Further Results on Steady-State Flow of a Navier-Stokes Liquid Around a Rigid Body. Existence of the Wake.

Giovanni P. Galdi & Ana L. Silvestre

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Abstract

A rigid body $\mathcal{R}$ is moving in a Navier-Stokes liquid $\mathcal{L}$ that fills the whole space. We assume that all data with respect to a frame, $\mathcal{F}$, attached to $\mathcal{R}$, namely, the body force acting on $\mathcal{L}$, the boundary conditions on $\mathcal{R}$ as well as the translational velocity, U, and the angular velocity, Ω, of $\mathcal{R}$ are independent of time. We assume $\Omega \neq 0$ (the case $\Omega = 0$ being already known) and take, without loss of generality, Ω parallel to the base vector e_1 in $\mathcal{F}$. We show that, if the magnitude of these data is not "too large", there exists at least one steady motion of $\mathcal{L}$ in $\mathcal{F}$, such that the velocity field and its gradient decay like $(1+|x|)^{-1}(1+2\mathrm{Re}\, s(x))^{-1}$ and $(1+|x|)^{-\frac{3}{2}}(1+2\mathrm{Re}\, s(x))^{-\frac{3}{2}}$, respectively, where Re is the Reynolds number and $s(x) := |x| + x_1$ is representative of the "wake" behind the body. This motion is unique in the (larger) class of motions having velocity field decaying like $|x|^{-1}$. Since Re is proportional to $|U \cdot e_1|$, the above formulas show that the $\mathcal{L}$ exhibits a wake behind $\mathcal{R}$ if and only if U is not orthogonal to Ω.

1 Introduction

Let $\mathcal{R}$ be a rigid body moving through an incompressible Navier-Stokes liquid $\mathcal{L}$ that fills the whole three-dimensional space exterior to $\mathcal{R}$. We assume that with respect to a frame, $\mathcal{F}$, attached to $\mathcal{R}$, the translational and the angular velocities of $\mathcal{R}$, the boundary velocity of $\mathcal{L}$ and the body force acting on $\mathcal{L}$ are time-independent. The steady motions of $\mathcal{L}$ with respect to $\mathcal{F}$ are then described by the following set of equations (see, e.g., [11])

$$\left.\begin{aligned} &\nu\Delta u = u\cdot\nabla u - U\cdot\nabla u - \Omega\times x\cdot\nabla u + \Omega\times u + \nabla p + f \\ &\nabla\cdot u = 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$u = u_*, \text{ at } \partial\mathcal{D} \qquad (1.1)$$
$$\lim_{|x|\to\infty} u(x) = 0\,.$$

Here, $\mathcal{D} = \mathbb{R}^3\backslash\mathcal{R}$, with boundary $\partial\mathcal{D}$, is the domain occupied by $\mathcal{L}$. Furthermore, ν , u and p are the kinematical viscosity coefficient, velocity and pressure fields of $\mathcal{L}$, respectively, $\overline{u}(x) = U + \Omega\times x$ is the velocity field of $\mathcal{R}$, f is the external force acting on $\mathcal{L}$ and u_* is a prescribed velocity distribution at $\partial\mathcal{R} \equiv \partial\mathcal{D}$.

Over the last few years, the study of well-posedness of the boundary value problem (1.1) –and of the associated initial-boundary value problem– has attracted the attention of several authors; see, e.g., [2, 3, 8, 12, 13, 14, 16, 17, 21, 22, 23, 24]. This is due, on the one hand, to the intrinsic mathematical interest and challenge associated with it, and, on the other hand, to the fact that problem (1.1) is at the foundation of several important engineering applications for which we refer the interested reader to [11] and to the literature cited therein.

The knowledge of a sufficiently detailed behavior at large distances ensures existence of solutions satisfying basic physical requirements, at least for small data, such as uniqueness, validity of a global energy balance and nonlinear stability in the sense of Liapunov, that is, the solution is "physically reasonable" in the sense of Finn [9]; see, e.g., [2], [11]. Moreover, it is also fundamental in numerical computations, especially in evaluating the error made by approximating the infinite region of flow with a bounded domain. For this type of problems related to (1.1) and, in particular, to the "classical" Navier-Stokes equations, we refer to [1], [19], [4].

Concerning the behavior at large distances of solutions to (1.1), in [12] it is shown that, if $U = 0$ (rotating body), then, provided the magnitude of the data is suitably restricted, (1.1) has one and only one solution (u, p) such that u and ∇u decay like $(1 + |x|)^{-1}$ and $(1+|x|)^{-2}$, respectively, while p and ∇p decay like $(1+|x|)^{-2}$ and $(1+|x|)^{-3}$. In particular, (u, p) is "physically reasonable" in the sense of Finn. Based on the same method, in [15] we have shown that, for nonzero U and Ω, (1.1) possesses one and only solution (u, p) such that u decays like $(1 + |x|)^{-1}$, provided the data are sufficientlt "small".

In this paper we complete our results of [15], by giving a more detailed analysis of the asymptotic behavior of solutions. In fact, not only do we provide a spatial decay of the first derivatives of u but, more importantly, we show that, if $\Omega \neq 0$ and $U \cdot \Omega \neq 0$, [1] the liquid exhibits a "wake" behind the body. Specifically, we prove that, under these conditions on U and Ω and if the magnitude of the data are not "too large", there exists at least one solution to problem (1.1), such that

$$\begin{aligned} |u(x)| &\leq M\,(1 + |x|)^{-1}(1 + 2\mathrm{Re}\, s(x))^{-1} \\ |\nabla u(x)| &\leq M\,(1 + |x|)^{-\frac{3}{2}}(1 + 2\mathrm{Re}\, s(x))^{-\frac{3}{2}}\,. \end{aligned} \tag{1.2}$$

In these inequalities, M is a constant depending on the data, Re is the Reynolds number, proportional to $|U \cdot \Omega|$, and $s(x) := |x| + x_1$ is representative of the "wake" behind the body, where x_1 is in the direction of Ω. This solution is then proved to be unique in the class of those solutions whose velocity field decays like $(1 + |x|)^{-1}$. As a corollary, we then find that the solutions determined in [15] possess the asymptotic behavior (1.2).

As in [15], besides "smallness", *no other condition is imposed on* U *and* Ω. In particular, their direction can be completely arbitrary.

Since $\Omega \neq 0$, without loss of generality, we take $\Omega/|\Omega| = e_1$, with $\{e_1, e_2, e_3\}$ the canonical basis in $\mathbb{R}^3$. By an appropriate change of coordinates and nondimensionalization,

[1] For the cases when $\Omega = 0$ or $\Omega \neq 0$ and $U \cdot \Omega = 0$ see the remark at the end of this section.

see [15], problem (1.1) can be equivalently rewritten as

$$
\left.
\begin{aligned}
&\Delta u = \mathrm{Re}\left(u\cdot\nabla u - \frac{\partial u}{\partial x_1}\right)\\
&\qquad\qquad +\mathrm{Ta}^{\frac{1}{2}}\left(e_1\times u - e_1\times x\cdot\nabla u\right)+\nabla p + \mathrm{Re}\, f\\
&\nabla\cdot u = 0
\end{aligned}
\right\}\ \text{in } \mathcal{D}
\tag{1.3}
$$

$$
\begin{aligned}
&u = u_*,\ \text{ at } \partial\mathcal{D}\\
&\lim_{|x|\to\infty} u(x) = 0
\end{aligned}
$$

where $\mathrm{Re} = |U\cdot e_1|\,\delta(\mathcal{R})/\nu$, with $\delta(\mathcal{R})$ the diameter of $\mathcal{R}$, is the Reynolds number and $\mathrm{Ta} = |\Omega|^2\delta(\mathcal{R})^4/\nu^2$ is the Taylor number.

