ \overline{c}_p\Big(\partial_t \Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big) - \mathrm{div}_x(\overline{\kappa}\nabla_x\Theta) = 0, \end{aligned}\right\} \tag{3.2}$$

where $\mathbf{U}(t,x)$ is the velocity at time $t \in (0,T)$ and position $x \in \Omega$, $\Theta(t,x)$ is the temperature, the symbol P denotes the normal stress (pressure), and $\mathbf{j} = [0,0,1]$ is the unit vector in the x_3 direction. Similarly to the above, the viscous stress tensor $\mathbb{S}$ is given through Newton's rheological law

$$\mathbb{S} = \overline{\mu}\Big(\nabla_x \mathbf{u} + \nabla_x^{\perp}\mathbf{u}\Big), \tag{3.3}$$

where the viscosity coefficient $\overline{\mu}$ as well as the heat conductivity coefficient $\overline{\kappa}$, and the specific heat at constant pressure $\overline{c}_p$ are evaluated at constant density $\overline{\varrho}$ and constant temperature $\overline{\vartheta} = \frac{1}{|\Omega|}\int_\Omega \Theta \ \mathrm{d}x$. Consistent with the Boussinesq approximation, the temperature-dependent density $r = r(\Theta)$ appears only in the momentum equation and is assumed to vary with temperature as

$$r + \overline{\varrho}\ \overline{\alpha}(\Theta - \overline{\vartheta}) = 0, \tag{3.4}$$

where $\overline{\alpha}$ stands for the coefficient of thermal expansion (see Zeytounian [42] or Rajagopal et al. [37] for more details on the physical background of the problem).

Consistently with Section 2, we consider the periodic boundary conditions with respect to the spatial coordinates x_1, x_2, together with the conservative boundary conditions

$$\mathbf{u}\cdot\mathbf{j} = 0, \ (\mathbb{S}\mathbf{j})\times\mathbf{j} = 0, \ \nabla_x\Theta\cdot\mathbf{j} = F_b \text{ on the lateral boundary} \Gamma_B \cup \Gamma_T. \tag{3.5}$$

Under these circumstances, the total amount of thermal energy

$$E_{\rm th} = \int_\Omega \Theta \ {\rm d}x = \overline{\vartheta}|\Omega| \ \text{is a constant of motion.}$$

Following [18] we shall show that the system of equations (3.2 - 3.4) supplemented with the boundary conditions (3.5) can be obtained as an asymptotic limit of the full Navier-Stokes-Fourier system (1.1 - 1.8) provided the Mach and Froude numbers tend to zero. Note that the Oberbeck-Boussinesq approximation is used when the density is nearly constant but the density differences exist due to temperature changes, causing an imbalance of the hydrostatic equilibrium. Such a situation occurs with many convection problems where the temperature differences are introduced independent of the flow dynamics. The requirement that the Mach number tends to zero is needed, allowing the density ϱ to approach a constant $\overline{\varrho}$ except in the gravitational body force rescaled by a suitable choice of the Froude number. Accordingly, the temperature differences are not caused by the flow but exist independent of the flow.

In order to be more specific, we introduce a small paremeter $\varepsilon > 0$ and set

$$\text{Ma} = \varepsilon, \ \text{Fr} = \sqrt{\varepsilon}$$

in Navier-Stokes-Fourier system (1.1 - 1.8). Furthermore, we take the initial conditions in the form

$$\varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0}, \ \mathbf{u}_0 = \mathbf{u}_{\varepsilon,0}, \ \vartheta_{\varepsilon,0} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{\varepsilon,0} \tag{3.6}$$

where

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_{\varepsilon,0} \ {\rm d}x > 0, \ \overline{\vartheta} = \frac{1}{|\Omega|} \int_\Omega \vartheta_{\varepsilon,0} \ {\rm d}x > 0, \tag{3.7}$$

with the quantities $\varrho^{(1)}_{\varepsilon,0}$, $\mathbf{u}_{\varepsilon,0}$, $\vartheta^{(1)}_{\varepsilon,0}$ bounded uniformly with respect to $\varepsilon \to 0$.

