Anisotropic L^2-estimates of weak solutions to the stationary Oseen-type equations in $\mathbb{R}^3$ for a rotating body

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Abstract

We study the Oseen problem with rotational effect in the whole three-dimensional space. Using a variational approach we prove existence and uniqueness theorems in anisotropically weighted Sobolev spaces. As the main tool we derive and apply an inequality of the Friedrichs-Poincaré type.

1 Introduction

1.1 A formulation of the problem

In a three-dimensional exterior domain in $\mathbb{R}^3$, the classical Oseen problem [19] describes the velocity vector $\mathbf{u}$ and the associated pressure p by a linearized version of the incompressible Navier-Stokes equations as a perturbation of $\mathbf{v}_\infty$ the velocity at infinity; $\mathbf{v}_\infty$ is generally assumed to be constant in a fixed direction, say the first axis, $\mathbf{v}_\infty = |\mathbf{v}_\infty|\,\mathbf{e}_1$. In the next we denote $|\mathbf{v}_\infty|$ by k, and we will write the Oseen operator $k\,\partial_1\mathbf{v}$. On the other hand it is known that for various flows past a rotating obstacle, the Oseen operator appears with some concrete non-constant coefficient functions, e.g. $\mathbf{a}\,(\mathbf{x}) = \omega \times \mathbf{x}$, where ω is a given vector, see [10, 18]; in view of industrial applications $\mathbf{a}(\mathbf{x})$ can also play the role of an "experimental" known velocity field, see [?].

This paper is devoted to the study of the following problem in $\mathbb{R}^3$ for vector function $\mathbf{u} = \mathbf{u}\,(\mathbf{x})$ and scalar function $p = p\,(\mathbf{x})$:

$$-\nu\,\Delta\mathbf{u} + k\partial_1\mathbf{u} - (\omega \times \mathbf{x}) \cdot \nabla\mathbf{u} + \omega \times \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = g \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

$$\mathbf{u} \to \mathbf{0} \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{1.3}$$

where $\omega = (\widetilde{\omega},\, 0,\, 0)$ is a constant vector, ν, k and $\widetilde{\omega}$ are some positive constants, and $\mathbf{f} = \mathbf{f}(\mathbf{x})$ a given vector function, $g = g\,(\mathbf{x})$ a given scalar function.

We examine the problem assuming conditions which are necessary for an extension of existence and uniqueness results also on the case of exterior domains (and solenoidal solutions), see a forthcoming paper [13]. In particular, we need the assumption of a non-zero divergence in general. We restrict ourselves to the assumption of compact support of g, it is sufficient for this aim. For this reason we will also prove two different uniqueness theorems, see Section 3.

The system arises from the Navier-Stokes system modelling viscous incompressible fluid around a rotating body which is moving with a non-zero velocity in the direction of its axis of rotation. An appropriate coordinate transform and a linearization yield in the stationary case equations (1.1) and (1.2), for details see [3, 10].

Let us begin with some comment and relevant process of analysis of the problem (1.1)–(1.3). The governing fluid motion is linear, but we are concerned in this paper with $\mathbb{R}^3$ and the convective operators, $k\,\partial_1$ and $(\omega \times \mathbf{x}) \cdot \nabla$, cannot be treated as perturbations of lower order of the Laplacian, this is well known.

A common approach for studying the asymptotic properties of the solutions to the Dirichlet problem of the classical steady Oseen flow is to use convolutions with Oseen fundamental tensor and its first and second gradients for the velocity (or with the fundamental solution of Laplace equation for the pressure): the L^q estimates in anisotropically weighted Sobolev spaces can be derived, see e.g. [2, 12, 14, 15]. The fundamental solution to rotating Oseen problem in the time dependent case is known, see [21], but, unfortunately, the respective stationary kernel is not seem to be of Calderon-Zygmund type. The Littlewood-Paley theory offers another approach for an L^q-analysis: Thus, L^q estimates in non-weighted spaces were derived for the rotating Stokes problem by T. Hishida [10, 11], and for the rotating Oseen problem in $\mathbb{R}^3$ by R. Farwig, T. Hishida and D. Müller [5], see also [3, 4]. Looking for estimates in anisotropically weighted spaces, see [6], this approach generates some technical difficulties. Another approach using non-stationary equations in both the linear and also non-linear cases is proposed by G.P. Galdi and A.L. Silvestre in [9].