As already noticed in [15], the study of problem (1.1) is interesting only if $\Omega\neq 0$ and U has a nonzero component along the direction of Ω, that is, $\mathrm{Re}>0$; see also the remark at the end of this section. Therefore, in what follows we shall take $\mathrm{Re}>0$.

In order to state our result, we introduce some notation. By $L^q(\mathcal{A})$, $W^{m,q}(\mathcal{A})$, etc., we denote the usual Lebesgue and Sobolev spaces on the domain $\mathcal{A}$, with norms $\|.\|_{q,\mathcal{A}}$ and $\|.\|_{m,q,\mathcal{A}}$, respectively. [2] By $W^{m-\frac{1}{q},q}(\partial\mathcal{A})$ we indicate the trace space on the boundary $\partial\mathcal{A}$ of $\mathcal{A}$, for functions from $W^{m,q}(\mathcal{A})$, equipped with the usual norm $\|.\|_{m-\frac{1}{q},q,\partial\mathcal{A}}$. The homogeneous Sobolev space of order (k,q), $k\geq 1$, $1<q<\infty$, on $\mathcal{A}$ is denoted by $D^{k,q}(\mathcal{A})$ with associated seminorm $|u|_{m,q,\mathcal{A}} = \sum_{i=|k|}\|D^i u\|_q$. If G is a vector or a second-order tensor field and α, β are nonnegative reals, then $[\![G]\!]_{\alpha,\beta,\mathcal{A}} := \sup_{x\in\mathcal{A}}[(1+|x|)^\alpha(1+2\mathrm{Re}\, s(x))^\beta|G(x)|]$. Whenever confusion will not arise, we shall omit the subscript $\mathcal{A}$ in the above norms and seminorms.

Our main result reads as follows.

Theorem 1 *Let $\mathcal{D}$ be an exterior domain of class C^2 and let* $\mathrm{Re}\in]0,B_1]$, $\mathrm{Ta}\in[0,B_2]$ *for given $B_1, B_2>0$. Moreover, let $u_*\in W^{\frac{3}{2},2}(\partial\mathcal{D})$ with $\int_{\partial\mathcal{D}} u_*\cdot n = 0$, and $f\in L^\infty(\mathcal{D})$ with $[\![f]\!]_{\frac{5}{2},\frac{5}{2}}<\infty$. Then, the following properties hold.*

(i) Existence. *There exists a positive constant $K_1 = K_1(\mathcal{D},B_1,B_2)$, such that if*

$$
\mathrm{Re}^{\frac{1}{2}}\left(\|u_*\|_{\frac{3}{2},2}+[\![f]\!]_{\frac{5}{2},\frac{5}{2}}\right) < K_1,
\tag{1.4}
$$

then problem (1.3) *admits a solution (u,p) satisfying*

$$
\begin{aligned}
&u\in W^{2,2}_{\mathrm{loc}}(\overline{\mathcal{D}})\cap D^{2,2}(\mathcal{D})\cap D^{1,2}(\mathcal{D}),\quad [\![u]\!]_{1,1}+[\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}}<\infty,\\
&p\in W^{1,2}(\mathcal{D}).
\end{aligned}
$$

This solution satisfies the estimate

$$
|u|_{2,2}+|u|_{1,2}+[\![u]\!]_{1,1}+[\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}}+|p|_{1,2}+\|p\|_2\leq C\left(\|u_*\|_{\frac{3}{2},2}+[\![f]\!]_{\frac{5}{2},\frac{5}{2}}\right),
$$

[2]Throughout the paper we shall use the same font style to denote scalar, vector and tensor-valued functions and corresponding function spaces.

with $C = C(\mathcal{D}, B_1, B_2) > 0$, *and the* energy equation

$$
\begin{aligned}
2\|D(u)\|_2^2 - & \int_{\partial\mathcal{D}} u_* \cdot T(u,p) \cdot n + \frac{\mathrm{Re}}{2} \int_{\partial\mathcal{D}} |u_*|^2 u_* \cdot n \\
- & \ \frac{1}{2} \int_{\partial\mathcal{D}} |u_*|^2 (\mathrm{Re}\, e_1 + \mathrm{Ta}^{\frac{1}{2}}\, e_1 \times x) \cdot n + \mathrm{Re} \int_{\mathcal{D}} f \cdot u = 0\,,
\end{aligned}
$$

where $T(u,p) = 2D(u) - pI$ *(Cauchy stress tensor),* I *is the identity matrix,* $2D(u) = \nabla u + (\nabla u)^\top$ *and "*$\top$*" denotes transpose.*

(ii) Uniqueness. *Let* (u_1, p_1) *be another solution to* (1.3) *with*

$$
u_1 \in W^{2,2}_{loc}(\overline{\mathcal{D}}) \cap D^{1,2}(\mathcal{D}),\ [\![u]\!]_{1,0} < \infty,\ p_1 \in W^{1,2}_{loc}(\overline{\mathcal{D}}) \cap L^2(\mathcal{D}) \tag{1.5}
$$

If condition (1.4) *holds and* $\mathrm{Re} \le C^{-2}$ *, then* $u = u_1$ *and* $p = p_1$.

Remarks.

(a) Theorem 1 is relevant only if $\Omega \neq 0$ and, moreover, $\Omega \cdot U \neq 0$ (that is, $\mathrm{Re} > 0$). Actually, if $\Omega = 0$, Theorem 1 has been proved in [9] while, if $\Omega \neq 0$ and $U \cdot \Omega = 0$, (1.1) reduces, formally, to the problem of the steady motion of a Navier-Stokes liquid around a body that rotates without translating, which was already solved in [12].

(b) One significant consequence of our result is that, under the hypothesis (1.4), there exists one and only one solution in the class (1.5) and this solution must decay, uniformly for large x, as $(1+|x|)^{-1}(1+2\mathrm{Re}\, s(x))^{-1}$. Hence, if $\mathrm{Re} > 0$, that is, if $\Omega \cdot U \neq 0$, there is a formation of a "wake" behind $\mathcal{R}$ whose "width" will depend on the angle between Ω and U. More specifically, there is an infinite paraboloidal region within which u decays like $(1+|x|)^{-1}$ and outside of which u decays even faster. However, if $\Omega \neq 0$ and Ω and U are orthogonal there is no wake behind $\mathcal{R}$.

2 Preliminary results

Our results rely upon some suitable estimates of the fundamental solution (E, Q) of the following *unsteady* Oseen system (see [20, §7])

$$
\left\{
\begin{aligned}
& \frac{\partial E_{ij}}{\partial t} = \Delta E_{ij} + \mathrm{Re}\frac{\partial E_{ij}}{\partial x_1} - \frac{\partial Q_i}{\partial x_j} + \delta_{ij}\delta(t)\delta(x) \\
& \frac{\partial E_{ij}}{\partial x_j} = 0
\end{aligned}
\right. \quad , \quad i,j = 1,2,3
$$

where δ_{ij} is the Kronecker delta and $\delta(.)$ is the Dirac delta distribution. The pair (E, Q) is given by

$$
\begin{aligned}
E_{ij}(x,t) &= \delta_{ij}\Gamma(x,t) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} \frac{\Gamma(x-y,t)}{|y|} dy \\
Q_i(x,t) &= \frac{x}{4\pi|x|^3}\delta(t)
\end{aligned}
\tag{2.6}
$$

where $\Gamma(x,t) = (4\pi t)^{-\frac{3}{2}} \exp\left(-|x + \mathrm{Re}\, t\, e_1|^2/4t\right)$.