Relation (3.6) reveals a crucial aspect of the problem: If we desire to recover the Oberbeck-Boussinesq system (3.2 - 3.4) as the asymptotic limit of the complete system (1.1 - 1.8), then we have to deal with the so-called ill-prepared initial data, that means, the functions $\varrho^{(1)}_{\varepsilon,0}$, $\vartheta^{(1)}_{\varepsilon,0}$ *must not vanish* in the asymptotic limit for $\varepsilon \to 0$. In particular, the solutions develop high frequency acoustic waves considered "harmless" in the asymptotic limit but still producing large amplitude velocity field oscillations in the original system (cf. the survey paper by Schochet [38]). In other words, unless we are satisfied with local solutions existing only on a possibly very short time interval (see Alazard [2] or Danchin [8] for relevant results and techniques), we have to consider global-in-time large data solutions of the full Navier-Stokes-Fourier system, the existence of which was stated in Theorem 2.1.

If the spatial domain is flat, specifically if Φ_B, Φ_T are constant as in (3.1), solutions of (1.1 - 1.8) are invariant with respect to the symmetry transformations:

$$\left\{ \begin{array}{c} \varrho(t, x_1, x_2, -x_3) = \varrho(t, x_1, x_2, x_3), \\[2mm] \vartheta(t, x_1, x_2, -x_3) = \vartheta(t, x_1, x_2, x_3), \\[2mm] u_1(t, x_1, x_2, -x_3) = u_1(t, x_1, x_2, x_3), \ u_2(t, x_1, x_2, -x_3) = u_2(t, x_1, x_2, x_3), \\[2mm] u_3(t, x_1, x_2, -x_3) = -u_3(t, x_1, x_2, x_3). \end{array} \right\} \tag{3.8}$$

Accordingly, the boundary conditions (2.11 - 2.13) can be conveniently recast in terms of the additional symmetry properties specified in (3.8) provided all quantities are considered periodic also in x_3-variable, that means, one can identify

$$\Omega \equiv \mathcal{T}^3 = \Big([-\pi, \pi]|_{\{-\pi,\pi\}} \Big)^3,$$

with the potential $F = |x_3|$ in (1.3) (cf. Ebin [13]).

In order to collect all the preliminary material, we introduce a concept of variational solutions to the Oberbeck-Boussinesq system.

Definition 3.1 *We shall say that functions* $\{r, \mathbf{U}, \Theta\}$ *represent a* **variational solution** *of system (3.2 - 3.4), supplemented with the boundary conditions (3.5) and the initial conditions*

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0,\ \Theta(0,\cdot) = \Theta_0, \tag{3.9}$$

if the following conditions are met:

- $$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3)),\ r \in L^\infty(0,T;L^2(\Omega)),$$

  $$\mathrm{div}_x \mathbf{U} = 0 \ \textit{a.a. on}\ (0,T)\times\Omega,\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

  and the integral identity

  $$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi \Big)\ \mathrm{d}x\mathrm{d}t = \tag{3.10}$$

  $$\int_0^T \int_\Omega \Big(\overline{\mu}(\nabla_x\mathbf{U} + \nabla_x^\perp\mathbf{U}) : \nabla_x\varphi + r\nabla_x x_3\cdot\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{U}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

  holds for any test function

  $$\varphi \in \mathcal{D}([0,T)\times\overline{\Omega};R^3),\ \mathrm{div}_x\ \varphi = 0,\ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0;$$

- $$\Theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),$$

  $$r + \overline{\varrho}\ \overline{\alpha}\Big(\Theta - \frac{1}{|\Omega|}\int_\Omega \Theta\ \mathrm{d}x\Big) = 0\ \textit{a.a. on}\ (0,T)\times\Omega, \tag{3.11}$$

  and the integral identity

  $$\int_0^T \int_\Omega \Big(\overline{\varrho}\ \overline{c}_p(\Theta\ \partial_t\varphi + \Theta\mathbf{U}\cdot\nabla_x\varphi) - \overline{\kappa}\nabla_x\Theta\cdot\nabla_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t = \tag{3.12}$$

  $$\overline{\kappa}\Big(\int_{\{x_3=0\}} F_b\ \varphi\ \mathrm{d}\sigma - \int_{\{x_3=\pi\}} F_b\ \varphi\ \mathrm{d}\sigma \Big) - \int_\Omega \overline{\varrho}\ \overline{c}_p\Theta_0\varphi(0,\cdot)\ \mathrm{d}x$$

  is satisfied for any test function $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$.

We are ready to formulate the main result to be discussed in this section.