In this paper we will prefer a variational approach. The same variational viewpoint has been already applied in [16, 17] by S. Kračmar and P. Penel to solve the following generic scalar model equation with a given non-constant and, in general, non-solenoidal vector function $\mathbf{a}$ in an exterior domain Ω

$$-\nu\,\Delta u + k\partial_1 u - \mathbf{a} \cdot \nabla u = f \quad \text{in } \Omega$$

together with boundary conditions $u = 0$ on $\partial\Omega$ and $u \to 0$ as $|\mathbf{x}| \to \infty$.

To reflect the decay properties near the infinity we introduce the following weight functions:

$$w(x) = \eta_\beta^\alpha(\mathbf{x}) = \eta_{\beta,\varepsilon}^{\alpha,\delta}(\mathbf{x}) = (1 + \delta r)^\alpha (1 + \varepsilon s)^\beta,$$

with $r = r(\mathbf{x}) = |\mathbf{x}| = (\sum_{i=1}^3 x_i^2)^{1/2}$, $s = s(\mathbf{x}) = r - x_1$, $\mathbf{x} \in \mathbb{R}^3$, $\varepsilon, \delta > 0$, $\alpha, \beta \in \mathbb{R}$. Discussing the range of the exponents α and β the corresponding weighted spaces $L^q(\mathbb{R}^3; w)$ give the appropriate framework to test the solutions to (1.1)-(1.3). This paper is concerned with $q = 2$.

Let us mention that η_β^α belongs to the Muckenhoupt class A_2 of weights in $\mathbb{R}^3$ if $-1 < \beta < 1$ and $-3 < \alpha + \beta < 3$.

1.2 Basic notations and elementary properties

Let us outline our notations. Let

$$D^{m,q} = D^{m,q}\left(\mathbb{R}^3\right) = \left\{u \in L^1_{loc}\left(\mathbb{R}^3\right) : \ D^l u \in L^q\left(\mathbb{R}^3\right)\right\}$$

with the seminorm $|u|_{m,q} = \left(\sum_{|l|=m} \int_{\mathbb{R}^3} |u|^q \right)^{1/q}$. It is known that $D^{m,q}$ is a Banach space (and if $q = 2$ a Hilbert space), provided we identify two functions u_1, u_2 whenever $|u_1 - u_2|_{m,q} = 0$, i.e. u_1, u_2 differ (at most) on the polynomial of the degree $m-1$. As usual, we denote by $D_0^{m,q}$ the closure of $C_0^\infty = C_0^\infty(\mathbb{R}^3)$ in $D^{m,q}$.

Let $(L^2(\mathbb{R}^3;\, w))^3$ be the set of measurable vector functions $\mathbf{f} = (f_1,\, f_2,\, f_3)$ on $\mathbb{R}^3$ such that

$$\|\mathbf{f}\|^2_{2,\mathbb{R}^3;\, w} = \int_{\mathbb{R}^3} |\mathbf{f}|^2\, w\, d\mathbf{x} < \infty.$$

We will use the notation $\mathbf{L}^2_{\alpha,\beta}$ instead of $\left(L^2\left(\mathbb{R}^3;\, \eta^\alpha_\beta\right)\right)^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{\left(L^2\left(\mathbb{R}^3;\eta^\alpha_\beta\right)\right)^3}$. Let us define the weighted Sobolev space $\mathbf{H}^1\left(\mathbb{R}^3; \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ as the set of functions $\mathbf{u} \in \mathbf{L}^2_{\alpha_0,\beta_0}$ with the weak derivatives $\partial_i \mathbf{u} \in \mathbf{L}^2_{\alpha_1,\beta_1}$. The norm of $\mathbf{u} \in \mathbf{H}^1\left(\mathbb{R}^3;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ is given by

$$\|\mathbf{u}\|_{\mathbf{H}^1\left(\mathbb{R}^3;\eta^{\alpha_0}_{\beta_0},\eta^{\alpha_1}_{\beta_1}\right)} = \left(\int_{\mathbb{R}^3} |\mathbf{u}|^2\, \eta^{\alpha_0}_{\beta_0}\, d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, \eta^{\alpha_1}_{\beta_1}\, d\mathbf{x} \right)^{1/2}.$$