In what follows

$$\mathcal{E}_0(x) := \int_0^\infty |E(x,s)| ds\,, \quad \mathcal{E}_1(x) := \int_0^\infty |\nabla E(x,s)| ds.$$

and

$$s(x) := |x| + x_1.$$

We recall some estimates for $\mathcal{E}_0$ and $\mathcal{E}_1$ that we have shown in [15].

Lemma 1 *Let $0 < \theta < \frac{1}{2}$. Then*

$$\mathcal{E}_0(x) \le 2|x|^{-1}(1 + 2\,\mathrm{Re}\; s(x))^{-1}, \quad \forall x \in \mathbb{R}^3,$$
$$\mathcal{E}_1(x) \le C(\theta) \begin{cases} \mathrm{Re}^{\frac{1}{2}} |x|^{-\frac{3}{2}} (1 + 2\mathrm{Re}\; s(x))^{-\frac{3}{2}}, & |x| \ge \theta/\mathrm{Re}, \\ |x|^{-2}, \quad 0 < |x| < \theta/\mathrm{Re}. & \end{cases}$$

In order to simplify the presentation of the next results, we introduce additional notation. For each $\delta > 0$ and $a \in \mathbb{R}^3$, we set $B_\delta(a) := \{x \in \mathbb{R}^n : |x - a| < \delta\}$ and $B^\delta(a) := \{x \in \mathbb{R}^n : |x - a| > \delta\}$. By $\chi_{\mathcal{A}}$ we denote the characteristic function of the set $\mathcal{A}$ and

$$\eta^\gamma_{\alpha,\beta}(x) := (1 + |x|)^\alpha (1 + \gamma s(x))^\beta,$$
$$\mu^\gamma_{\alpha,\beta}(x) := |x|^\alpha (1 + \gamma s(x))^\beta,$$

for $\alpha, \beta \in \mathbb{R}$ and $\gamma \ge 0$.

Using these functions, we can write estimates of Lemma 1 in the form

$$\mathcal{E}_0(x) \le 2\,\mu^{2\mathrm{Re}}_{-1,-1}(x),\ \forall x \in \mathbb{R}^3,$$
$$\mathcal{E}_1(x) \le C(\theta) \left(\chi_{B_{\theta/\mathrm{Re}}(0)\setminus\{0\}}(x)\, \mu^{2\mathrm{Re}}_{-2,0}(x) + \mathrm{Re}^{\frac{1}{2}}\, \chi_{\overline{B^{\theta/\mathrm{Re}}(0)}}(x)\, \mu^{2\mathrm{Re}}_{-\frac{3}{2},-\frac{3}{2}}(x) \right),\ \forall x \in \mathbb{R}^3.$$

Next we establish two fundamental inequalities for convolutions involving $\mathcal{E}_0$ and $\mathcal{E}_1$, which will play an important role in the proof of our result on the asymptotic behavior of the velocity.

Lemma 2 *Let $K = \max\{1, \mathrm{Re}\}$ and $\rho > 0$.*

1. *There exists positive constants C_1, C_2, independent of* Re*, such that*

$$\mathcal{E}_0 * \eta^{2\mathrm{Re}}_{-\frac{5}{2},-\frac{5}{2}} \le C_1\, K\, \mathrm{Re}^{-\frac{1}{2}}\, \eta^{2\mathrm{Re}}_{-1,-1},$$
$$\mathcal{E}_1 * \eta^{2\mathrm{Re}}_{-\frac{5}{2},-\frac{5}{2}} \le C_2\, K^{\frac{3}{2}}\, \eta^{2\mathrm{Re}}_{-\frac{3}{2},-\frac{3}{2}}.$$

2. *There exists positive constants $C_i = C_i(\rho)$, $i = 3, 4$, independent of* Re*, such that*

$$\mathcal{E}_0 * \chi_{B_\rho(0)} \le C_3\, K\, \eta^{2\mathrm{Re}}_{-1,-1},$$
$$\mathcal{E}_1 * \chi_{B_\rho(0)} \le C_4\, K^2\, \eta^{2\mathrm{Re}}_{-\frac{3}{2},-\frac{3}{2}}.$$

Proof. For the first convolution in 1, we write

$$
\begin{aligned}
&\mathrm{Re}^{\frac{1}{2}}\mathcal{E}_0 * \eta^{2\mathrm{Re}}_{-\frac{5}{2},-\frac{5}{2}}(x) = \\
&\quad = \int_{B_{\theta/\mathrm{Re}}(0)} \mathrm{Re}^{\frac{1}{2}}|x-y|^{-1}(1+2\mathrm{Re}\, s(x-y))^{-1}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&\quad + \int_{B^{\theta/\mathrm{Re}}(0)} \mathrm{Re}^{\frac{1}{2}}|x-y|^{-1}(1+2\mathrm{Re}\, s(x-y))^{-1}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&\quad := I_1(x) + I_2(x).
\end{aligned}
$$

Let us consider $I_1(x)$ when $|x| < 2\theta/\mathrm{Re}$. By a classical estimate on weakly singular integrals (see, e.g., Lemma II.7.2 of [10]), we get

$$
I_1(x) \le \mathrm{Re}^{\frac{1}{2}} \int_{\mathbb{R}^3} |x-y|^{-1}(1+|y|)^{-\frac{5}{2}}dy \le C\mathrm{Re}^{\frac{1}{2}}|x|^{-\frac{1}{2}}, \quad 0 < |x| < 2\theta/\mathrm{Re},
$$

and, since $\mathrm{Re}|x| < 2\theta$, it follows that

$$
I_1(x) \le C(\theta)|x|^{-1}, \quad 0 < |x| < 2\theta/\mathrm{Re}.
$$

However, for $x = 0$, it is

$$
I_1(0) \le \mathrm{Re}^{\frac{1}{2}} \int_{\mathbb{R}^3} |y|^{-1}(1+|y|)^{-\frac{5}{2}}dy \le C\mathrm{Re}^{\frac{1}{2}} \le CK,
$$

and taking into account that

$$
\frac{1}{8\theta+1} \le \frac{1}{1+2\mathrm{Re}\, s(x)} \le 1, \quad \forall x \in B_{2\theta/\mathrm{Re}}(0), \tag{2.7}
$$

we find

$$
I_1(x) \le C(\theta)K(1+|x|)^{-1} \le C(\theta)K(1+|x|)^{-1}(1+2\mathrm{Re}\, s(x))^{-1}, \quad |x| < 2\theta/\mathrm{Re}.
$$

When $|x| \ge 2\theta/\mathrm{Re}$, we have $|x-y| \ge \theta/\mathrm{Re}$, for $|y| \le \theta/\mathrm{Re}$. Hence,

$$
\begin{aligned}
&I_1(x) \le \\
&\le C(\theta)\mathrm{Re}^{\frac{3}{2}} \int_{B_{\theta/\mathrm{Re}}(0)} (1+|x-y|)^{-1}(1+2\mathrm{Re}\, s(x-y))^{-1}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&\quad \le C(\theta)\mathrm{Re} \int_{B_\theta(0)} (1+|\mathrm{Re}\, x-\xi|)^{-1}(1+2s(\mathrm{Re}\, x-\xi))^{-1}|\xi|^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi \\
&\quad \le C(\theta)\mathrm{Re}\, \eta^1_{-1,-1} * \eta^1_{-\frac{5}{2},-\frac{5}{2}}(\mathrm{Re}\, x), \quad |x| \ge 2\theta/\mathrm{Re}.
\end{aligned}
$$