Theorem 3.1 *In addition to the hypotheses of Theorem 2.1, assume that Ω is flat, that means, the functions Φ_B, Φ_T appearing (2.8) satisfy (3.1). Furthermore, suppose that $\beta = 1$ in (2.1), and $F = |x_3|$ in (1.2). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of variational solutions to system (1.1 - 1.8) supplemented with the boundary conditions (2.11 - 2.13) in the sense of Definition 2.1, with*

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}, \ a = \varepsilon.$$

Assume the the solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ emanates from the initial state

$$\varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0}, \ \mathbf{u}_{\varepsilon,0}, \ \vartheta_{\varepsilon,0} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{\varepsilon,0}$$

satisfying (3.7), where

$$\varrho^{(1)}_{\varepsilon,0} \to \varrho^{(1)}_0, \ \mathbf{u}_{\varepsilon,0} \to \mathbf{u}_0, \ \vartheta^{(1)}_{\varepsilon,0} \to \vartheta^{(1)}_0 \ \textit{weakly-(*) in } L^\infty(\Omega).$$

Then

$$\varrho_\varepsilon \to \overline{\varrho} \ \textit{in } C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{\frac{5}{3}}(\Omega)),$$
$$\vartheta_\varepsilon \to \overline{\vartheta} \ \textit{in } L^2(0,T; W^{1,2}(\Omega)),$$

and, passing to a subsequence if necessary,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{3.13}$$
$$\varrho^{(1)}_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \ \textit{weakly-(*) in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)),$$
$$\vartheta^{(1)}_\varepsilon = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \ \textit{weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

where the trio

$$\mathbf{U}, \ r = \varrho^{(1)} + \overline{\varrho}\Big(\frac{1}{\partial_\varrho p_F(\overline{\varrho}, \overline{\vartheta})} - \frac{\overline{\vartheta} \ \overline{\alpha}^2}{\overline{c}_p}\Big)(x_3 - \frac{\pi}{2}),$$
$$\Theta = \overline{\vartheta} + \vartheta^{(1)} + \frac{\overline{\vartheta} \ \overline{\alpha}}{\overline{c}_p}(x_3 - \frac{\pi}{2})$$

represents a (weak) solution of system (3.2 - 3.5) in the sense of Definition 3.1, with the initial data

$$\mathbf{U}_0 = \mathbf{H}[\mathbf{u}_0], \ \Theta_0 = \overline{\vartheta} + \frac{\overline{c}_v}{\overline{c}_p}\vartheta^{(1)}_0 - \frac{2}{3}\frac{\overline{\vartheta} \ \overline{c}_v}{\overline{\varrho} \ \overline{c}_p}\varrho^{(1)}_0,$$

and the heat flux through the boundary given by

$$F_b = \frac{\overline{\vartheta} \ \overline{\alpha}}{\overline{c}_p} \ \textit{on the lateral boundary } \Gamma_B \cup \Gamma_T.$$

Here the symbol $\mathbf{H}$ stands for the Helmholtz projection onto the space of divergenceless vector fields.

Remark 1.1 As already observed, one can get rid of the boundary conditions (3.5) extending all relevant quantities as periodic functions defined on the torus $\mathcal{T}^3$ in such a way that they belong to the symmetry class specified in (3.8). Accordingly, the Helmholtz projection can be defined through formula

$$\mathbf{H}[\mathbf{v}] = \mathbf{v} - \nabla_x \Delta^{-1}[\mathrm{div}_x\ \mathbf{v}],\ \mathbf{H}^{\perp} = \nabla_x \Delta^{-1}[\mathrm{div}_x\ \mathbf{v}] \text{ for } \mathbf{v} \in L^2(\mathcal{T}^3),$$

where the symbol Δ stands for the Laplace operator considered on the space of spatially periodic functions with zero mean. Introducing the Fourier coefficients

$$[v]_{\mathbf{k}} = \frac{1}{2\pi} < v, \exp(-\mathrm{i}\mathbf{k}\cdot x) >,\ \mathbf{k} \in Z^3,\ \text{for any } v \in \mathcal{D}'(\mathcal{T}^3),$$

we have

$$\Big[\mathbf{H}[\mathbf{v}]\Big]_{\mathbf{k}} = [\mathbf{v}]_{\mathbf{k}} - \frac{\mathbf{k}}{|\mathbf{k}|^2}\mathbf{k}\cdot[\mathbf{v}]_{\mathbf{k}},\ \Big[\mathbf{H}^{\perp}[\mathbf{v}]\Big]_{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|^2}\mathbf{k}\cdot[\mathbf{v}]_{\mathbf{k}},\ \mathbf{k} \in Z^3.$$

Remark 1.2 Let us recall that the standard definition of the thermodynamics constants:

$$\overline{c}_v = c_v(\overline{\varrho}, \overline{\vartheta}) = \frac{\partial e_F(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta},\ \overline{\alpha} = \frac{1}{\overline{\varrho}}\Big(\frac{\partial_\vartheta p_F(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p_F(\overline{\varrho}, \overline{\vartheta})}\Big),\ \overline{c}_p - \overline{c}_v = \frac{2}{3}\overline{c}_v\overline{\vartheta}\overline{\alpha}.$$

The proof of Theorem 3.1 can be found in [18, Theorem 1.1]. There are two main issues to be addressed in the proof: **(i)** suitable *uniform estimates* independent of $\varepsilon \to 0$, **(ii)** a precise description of the time oscillations of the "gradient part" $\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ of the momentum.

The former difficulty can be overcome by means of the **dissipation equality**:

$$\frac{1}{\varepsilon^2}\Big(E_\varepsilon(\tau) - \overline{\vartheta}\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(\tau)\ \mathrm{d}x\Big) + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0,\tau)\times\Omega\Big] =$$

$$\frac{1}{\varepsilon^2}\Big(E_\varepsilon(0) - \overline{\vartheta}\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0)\ \mathrm{d}x\Big),$$

where

$$E_\varepsilon(\tau) = \int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon\varrho_\varepsilon F\Big)\ \mathrm{d}x$$

that can be easily deduced from (1.3), (1.4). Here σ_ε is a positive measure expressing the rate of entropy production in accordance with (1.3). Using convexity of the nonlinear quantity on the left-hand side (Helmholtz free energy), one can deduce the desired uniform estimates necessary for passing to the limit as stated in the conclusion of Theorem 3.1 (see Section 2 in [18]).

The answer to the latter question is provided by the **acoustic equation**

$$\varepsilon\partial_t(\mathbf{H}^{\perp}[\varrho_\varepsilon\mathbf{u}_\varepsilon]) + \nabla_x V_\varepsilon = \varepsilon\mathbf{H}^{\perp}[\mathrm{div}_x\mathbb{S}_\varepsilon] + \varepsilon\mathbf{G}_1^\varepsilon \tag{3.14}$$

$$\varepsilon\partial_t V_\varepsilon + M_0 \, \mathrm{div}_x(\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]) = S_0\sigma_\varepsilon + \varepsilon G_2^\varepsilon \tag{3.15}$$

that can be deduced from (1.1 - 1.3). Here the quantities $\mathbf{G}_1^\varepsilon$, G_2^ε are uniformly bounded in suitable function spaces, M_0, S_0 are positive constants, and

$$V_\varepsilon = \Lambda_1\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big) + \Lambda_2\Big(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big)$$

for certain constants Λ_1, Λ_2. System (3.14), (3.15) represents a linear wave equation that can be solved explictly yielding a precise description of possible time oscillations of $\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]$. Note, however, that (3.15) contains a measure term - the entropy production σ_ε as a source. This fact makes the analysis of solutions quite delicate (see Sections 4,5 in [18]).

3.2 The Dirichlet boundary conditions

As we have seen in the preceding section, the motion of a compressible viscous fluid occupying the domain between two paralel plates features the non-linear interaction of fast acoustic waves and slow shear motion. Under appropriate constitutive assumptions, with the relative sound speed approaching infinity, the fluid is driven toward incompressibility. In the case of flat boundaries, however, the convergence of the velocity field takes place only in the *weak* topology due to possible large amplitude fast oscillations of the acoustic waves (cf. (3.13)). The main issue to be discussed in this section is the interaction of fast acoustic waves with a boundary layer caused by a wavy bottom of the physical domain, resulting in the *strong* convergence of the velocity field.