Lemma 3.1 of [5] (see also Theorem 3.1 of [18]) furnishes

$$
I_1(x) \le C(\theta)\mathrm{Re}\, \eta^1_{-1,-1}(\mathrm{Re}\, x) = C(\theta)\mathrm{Re}(1+|\mathrm{Re}\, x|)^{-1}(1+\mathrm{Re}\, s(x))^{-1},
$$

and therefore, we get

$$
I_1(x) \le C(\theta)K(1+|x|)^{-1}(1+2\mathrm{Re}\, s(x))^{-1}, \quad |x| \ge 2\theta/\mathrm{Re}.
$$

Concerning the integral $I_2(x)$,

$$
\begin{aligned}
&I_2(x) \le \\
&\le C(\theta)\mathrm{Re}^3 \int_{B^{\theta/\mathrm{Re}}(0)} |x-y|^{-1}(1+2\mathrm{Re}\, s(x-y))^{-1}(1+\mathrm{Re}\,|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&= C(\theta)\mathrm{Re} \int_{B^{\theta}(0)} |\mathrm{Re}\, x-\xi|^{-1}(1+2s(\mathrm{Re}\, x-\xi))^{-1}(1+|\xi|)^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi \\
&\le C(\theta)\mathrm{Re}\, \eta^1_{-\frac{5}{2},-\frac{5}{2}} * \eta^1_{-1,-1}(\mathrm{Re}\, x) \le C(\theta)K(1+|x|)^{-1}(1+2\mathrm{Re}\, s(x))^{-1}.
\end{aligned}
$$

In the first inequality we have used

$$
\frac{1}{1+|y|} \le \frac{\theta+1}{\theta}\frac{\mathrm{Re}}{1+\mathrm{Re}\,|y|}, \quad |y| \ge \theta/\mathrm{Re}. \tag{2.8}
$$

In order to estimate the second convolution in 1, we consider the following partition

$$
\begin{aligned}
\mathbb{R}^3 &= (B_\lambda(0)\cap B_\lambda(a)) \cup (B_\lambda(0)\cap \overline{B^\lambda(a)}) \cup (\overline{B^\lambda(0)}\cap B_\lambda(a)) \cup (\overline{B^\lambda(0)}\cap \overline{B^\lambda(a)}) \\
&:= S_1(a,\lambda)\cup S_2(a,\lambda)\cup S_3(a,\lambda)\cup S_4(a,\lambda).
\end{aligned}
$$

where $a \in \mathbb{R}^3$ and $\lambda > 0$. Then, we can write

$$
\begin{aligned}
&\mathcal{E}_1 * \eta^{2\mathrm{Re}}_{-\frac{5}{2},-\frac{5}{2}}(x) \le \\
&C(\theta)\Bigg(\int_{S_1(x,\theta/\mathrm{Re})} |x-y|^{-2}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&+ \int_{S_2(x,\theta/\mathrm{Re})} \mathrm{Re}^{\frac{1}{2}}\, |x-y|^{-\frac{3}{2}}(1+2\mathrm{Re}\, s(x-y))^{-\frac{3}{2}}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&+ \int_{S_3(x,\theta/\mathrm{Re})} |x-y|^{-2}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy \\
&+ \int_{S_4(x,\theta/\mathrm{Re})} \mathrm{Re}^{\frac{1}{2}}\, |x-y|^{-\frac{3}{2}}(1+2\mathrm{Re}\, s(x-y))^{-\frac{3}{2}}(1+|y|)^{-\frac{5}{2}}(1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}}dy\Bigg) \\
&:= C(\theta)\left(J_1(x)+J_2(x)+J_3(x)+J_4(x)\right).
\end{aligned}
$$

By Lemma II.7.2 of [10] and (2.7),

$$
\begin{aligned}
J_1(x) &\le \chi_{B_{2\theta/\mathrm{Re}}(0)}(x)\int_{\mathbb{R}^3} |x-y|^{-2}(1+|y|)^{-\frac{5}{2}}dy \le C\chi_{B_{2\theta/\mathrm{Re}}(0)}(x)(1+|x|)^{-\frac{3}{2}} \\
&\le C(\theta)(1+|x|)^{-\frac{3}{2}}(1+2\mathrm{Re}\, s(x))^{-\frac{3}{2}}, \quad \forall x\in\mathbb{R}^3.
\end{aligned}
$$

Again, by Lemma of [5] (or Theorem 3.2 of [18]), we get

$$
\begin{aligned}
&J_2(x) \le \\
&\le \frac{C(\theta)}{\mathrm{Re}} \int_{S_2(\mathrm{Re}\,|x|,\theta)} (1+|\mathrm{Re}\,x-\xi|)^{-\frac{3}{2}}(1+2s(\mathrm{Re}\,x-\xi))^{-\frac{3}{2}}(1+|\xi|/\mathrm{Re})^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi \\
&\le C(\theta)\mathrm{Re}^{\frac{3}{2}} \int_{B_\theta(0)} (1+|\mathrm{Re}\,x-\xi|)^{-\frac{3}{2}}(1+2s(\mathrm{Re}\,x-\xi))^{-\frac{3}{2}}|\xi|^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi \\
&= C(\theta)\mathrm{Re}^{\frac{3}{2}} \int_{B_\theta(\mathrm{Re}\,x)} (1+|\xi|)^{-\frac{3}{2}}(1+2s(\xi))^{-\frac{3}{2}}|\mathrm{Re}\,x-\xi|^{-\frac{5}{2}}(1+2s(\mathrm{Re}\,x-\xi))^{-\frac{5}{2}}d\xi \\
&\le C(\theta)\mathrm{Re}^{\frac{3}{2}}\,\eta^1_{-\frac{3}{2},-\frac{3}{2}} * \eta^1_{-\frac{5}{2},-\frac{5}{2}}(\mathrm{Re}\,x) \le C(\theta)\mathrm{Re}^{\frac{3}{2}}\,\eta^1_{-\frac{3}{2},-\frac{3}{2}}(\mathrm{Re}\,x) \\
&\le C(\theta)K^{\frac{3}{2}}(1+|x|)^{-\frac{3}{2}}(1+2\mathrm{Re}\,s(x))^{-\frac{3}{2}}, \quad \forall x \in \mathbb{R}^3.
\end{aligned}
$$

Using (2.8) and Lemma 3.1 of [5], we deduce

$$
\begin{aligned}
&J_3(x)+J_4(x) \le \\
\le\;& C(\theta)\mathrm{Re}^{\frac{3}{2}}\Bigg(\int_{B_\theta(\mathrm{Re}\,x)} |\mathrm{Re}\,x-\xi|^{-2}(1+|\xi|)^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi \\
&+\int_{B^\theta(\mathrm{Re}\,x)} (1+|\mathrm{Re}\,x-\xi|)^{-\frac{3}{2}}(1+s(\mathrm{Re}\,x-\xi))^{-\frac{3}{2}}(1+|\xi|)^{-\frac{5}{2}}(1+2s(\xi))^{-\frac{5}{2}}d\xi\Bigg) \\
\le\;& C(\theta)\mathrm{Re}^{\frac{3}{2}}\eta^1_{-\frac{3}{2},-\frac{3}{2}}(\mathrm{Re}\,x) \le C(\theta)K^{\frac{3}{2}}(1+|x|)^{-\frac{3}{2}}(1+2\mathrm{Re}\,s(x))^{-\frac{3}{2}}, \quad \forall x \in \mathbb{R}^3.
\end{aligned}
$$