In what follows, we take

$$\Phi_B = \Phi_B(x_1, x_2), \ \Phi_T = \pi$$

in (2.8) and replace (2.13) by the no-slip boundary conditions (2.14). In order to eliminate fast oscillations in the velocity field, we consider spatial domains with wavy bottoms, specifically, we assume

$$|\Phi_B(x_1, x_2)| < \pi, \ \Phi_B(-x_1, x_2) = -\Phi_B(x_1, x_2) \text{ for all } (x_1, x_2) \in \mathcal{T}^2. \tag{3.16}$$

A rather surprising damping effect resulting from the interaction of fast acoustic waves with boundaries was discovered in a truly pioneering paper by Desjardins et al. [9] dealing with a simplified isentropic model. In particular, they showed strong convergence of the velocity field in the low Mach number regime provided the following overdetermined eigenvalue problem

$$\Delta w = \lambda w \text{ in } \Omega, \ w|_{\partial\Omega} = const, \ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{3.17}$$

admits only the trivial solution $\lambda = 0$, $w = const$.

Solvability of (3.17), being equivalent to the so-called Pompeiu problem, has been studied by several authors. In particular, it is known that for a bounded simply connected domain $\Omega \subset R^2$, with Lipschitz but not real analytic boundary, problem (3.17) admits only the trivial solution (see Garofalo and Segala [24]). The same is true for an arbitrary bounded Lipschitz domain in R^N, $N \geq 2$, with $\partial\Omega$ homeomorphic to the unit sphere but not real analytic (see Williams [40]). Similar results for ellipsoids and certain tori in R^3 were obtained by Dalmasso [7]. On the other hand, it is relatively easy to check that for balls as well as flat "periodic" objects like $\mathcal{T}^2 \times (0,\pi)$, problem (3.17) admits non-constant solutions.

Here we claim that for domains with wavy bottoms whose boundary is determined by (2.8), (3.16) problem (3.17) admits only the trivial solution as soon as $\Phi_B \neq 0$. More precisely, the following result was proved in [21, Proposition 5.1].

Proposition 3.1 *Let $\Omega \subset R^3$ be given by (2.8), where the "bottom" part $\Gamma_B \subset \partial\Omega$ is determined by a function $\Phi_B \in C^3(\mathcal{T}^2)$ satisfying (3.16), and $\Phi_T \equiv \pi$. Assume there is a function $w \not\equiv const$ solving the overdetermined eigenvalue problem*

$$\Delta w = \lambda w \ \text{in}\ \Omega,\ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0,\ w = c_T \ \text{on}\ \Gamma_T,\ w = c_B \ \text{on}\ \Gamma_B. \tag{3.18}$$

Then $\Phi_B \equiv 0$.

The principal idea in the pioneering work by Desjardins et al. [9] is to show that non-flat boundaries combined with the no-slip boundary conditions for the velocity lead to creation of a boundary layer resulting in a faster decay of the acoustic waves of order $\exp(-\sqrt{\varepsilon}t)$ provided $\mathrm{Ma} \approx \varepsilon$. This, in turn, leads to a complete anihilation of fast sound waves described by the acoustic equation (3.14), (3.15) (see Proposition 2 in [9]) and strong convergence of the velocity field. This observation together with Proposition 3.1 makes it possible to show the following result (Theorem 3.1 in [21]):

Theorem 3.2 *In addition to the hypotheses of Theorem 2.1, assume that $\Omega \subset R^3$ is given by (2.8), where the "bottom" part $\Gamma_B \subset \partial\Omega$ is determined by a function $\Phi_B \in C^3(\mathcal{T}^2)$ satisfying (3.16), $\Phi_T \equiv \pi$, and $\Phi_B \neq 0$. Furthermore, suppose that $\beta = 1$ in (2.1), and $F = |x_3|$ in (1.2). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of variational solutions to system (1.1 - 1.8) supplemented with the boundary conditions (2.11), (2.12), and 2.14) in the sense of Definition 2.1, with*

$$\mathrm{Ma} = \varepsilon,\ \mathrm{Fr} = \sqrt{\varepsilon},\ a = \varepsilon.$$

Assume the the solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ emanates from the initial state

$$\varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_{\varepsilon,0},\ \vartheta_{\varepsilon,0} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{\varepsilon,0}$$

satisfying (3.7), where

$$\varrho^{(1)}_{\varepsilon,0} \to \varrho^{(1)}_0,\ \mathbf{u}_{\varepsilon,0} \to \mathbf{u}_0,\ \vartheta^{(1)}_{\varepsilon,0} \to \vartheta^{(1)}_0 \ \text{weakly-(*) in}\ L^\infty(\Omega).$$

Then

$$\varrho_\varepsilon \to \overline{\varrho} \ \text{in}\ C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$
$$\vartheta_\varepsilon \to \overline{\vartheta} \ \text{in}\ L^2(0,T;W^{1,2}(\Omega)),$$

and, passing to a subsequence if necessary,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)) \text{ and (strongly) in } L^2(0,T;L^2(\Omega;R^3)),$$

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

where the trio

$$\mathbf{U},\ r = \varrho^{(1)} + \overline{\varrho}\Big(\frac{1}{\partial_\varrho p_F(\overline{\varrho},\overline{\vartheta})} - \frac{\overline{\vartheta}\ \overline{\alpha}^2}{\overline{c}_p}\Big)(x_3 - \frac{\pi}{2}),$$

$$\Theta = \overline{\vartheta} + \vartheta^{(1)} + \frac{\overline{\vartheta}\ \overline{\alpha}}{\overline{c}_p}(x_3 - \frac{\pi}{2})$$

represents a (weak) solution of system (3.2 - 3.5) in the sense of Definition 3.1, with the initial data

$$\mathbf{U}_0 = \mathbf{H}[\mathbf{u}_0],\ \Theta_0 = \overline{\vartheta} + \frac{\overline{c}_v}{\overline{c}_p}\vartheta_0^{(1)} - \frac{2}{3}\frac{\overline{\vartheta}\ \overline{c}_v}{\overline{\varrho}\ \overline{c}_p}\varrho_0^{(1)},$$

and the heat flux through the boundary given by

$$F_b = \frac{\overline{\vartheta}\ \overline{\alpha}}{\overline{c}_p} \text{ on the lateral boundary } \Gamma_B \cup \Gamma_T.$$

Here the symbol $\mathbf{H}$ *stands for the Helmholtz projection onto the space of divergenceless vector fields.*

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4 Strongly stratified flows

The last part of this survey focuses on a qualitatively new situation, where both Mach and Froude numbers tend to zero at the same rate, specifically,

$$\mathrm{Ma} = \mathrm{Fr} = \varepsilon; \tag{4.1}$$

whence the limit flow is strongly stratified, that means, the density depends effectively on the vertical coordinate.

The results will be given only for the reduced *Navier-Stokes system* of equations governing the time evolution of the density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible viscous fluid:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.2}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho\nabla_x g, \tag{4.3}$$

where p denotes the pressure, $g = g(x) = -x_3$ represents the gravitational potential, and the symbol $\mathbb{S}$ stands for the viscous stress tensor assumed to be given through

Newton's rheological relation (1.6), with $\mu > 0$ constant and $\eta \equiv 0$. In particular, the effect of temperature changes is neglected.

Keeping in mind possible applications to atmospheric flows we consider Ω as in (2.8) with a flat boundary determined through (3.1), where

$$\mathbf{u} \cdot \mathbf{n} = 0, \ (\mathbb{S}\mathbf{n}) \times \mathbf{n} = 0 \text{ on the lateral boundary } \Gamma_B \cup \Gamma_T. \tag{4.4}$$

As we have observed in Section 3, such a setting is "equivalent" to the purely periodic boundary conditions, and may be viewed as a suitable compromise between physical interpretation and mathematical simplicity of the model.

Similarly to Section 3, we take the initial data

$$\varrho(0,\cdot) = \varrho_{\varepsilon,0} = \varrho_s + \varepsilon \varrho^{(1)}_{\varepsilon,0}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{\varepsilon,0}, \tag{4.5}$$

and consider the pressure term in the form and

$$p = p_\varepsilon(\varrho) = \varrho + \varepsilon p_d(\varrho). \tag{4.6}$$

The zeroth order term ϱ_s in (4.5) stands for the (unique) solution of the static problem

$$\nabla_x p_\varepsilon(\tilde{\varrho}) = \tilde{\varrho} \nabla_x g \text{ in } \Omega, \ \int_\Omega \tilde{\varrho} \ \mathrm{d}x = m, \tag{4.7}$$

or, with the $\varepsilon-$dependent perturbation in (4.6) neglected,

$$\tilde{\varrho} = \tilde{\varrho}(x_3) = k \exp(-x_3), \tag{4.8}$$

where the constant $k > 0$ is uniquely determined by the total mass constraint $\int_\Omega \tilde{\varrho} \ \mathrm{d}x = m$. Without loss of generality, we shall always assume that m was fixed so that $k = 1$. Moreover, we take, for simplicity,

$$p_d \equiv 0 \text{ on } [0, \overline{\varrho}], \quad \text{where } \overline{\varrho} > \sup_{x \in \Omega} \tilde{\varrho}(x). \tag{4.9}$$