It remains to estimate the integrals $\int_{B_\rho(0)} \mathcal{E}_j(x-y)dy$, $j=0,1$. From Lemma 1, we have

$$
\begin{aligned}
\int_{B_\rho(0)} \mathcal{E}_0(x-y)dy &\le 2\int_{B_\rho(0)} |x-y|^{-1}(1+2\mathrm{Re}\,s(x-y))^{-1}dy, \\
\int_{B_\rho(0)} \mathcal{E}_1(x-y)dy &\le (1+2\rho)^{\frac{3}{2}}\int_{B_\rho(0)} |x-y|^{-2}(1+2\mathrm{Re}\,s(x-y))^{-\frac{3}{2}}dy \\
&\quad +\mathrm{Re}^{\frac{1}{2}}\int_{B_\rho(0)} |x-y|^{-\frac{3}{2}}(1+2\mathrm{Re}\,s(x-y))^{-\frac{3}{2}}dy.
\end{aligned}
$$

Since, for $y \in B_\rho(0)$, it is

$$
\begin{aligned}
1+2\mathrm{Re}\,s(x) &\le 1+2\mathrm{Re}\,s(x-y))+2\mathrm{Re}\,s(y) \\
&\le (1+4\mathrm{Re}\rho)(1+2\mathrm{Re}\,s(x-y)) \\
&\le C(\rho)\,K\,(1+2\mathrm{Re}\,s(x-y))
\end{aligned}
$$

we get

$$
\begin{aligned}
\int_{B_\rho(0)} \mathcal{E}_0(x-y)dy &\le C(\rho)\,K\,(1+2\mathrm{Re}\,s(x))^{-1}\int_{B_\rho(0)} |x-y|^{-1}dy, \\
\int_{B_\rho(0)} \mathcal{E}_1(x-y)dy &\le C(\rho)\,K^2\,(1+2\mathrm{Re}\,s(x))^{-\frac{3}{2}}\int_{B_\rho(0)} (|x-y|^{-\frac{3}{2}}+|x-y|^{-2})dy.
\end{aligned}
$$

Now, we obtain estimates for the integrals $\int_{B_\rho(0)} |x-y|^{-\kappa} dy$, $\kappa = 1, 3/2, 2$. If $|x| \leq 2\rho + 1$ then

$$\int_{B_\rho(0)} |x-y|^{-\kappa} dy \leq \int_{B_{3\rho+1}(0)} |y|^{-\kappa} dy = C(\rho,\kappa) \leq C'(\rho,\kappa)(1+|x|)^{-\kappa}.$$

If $|x| \geq 2\rho+1$ then $|x-y| \geq (1+|x|)/2$, for $|y| \leq \rho$. This implies

$$\int_{B_\rho(0)} |x-y|^{-\kappa} dy \leq 2^\kappa (1+|x|)^{-\kappa} \int_{B_\rho(0)} dy = C(\rho,\kappa)(1+|x|)^{-\kappa},$$

for $|x| \geq 2\rho+1$, and the lemma is proved. ■

3 The linear problem in the whole space

In this section, we shall consider existence and uniqueness, in suitable function spaces, of solutions to the following linear problem in $\mathbb{R}^3$

$$\begin{aligned} &\Delta u = -\mathrm{Re}\frac{\partial u}{\partial x_1} + \mathrm{Ta}^{\frac{1}{2}}\left(e_1 \times u - e_1 \times x \cdot \nabla u\right) + \nabla p + \mathrm{Re}\, f + b \\ &\nabla \cdot u = 0\,. \end{aligned} \tag{3.1}$$

with b of compact support. Specifically, we shall show the following result.

Theorem 2 *Let $f, b \in L^\infty(\mathbb{R}^3)$ with $[\![f]\!]_{\frac{5}{2},\frac{5}{2}} < \infty$ and $\mathrm{supp}(b) \subset B_\rho(0)$, and let $\mathrm{Re} \in]0, B_1]$, for some $B_1 > 0$. Then, the problem (3.1) has at least one solution such that*

$$\begin{aligned} &u \in W^{2,2}_{loc}(\mathbb{R}^3) \cap D^{2,2}(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3),\ [\![u]\!]_{1,1} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}} < \infty, \\ &p \in W^{1,2}_{loc}(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \end{aligned}$$

This solution satisfies the estimate

$$|u|_{2,2} + |u|_{1,2} + [\![u]\!]_{1,1} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}} + |p|_{1,2} + \|p\|_2 \leq C\left(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty\right),$$

where $C = C(B_1, \rho) > 0$. Moreover, if (u_1, p_1) is another solution to (3.1) with

$$u_1 \in W^{2,2}_{loc}(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3) \cap L^6(\mathbb{R}^3),\ p_1 \in W^{1,2}_{loc}(\mathbb{R}^3)$$

then $u = u_1$ and $p = p_1 + const.$

This theorem will be proved with the aid of the unsteady problem

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} = \Delta v + \mathrm{Re}\frac{\partial v}{\partial x_1} - \mathrm{Ta}^{\frac{1}{2}}(e_1 \times v - e_1 \times x \cdot \nabla v) - \nabla q - \mathrm{Re}\, f - b \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times]0,\infty[$$
$$v(x,0) = 0, \quad x \in \mathbb{R}^3\,. \tag{3.2}$$

For this problem we shall prove that (i) $[\![v(x,t)]\!]_{1,1}$ and $[\![\nabla v(x,t)]\!]_{\frac{3}{2},\frac{3}{2}}$ are uniformly bounded in time, and that (ii) $v(x,t)$ and $\nabla v(x,t)$ converge as $t \to \infty$ to $u(x)$ and $\nabla u(x)$, respectively, in appropriate norms. The asymptotic (spatial) behavior of u will be shown to be the same as that of v.

In the next lemma we shall use the following standard notation. If X is a Banach space with norm $\|.\|_X$, we denote by $L^p(a,b;X)$, $W^{1,p}(a,b;X)$, etc., the class of all measurable functions $u:(a,b) \to X$ such that $\int_a^b \|u(t)\|_X^q dt < \infty$, $\int_a^b (\|u(t)\|_X^q + \|\frac{du}{dt}(t)\|_X^q) dt < \infty$.

Lemma 3 *Let $f, b \in L^\infty(\mathbb{R}^3)$ with $[\![f]\!]_{\frac{5}{2},\frac{5}{2}} < \infty$ and $\operatorname{supp}(b) \subset B_\rho(0)$, and let $\mathrm{Re} \in]0, B_1]$, for some $B_1 > 0$. Then, the Cauchy problem* (3.2) *has one and only one solution such that*

$$
\begin{gathered}
(v, \nabla q) \in W^{1,2}(0,T; L^2(B_R(0))) \cap L^2(0,T; W^{2,2}(\mathbb{R}^3)) \times L^2(0,T; L^2(\mathbb{R}^3))\,, \\
\textit{for all } R > \delta(\mathcal{R}) \ \textit{ and for all } T > 0.
\end{gathered}
\tag{3.3}
$$