Consequently, the unique solution of (4.7) is independent of ε and given through formula (4.8) with $k = 1$.

Similarly to the weak solutions of the incompressible Navier-Stokes system introduced by Leray [28], we restrict our consideration to a class of weak solutions to problem (4.1 - 4.5) satisfying the energy inequality

$$E(\tau) + \mathrm{Ma}^2 \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq E_0 \text{ for a.a } \tau > 0, \tag{4.10}$$

where

$$E(\tau) \equiv \int_\Omega \Big(\mathrm{Ma}^2 \varrho |\mathbf{u}|^2 + P(\varrho) + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho x_3 \Big)(\tau) \ \mathrm{d}x,$$

$$E_0 \equiv \int_\Omega \Big(\mathrm{Ma}^2 \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho_0 x_3 \Big) \ \mathrm{d}x,$$

with

$$P(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z. \tag{4.11}$$

The *existence* of solutions of this type was proved by Lions [29] for $p_d(\varrho) \approx \varrho^\gamma$, $\gamma \geq \frac{9}{5}$, and this result was later extended for $\gamma > \frac{3}{2}$ in [20].

In order to state our result, we need a "weighted" Helmholtz projection that can be defined as follows. For any $\mathbf{v} \in W^{1,p}(\Omega; R^3)$, $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$, we set

$$\mathbf{H}_{\rm w}[\mathbf{v}] = \mathbf{v} - \tilde{\varrho}\nabla_x \Psi \text{ and } \mathbf{H}_{\rm w}^{\perp}[\mathbf{v}] = \tilde{\varrho}\nabla_x \Psi \tag{4.12}$$

where Ψ is the unique solution of the Neumann problem

$${\rm div}_x(\tilde{\varrho}\nabla_x \Psi) = {\rm div}_x \mathbf{v} \text{ in } \Omega, \ \nabla_x \Psi \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \int_\Omega \Psi {\rm d}x = 0. \tag{4.13}$$

Note that $\mathbf{H}_{\rm w}[\mathbf{v}]$ and $\mathbf{H}_{\rm w}^{\perp}[\mathbf{v}]$ are orthogonal in the weighted Hilbert space $L^2_{1/\tilde{\varrho}}$ endowed with the scalar product

$$< \mathbf{u}, \mathbf{v} >_{L^2_{1/\tilde{\varrho}}} = \int_\Omega \mathbf{u}(x) \cdot \mathbf{v}(x) \frac{{\rm d}x}{\tilde{\varrho}}.$$

Similarly to the properties of the standard Helmholtz projection, it can be shown that $\mathbf{H}_{\rm w}$ is a bounded linear operator on $W^{1,p}_n(\Omega; R^3)$ as well as on $L^p(\Omega; R^3)$ for any $1 < p < \infty$, provided that we identify, in the latter case, ${\rm div}_x \mathbf{v}$ with a linear form $\phi \mapsto \int_\Omega \mathbf{v} \cdot \nabla_x \phi \ {\rm d}x$ bounded on $W^{1,p}(\Omega)$. Here, we have set

$$W^{1,p}_n(\Omega; R^3) =$$

$$\left\{ \mathbf{v} \in W^{1,p}(\mathcal{T}^2 \times (0,\pi)) \ \Big| \ v_3(x_1, x_2, 0) = v_3(x_1, x_2, \pi) = 0 \text{ for a.a. } (x_1, x_2) \in \mathcal{T}^2 \right\}.$$

The necessity to work with a "weighted" Helmholtz decomposition reflects the fact that we have to deal with an acoustic equation, similar to system (3.14), (3.15), where the wave speed depends on the vertical coordinate.

Having collected all the preliminary material, we are ready to formulate our main result (see Theorem 1.1 in [17]).