Moreover, this solution satisfies

$$
\operatorname*{ess\,sup}_{t \geq 0} \left([\![v(t)]\!]_{1,1} + [\![\nabla v(t)]\!]_{\frac{3}{2},\frac{3}{2}} \right) \leq C \left(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty \right)
$$

where $C = C(B_1, \rho) > 0$. Finally,

$$
\begin{aligned}
&\lim_{t \to \infty} \|v(t) - u\|_\sigma = 0, \ \textit{for all } \sigma > 6, \\
&\lim_{t \to \infty} \|\nabla(v(t) - u)\|_6 = 0,
\end{aligned}
\tag{3.4}
$$

where u is the unique solution to the steady problem (3.1)*.*

Proof. We make a change of variables that brings (3.2) into an appropriate Oseen system. Specifically, we define

$$
\begin{aligned}
&\chi = Q(t) \cdot x, \quad w(\chi, t) = Q(t) \cdot v(Q^\top(t) \cdot \chi, t), \quad \pi(\chi, t) = q(Q^\top(t) \cdot \chi, t)\,, \\
&\mathcal{F}(\chi, t) = Q(t) \cdot f(Q^\top(t) \cdot \chi, t), \quad \mathcal{B}(\chi, t) = Q(t) \cdot b(Q^\top(t) \cdot \chi, t)
\end{aligned}
\tag{3.5}
$$

where, for $t \geq 0$,

$$
Q(t) = \exp(\mathrm{Ta}^{\frac{1}{2}}\, W(e_1) t), \quad W(e_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

Since Q is an orthogonal matrix and $Q(t) \cdot e_1 = e_1$, the following identities hold

$$
|x| = |\chi|,\ s(x) = s(\chi),\ |v(x,t)| = |w(\chi,t)|,\ |\nabla v(x,t)| = |\nabla w(\chi,t)|,\ |p(x,t)| = |\pi(\chi,t)|.
\tag{3.6}
$$

The pair (w, π) satisfies

$$
\left.
\begin{aligned}
&\frac{\partial w}{\partial t} = \Delta w + \mathrm{Re} \frac{\partial w}{\partial \chi_1} - \nabla \pi - \mathrm{Re}\, \mathcal{F} - \mathcal{B} \\
&\nabla \cdot w = 0
\end{aligned}
\right\} \text{ in } \mathbb{R}^3 \times]0, \infty[\,,
\tag{3.7}
$$
$$
w(\chi, 0) = 0, \quad \chi \in \mathbb{R}^3
$$

Now, we observe that $f \in L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with

$$\begin{aligned}
\|f\|_{\frac{6}{5}} + \|f\|_2 &\leq 2\mathrm{Re}^{-\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} \int_{\mathbb{R}^3} (1+|x|)^{-\frac{5}{2}} s(x)^{-\frac{1}{2}} dx \\
&\leq C\mathrm{Re}^{-\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} \int_0^\infty r^{\frac{3}{2}} (1+r)^{-\frac{5}{2}} dr \int_0^\pi \sin(\alpha)/(1+\cos(\alpha))^{\frac{1}{2}} d\alpha \\
&= C'\,\mathrm{Re}^{-\frac{1}{2}}\, [\![f]\!]_{\frac{5}{2},\frac{5}{2}}.
\end{aligned}$$

From (3.5) and (3.6), it is easily seen that $\mathcal{F}$ satisfies

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\geq 0} \|\mathcal{F}(t)\|_2 &\leq \|f\|_2, \\
\operatorname*{ess\,sup}_{t\geq 0} [\![\mathcal{F}(t)]\!]_{\frac{5}{2},\frac{5}{2}} &\leq [\![f]\!]_{\frac{5}{2},\frac{5}{2}}.
\end{aligned}$$

Moreover, $\mathcal{B}$ has compact support and

$$\operatorname*{ess\,sup}_{t\geq 0} \|\mathcal{B}(t)\|_\infty \leq \|b\|_\infty.$$

Therefore, the Cauchy problem (3.7) has one and only one solution such that, for all $T > 0$,

$$(w, \nabla\pi) \in W^{1,2}(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; W^{2,2}(\mathbb{R}^3)) \times L^2(0,T; L^2(\mathbb{R}^3))\,, \tag{3.8}$$

and this solution satisfies

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\geq 0} ([\![w(t)]\!]_{1,1} + [\![\nabla w(t)]\!]_{\frac{3}{2},\frac{3}{2}}) &\leq C(\mathrm{Re}^{\frac{1}{2}} \operatorname*{ess\,sup}_{t\geq 0} [\![\mathcal{F}(t)]\!]_{\frac{5}{2},\frac{5}{2}} + \operatorname*{ess\,sup}_{t\geq 0} \|\mathcal{B}(t)\|_\infty) \\
&\leq C(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty)
\end{aligned} \tag{3.9}$$

with $C = C(B_1, \rho) > 0$. The existence of a unique solution satisfying (3.8) is a classical result. In order to show (3.9), we make use of the volume potential representation

$$w(\chi, t) = \int_0^t \int_{\mathbb{R}^3} E(\chi - y, t-s) \cdot (\mathrm{Re}\,\mathcal{F}(y,s) + \mathcal{B}(y,s)) dy\, ds$$

with E given by (2.6). By Lemma 2, we deduce that

$$\begin{aligned}
|w(\chi,t)| &\leq \int_0^t \int_{\mathbb{R}^3} |E(\chi-y,s)| (\mathrm{Re}\,|\mathcal{F}(y,t-s)| + |\mathcal{B}(y,t-s)|) dy\, ds \\
&= \int_{\mathbb{R}^3} \int_0^t |E(\chi-y,s)| (\mathrm{Re}\,|\mathcal{F}(y,t-s)| + |\mathcal{B}(y,t-s)|) ds\, dy \\
&\leq \mathrm{Re} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} \int_{\mathbb{R}^3} |\mathcal{E}_0(\chi-y)| (1+|y|)^{-\frac{5}{2}} (1+2\mathrm{Re}\, s(y))^{-\frac{5}{2}} dy \\
&\quad + \|b\|_\infty \int_{B_\rho(0)} |\mathcal{E}_0(\chi-y)| dy \\
&\leq C(B_1,\rho)(\mathrm{Re}^{\frac{1}{2}}\, [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty)\, (1+|\chi|)^{-1} (1+2\mathrm{Re}\, s(\chi))^{-1},
\end{aligned}$$

for all $\chi \in \mathbb{R}^3$ and all $t \geq 0$. Moreover, for $i = 1, 2, 3$,

$$\begin{aligned}
\left|\frac{\partial w}{\partial \chi_i}(\chi, t)\right| &\leq \int_0^t \int_{\mathbb{R}^3} |\nabla E(\chi - y, s)||\mathrm{Re}\, \mathcal{F}(y, t-s) + \mathcal{B}(y, t-s)| dy\, ds \\
&\leq \mathrm{Re}\, [\![f]\!]_{\frac{5}{2},\frac{5}{2}} \int_{\mathbb{R}^3} |\mathcal{E}_1(\chi - y)|(1 + |y|)^{-\frac{5}{2}}(1 + 2\mathrm{Re}\, s(y))^{-\frac{5}{2}} dy \\
&\quad + \|b\|_\infty \int_{B_\rho(0)} |\mathcal{E}_1(\chi - y)| dy \\
&\leq C(B_1, \rho)(\mathrm{Re}\, [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty)(1 + |\chi|)^{-\frac{3}{2}}(1 + 2\mathrm{Re}\, s(\chi))^{-\frac{3}{2}}.
\end{aligned}$$

We now go back to system (3.2). By the change of variables (3.5), the systems (3.2) and (3.7) are equivalent and the summability properties of v and q are consequence of (3.6). The convergence of v to u when $t \to \infty$ was showed in [15]. ■

We are now in a position to complete the proof of the main result of this section.

Proof of Theorem 2. Concerning the existence of such a solution, in view of Lemma 3 of [15], it only remains to show the summability properties of the pressure and that u satisfies $[\![u]\!]_{1,1} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}} < \infty$ along with the corresponding estimate.