Theorem 4.1 *Assume that* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *is a sequence of finite energy weak solutions to problem (4.1 - 4.5) (defined in a similar way as in* Definition 2.1 *above), where*

$${\rm Ma} = {\rm Fr} = \varepsilon,$$

the initial data are given by (4.5), with

$$\int_\Omega \varrho^{(1)}_{\varepsilon,0} \ {\rm d}x = 0, \ \{\varrho^{(1)}_{\varepsilon,0}\}_{\varepsilon>0} \textit{ bounded in } L^\infty(\Omega) \textit{ and } \mathbf{u}_{\varepsilon,0} \to \mathbf{u}_0 \textit{ weakly in } L^2(\Omega; R^3),$$

and the pressure can be written as $p_\varepsilon(\varrho) = \varrho + \varepsilon p_d(\varrho)$*, with* $p_d \in C^1[0,\infty)$ *such that*

$$p_d' \geq 0 \textit{ on } [0,\infty)$$

$$p_d(\varrho) = \begin{cases} 0 \textit{ for } \varrho \in [0, \overline{\varrho}], \\ \varrho^{\frac{5}{3}} \textit{ for } \varrho \geq 2\overline{\varrho}, \end{cases}$$

where $\overline{\varrho} > \sup_{x\in\Omega} \tilde{\varrho}(x)$.

Then

$$\varrho_\varepsilon \to \tilde{\varrho} \ in \ L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

and, at least for a subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ weakly \ in \ L^2(0,T;W^{1,2}(\Omega;R^3)),$$

where $\tilde{\varrho}$ satisfies (4.7), and the limit velocity field $\mathbf{u}$ is a weak solution (in the sense of Definition 4.1 *below) to problem*

$$\partial_t(\tilde{\varrho}\mathbf{u}) + \mathrm{div}_x(\tilde{\varrho}\mathbf{u}\otimes\mathbf{u}) + \tilde{\varrho}\nabla_x p = \mu \mathrm{div}_x\Big(\nabla_x\mathbf{u} + \nabla_x\mathbf{u}^t - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big), \tag{4.14}$$

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{u}) = 0, \tag{4.15}$$

supplemented with the boundary conditions (4.4), and satisfying the initial conditions

$$\tilde{\varrho}\mathbf{u}(0,\cdot) = \mathbf{H}_{\mathrm{w}}[\tilde{\varrho}\mathbf{u}_0]. \tag{4.16}$$

System (4.14 - 4.16) is usually termed **anelastic approximation**. In the present setting, it can be viewed as a simple model of an isothermal atmosphere with the background temperature $\Theta \equiv 1$ (see Durran [11], Lipps and Hemler [32], among others). Related models and further discussion can be found in the monograph by Majda [33]. Note that a suitable definition of weak solutions of system (4.14 - 4.16) reads as follows:

Definition 4.1 *We shall say that a function $\mathbf{u} \in L^2(0,T;W^{1,2}_n(\Omega;R^3))$ is a weak solution of system (4.14 - 4.16) if the following conditions hold:*

- $$\mathrm{div}_x(\tilde{\varrho}\mathbf{u}) = 0 \ a.a. \ on \ (0,T)\times\Omega;$$

- *momentum equation (4.14), together with initial condition (4.16), are satisfied in the sense of distributions, more specifically, the integral identity*

  $$\int_0^T\int_\Omega \Big(\tilde{\varrho}\mathbf{u}\cdot\partial_t\varphi + \tilde{\varrho}\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t = \int_0^T\int_\Omega \mathbb{S} : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \tilde{\varrho}\mathbf{u}_0\cdot\varphi \ \mathrm{d}x$$

  holds for any test function

$$\varphi \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};R^3)), \ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \ \mathrm{div}_x(\tilde{\varrho}\varphi) = 0 \ in \ (0,T)\times\Omega. \tag{4.17}$$

This can be viewed as a natural generalization of the standard definition of a weak solution in the spirit of Leray's original paper [28] (see also Ladyzhenskaya [27] or Temam [39] for more recent exposition). Accordingly, the satisfaction of the initial conditions reduces to (4.16), reflecting our inability to control the pressure that appears only implicitly through the choice of test functions (4.17).

Let us note, on the point of conclusion, that a suitable generalization of the above result to the complete Navier-Stokes-Fourier system is far from being obvious and will be the main topic of future work.

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