Since

$$p(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \cdot (\mathrm{Re}\, f(y) + b(y))}{|x - y|^3} dy,$$

using Theorem 3.4 of [18], we obtain

$$\begin{aligned}
|p(x)| &\leq C\mathrm{Re}\, [\![f]\!]_{\frac{5}{2},\frac{5}{2}} \int_{\mathbb{R}^3} |x - y|^{-2}(1 + |y|)^{-\frac{5}{2}}(1 + 2\mathrm{Re}\, s(y))^{-\frac{1}{2}} dy + \int_{B_\rho(0)} |x - y|^{-2} b(y) dy \\
&\leq C(B_1, \rho)(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty)(1 + |x|)^{-2} \max\{1, \ln(|x|)\}.
\end{aligned}$$

Hence, $p \in L^2(\mathbb{R}^3)$ with

$$\|p\|_2 \leq C(B_1, \rho)(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty).$$

Moreover, since $\nabla(\frac{(x-y)}{|x-y|^3})$ is a Calderón-Zygmund kernel, we have

$$\frac{\partial p}{\partial x_k}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_k} \left(\frac{x - y}{|x - y|^3} \right) \cdot (\mathrm{Re}\, f(y) + b(y)) dy,$$

and therefore $\nabla p \in L^2(\mathbb{R}^3)$ with

$$\|\nabla p\|_2 \leq C(B_1, \rho)(\mathrm{Re}^{\frac{1}{2}} [\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty).$$

Now, from (3.4) we conclude

$$\lim_{t \to \infty} \|v(t) - u\|_{1,\infty} = 0.$$

By Lemma 3, we have

$$
\begin{aligned}
&\eta_{1,1}^{2\mathrm{Re}}(x)\,|u(x)| + \eta_{\frac{3}{2},\frac{3}{2}}^{2\mathrm{Re}}(x)\,|\nabla u(x)| \le \\
&\qquad \le \eta_{1,1}^{2\mathrm{Re}}(x)\,|v(x,t)-u(x)| + \eta_{1,1}^{2\mathrm{Re}}(x)\,|v(x,t)| \\
&\qquad\quad + \eta_{\frac{3}{2},\frac{3}{2}}^{2\mathrm{Re}}(x)\,|\nabla v(x,t)-\nabla u(x)| + \eta_{\frac{3}{2},\frac{3}{2}}^{2\mathrm{Re}}(x)\,|\nabla v(x,t)| \\
&\qquad \le \eta_{1,1}^{2\mathrm{Re}}(x)|v(x,t)-u(x)| + \eta_{\frac{3}{2},\frac{3}{2}}^{2\mathrm{Re}}(x)|\nabla v(x,t)-\nabla u(x)| + C\,(\mathrm{Re}^{\frac{1}{2}}[\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty).
\end{aligned}
$$

Passing to the limit $t \to \infty$, we find that

$$
\eta_{1,1}^{2\mathrm{Re}}(x)\,|u(x)| + \eta_{\frac{3}{2},\frac{3}{2}}^{2\mathrm{Re}}(x)\,|\nabla u(x)| \le C\,(\mathrm{Re}^{\frac{1}{2}}[\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|b\|_\infty).
$$

which completes the proof of existence. For the uniqueness part, see [15]. ■

4 The linear problem in exterior domains

The objective of this section is to prove existence, uniqueness and corresponding estimates of solutions to the exterior problem

$$
\left.
\begin{aligned}
&\Delta u + \mathrm{Re}\frac{\partial u}{\partial x_1} + \mathrm{Ta}^{\frac{1}{2}}(e_1\times x\cdot\nabla u - e_1\times u) = \nabla p + \mathrm{Re}\, f \\
&\nabla\cdot u = 0
\end{aligned}
\right\} \text{ in } \mathcal{D}\,,
$$

$$
\begin{aligned}
&u = u_*, \text{ at } \partial\mathcal{D}\,, \\
&\lim_{|x|\to\infty} u(x) = 0\,.
\end{aligned}
\tag{4.1}
$$

We take the origin of coordinates in $\mathcal{R} := \mathbb{R}^3 \setminus \mathcal{D}$, and, for each $R > \delta(\mathcal{R})$, set $\mathcal{D}_R := \mathcal{D}\cap B_R(0)$, $\mathcal{D}^R := \mathcal{D}\setminus\overline{B_R(0)}$. If $\delta(\mathcal{R}) < R_1 < R_2$, $\mathcal{D}_{R_1,R_2}$ denotes the set $\mathcal{D}_{R_2}\setminus\overline{B_{R_1}(0)}$.

Theorem 3 *Let $\mathcal{D}$ be an exterior domain of class C^2, and let $\mathrm{Re} \in]0,B_1]$, $\mathrm{Ta}\in[0,B_2]$, for some $B_1, B_2 > 0$. Assume that $u_* \in W^{\frac{3}{2},2}(\partial\mathcal{D})$ with $\int_{\partial\mathcal{D}} u_*\cdot n = 0$, and that $f \in L^\infty(\mathcal{D})$ with $[\![f]\!]_{\frac{5}{2},\frac{5}{2}} < \infty$. Then, problem (4.1) has at least one solution (u,p) satisfying*

$$
\begin{aligned}
&u \in W^{2,2}_{loc}(\overline{\mathcal{D}}) \cap D^{1,2}(\mathcal{D}) \cap D^{2,2}(\mathcal{D}), \quad [\![u]\!]_{1,1} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}} < \infty \\
&p \in W^{1,2}(\mathcal{D}),
\end{aligned}
$$

along with the following estimate

$$
\begin{aligned}
&|u|_{2,2} + |u|_{1,2} + [\![u]\!]_{1,1} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2}} + \|p\|_{1,2} \\
&\qquad \le C\left(\|u_*\|_{\frac{3}{2},2} + \mathrm{Re}^{\frac{1}{2}}[\![f]\!]_{\frac{5}{2},\frac{5}{2}}\right),
\end{aligned}
\tag{4.2}
$$

with $C = C(\mathcal{D}, B_1, B_2) > 0$. Moreover, if (u_1,p_1) is another solution to (4.1) with

$$
u_1 \in W^{2,2}_{loc}(\overline{\mathcal{D}}), \quad [\![u_1]\!]_{1,0} < \infty, \quad p_1 \in W^{1,2}_{loc}(\overline{\mathcal{D}}) \cap L^2(\mathcal{D})\,, \tag{4.3}
$$

then $(u,p) = (u_1,p_1)$.

Proof. We only have to show the existence part, uniqueness was shown in [15]. The details of the proof of existence can be found in [15]. Let (u, p) be the solution to (4.1) which satisfies

$$u \in D^{1,2}(\mathcal{D}) \cap W^{2,2}_{loc}(\overline{\mathcal{D}}), \quad p \in W^{1,2}_{loc}(\overline{\mathcal{D}})$$

given by Lemma 6 and Lemma 7 of [15].

Let $R > \delta(\mathcal{R})$ be fixed. By Theorem IV.4.1 of [10], we have

$$\begin{aligned} & \|u\|_{2,s,\mathcal{D}_{R,2R}} + \|p\|_{1,s,\mathcal{D}_{R,2R}} \\ \leq \quad & C(\mathrm{Re}\,\|f\|_{s,\mathcal{D}_{3R}} + \|u_*\|_{\frac{3}{2},2,\partial\mathcal{D}} + \|u\|_{s,\mathcal{D}_{3R}} + \|p\|_{s,\mathcal{D}_{3R}}), \quad s > 3, \end{aligned} \tag{4.4}$$

with $C = C(\mathcal{D}, R, B_1, B_2)$.

Let $\varphi_R = \varphi_R(|x|)$ be a smooth function such that $\varphi_R(x) = 0$, for $|x| < R$, $\varphi_R(x) = 1$, for $|x| \geq 2R$, $|\nabla\varphi_R| \leq M/R$, and $|D^2\varphi_R| \leq M/R^2$, with M independent of R. We set $(v, q) := (\varphi_R u + w, \varphi_R p)$, where w solves the following problem

$$\begin{aligned} & \nabla \cdot w = -\nabla\varphi_R \cdot u \quad \text{in } \mathcal{D}_{2R}\,, \\ & w \in W^{3,s}_0(\mathcal{D}_{2R})\,, \\ & \|w\|_{3,s} \leq C(R, \mathcal{D})\|u\|_{2,s,\mathcal{D}_{R,2R}}, \quad s > 3\,. \end{aligned}$$

Then the pair (v, p) satisfies

$$\left.\begin{aligned} & \Delta v + \mathrm{Re}\frac{\partial v}{\partial x_1} + \mathrm{Ta}^{\frac{1}{2}}(e_1 \times x \cdot \nabla v - e_1 \times v) = \nabla q + \mathrm{Re}\,\tilde{f} + \tilde{b} \\ & \nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\,,$$

$$\lim_{|x|\to\infty} v(x) = 0\,,$$

where

$$\begin{aligned} \tilde{b} \quad &= \quad (\Delta\varphi_R)u + 2\nabla\varphi_R \cdot \nabla u + \mathrm{Re}\frac{\partial\varphi_R}{\partial x_1}u + \mathrm{Ta}^{\frac{1}{2}}(e_1 \times x \cdot \nabla\varphi_R)u \\ & \quad\; -p\nabla\varphi_R + \Delta w + \mathrm{Re}\frac{\partial w}{\partial x_1} + \mathrm{Ta}^{\frac{1}{2}}(e_1 \times x \cdot \nabla w - e_1 \times w), \\ \tilde{f} \quad &= \quad \varphi_R f \end{aligned}$$

From (4.4), we get

$$\|\tilde{b}\|_\infty \leq C(R, \mathcal{D}, B_1, B_2)(\mathrm{Re}\,\|f\|_{s,\mathcal{D}_{3R}} + \|u\|_{s,\mathcal{D}_{3R}} + \|p\|_{s,\mathcal{D}_{3R}}), \quad s > 3. \tag{4.5}$$

and from (4.5) and Theorem 2 we obtain the estimates

$$\begin{aligned} & \|D^2 u\|_{2,\mathcal{D}^{2R}} + \|\nabla u\|_{2,\mathcal{D}^{2R}} + [\![u]\!]_{1,1,\mathcal{D}^{2R}} + [\![\nabla u]\!]_{\frac{3}{2},\frac{3}{2},\mathcal{D}^{2R}} + \|p\|_{2,\mathcal{D}^{2R}} + \|\nabla p\|_{2,\mathcal{D}^{2R}} \\ & \leq C\,(\|u_*\|_{\frac{3}{2},2,\partial\mathcal{D}} + \mathrm{Re}^{\frac{1}{2}}\,[\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \|u\|_{6,\mathcal{D}_{3R}} + \|p\|_{6,\mathcal{D}_{3R}}). \end{aligned}$$

Using classical arguments, we then show that

$$\|u\|_{6,\mathcal{D}_{3R}} + \|p\|_{6,\mathcal{D}_{3R}} \leq C(\|u_*\|_{\frac{3}{2},2,\partial\mathcal{D}} + \mathrm{Re}^{\frac{1}{2}}\,[\![f]\!]_{\frac{5}{2},\frac{5}{2}}),$$

and obtain the desired estimate. ■

5 The nonlinear problem in exterior domains

In this section we give a proof of our main result, Theorem 1. The proof of existence is based upon a fixed point argument that uses the results of Theorem 3. To this end, let

$$X = \{\psi \in W^{2,2}_{loc}(\overline{\mathcal{D}}) : \nabla \cdot \psi = 0, \text{ in } \mathcal{D}, \, \|D^2\psi\|_2 + \|\nabla\psi\|_2 + [\![\psi]\!]_{1,1} + [\![\nabla\psi]\!]_{\frac{3}{2},\frac{3}{2}} < \infty\}.$$

This linear space becomes a Banach space with the norm

$$\|\psi\|_X := \|D^2\psi\|_2 + \|\nabla\psi\|_2 + [\![\psi]\!]_{1,1} + [\![\nabla\psi]\!]_{\frac{3}{2},\frac{3}{2}}.$$

Proof of Theorem 1. Set

$$D := \|u_*\|_{\frac{3}{2},2} + [\![f]\!]_{\frac{5}{2},\frac{5}{2}}\,, \quad \tilde{K} := \max\{1, B_1{}^{\frac{1}{2}}\}$$

and let

$$X' = \{\psi \in X : \|\psi\|_X \le 2\,\tilde{K}\,C\,D$$

where C is the constant appearing in the estimate (4.2). The solution u will be found as a fixed point of the map

$$M : X' \to X',$$

such that, for each $\psi \in X$, the pair $(\overline{\psi}, p) := (M(\psi), p)$ is the solution to the problem

$$\left.\begin{aligned} &\Delta\overline{\psi} + \mathrm{Re}\frac{\partial\overline{\psi}}{\partial x_1} + \mathrm{Ta}^{\frac{1}{2}}(e_1 \times x \cdot \nabla\overline{\psi} - e_1 \times \overline{\psi}) = \nabla p + \mathrm{Re}\,(f + \psi\cdot\nabla\psi) \\ &\nabla \cdot \overline{\psi} = 0 \end{aligned}\right\} \text{ in } \mathcal{D}\,,$$

$$\overline{\psi} = u_*, \text{ at } \partial\mathcal{D}\,,$$

$$\lim_{|x|\to\infty} \overline{\psi}(x) = 0\,.$$

It is easily seen that under the condition

$$4\,C^2\,\tilde{K}\,\mathrm{Re}^{\frac{1}{2}}\,D \,<\, 1\,, \tag{5.1}$$

the pair $(\overline{\psi}, p)$ satisfies

$$\begin{aligned} \|\overline{\psi}\|_X + \|\nabla p\|_2 + \|p\|_2 &\le C\left(\|u_*\|_{\frac{3}{2},2} + \mathrm{Re}^{\frac{1}{2}}[\![f]\!]_{\frac{5}{2},\frac{5}{2}} + \mathrm{Re}^{\frac{1}{2}}\|\psi\|_X^2\right) \\ &\le C\,\tilde{K}\,D(1 + 4\,C^2\,\tilde{K}\,\mathrm{Re}^{\frac{1}{2}}\,D) \le 2\,\tilde{K}\,C\,D. \end{aligned}$$

Let $\psi_1, \psi_2 \in X'$ with $\overline{\psi}_i = M(\psi_i)$, $i = 1, 2$. Again by Theorem 3 we obtain

$$\|\overline{\psi}_1 - \overline{\psi}_2\|_X \le C\,\mathrm{Re}^{\frac{1}{2}}\,(\|\overline{\psi}_1\|_X + \|\overline{\psi}_2\|_X)\|\psi_1 - \psi_2\|_X$$

which, because of (5.1), ensures that M is a contraction and, consequently, it has a fixed point $u \in X'$, which, in addition, implies that u satisfies the estimate

$$\|u\|_X \le 2\,\tilde{K}\,C\left(\|u_*\|_{\frac{3}{2},2} + [\![f]\!]_{\frac{5}{2},\frac{5}{2}}\right)\,.$$

The other results of the theorem were proved in [15].

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