As usual, $\overset{\circ}{\mathbf{H}}{}^1\left(\mathbb{R}^3;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ will be the closure of $\mathbf{C}_0^\infty$ in $\mathbf{H}^1\left(\mathbb{R}^3;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$, where $\mathbf{C}_0^\infty$ is $\left(C_0^\infty(\mathbb{R}^3)\right)^3$.

For simplicity, we shall use the following abbreviations:

$\mathbf{L}^2_{\alpha,\beta}$	instead of	$\left(L^2\left(\mathbb{R}^3;\, \eta^\alpha_\beta\right)\right)^3$
$\|\cdot\|_{2,\alpha,\beta}$	instead of	$\|\cdot\|_{\left(L^2\left(\mathbb{R}^3;\eta^\alpha_\beta\right)\right)^3}$
$\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$	instead of	$\overset{\circ}{\mathbf{H}}{}^1\left(\mathbb{R}^3;\, \eta^{\alpha-1}_{\beta-1}, \eta^\alpha_\beta\right)$
$\mathbf{V}_{\alpha,\beta}$	instead of	$\overset{\circ}{\mathbf{H}}{}^1\left(\mathbb{R}^3;\, \eta^{\alpha-1}_{\beta}, \eta^\alpha_\beta\right)$

We shall use these last two Hilbert spaces for $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$. $\mathbf{H}^1$ and $\overset{\circ}{\mathbf{H}}{}^1$ mean, as usual, the non-weighted spaces $(H^1(\mathbb{R}^3;\, 1,\, 1))^3$ and $(\overset{\circ}{H}{}^1(\mathbb{R}^3;\, 1,\, 1))^3$, respectively.

Concerning the weight functions η^α_β, we will use two notations $\eta^\alpha_\beta(x)$ and $\eta^{\alpha,\delta}_{\beta,\varepsilon}(x)$ taking the advantages of the following remark:

Remark 1.1 Let us note that for $\eta^{\alpha,\delta}_{\beta,\varepsilon}$ and for any $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2 > 0$ one has

$$c_{\min} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2} \leq \eta^{\alpha,\delta_1}_{\beta,\varepsilon_1} \leq c_{\max} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2},$$

$c_{\min} = \min\left(1, (\delta_1/\delta_2)^\alpha\right) \cdot \min(1, (\varepsilon_1/\varepsilon_2)^\beta)$, $c_{\max} = \max\left(1, (\delta_1/\delta_2)^\alpha\right) \cdot \max(1, (\varepsilon_1/\varepsilon_2)^\beta)$. The parameters δ and ε are useful to rescale separately the isotropic and anisotropic parts of weight function η^α_β.

We also use the notation of sets $B_R = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| \leq R\}$, $B^R = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| \geq R\}$.

1.3 Main results

The weighted estimates of the solution to the stationary classical Oseen problem were firstly obtained by R. Finn 1959, see [7], and then improved by R. Farwig [1] in 1992, see [16] for other comments and references.

Let us assume for a moment that pressure p is known. In solving the problem (1.1)–(1.3) with respect to $\mathbf{u}$ and p by means of a pure variational approach, we shall deal with the following equation:

$$\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 w d\mathbf{x} + \nu \int_{\mathbb{R}^3} \mathbf{u} \nabla \mathbf{u} \cdot \nabla w \, d\mathbf{x} - \frac{k}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \partial_1 w d\mathbf{x} \tag{1.4}$$
$$-\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \operatorname{div} (w \, [\omega \times \mathbf{x}]) \, d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{f} \, \mathbf{u} \, w \, d\mathbf{x} - \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u} \, w \, d\mathbf{x}$$

as we get integrating formally the product of (1.1) and $w \, \mathbf{u}$ with w an appropriate weight function. First, let us note that $\operatorname{div} \left(\eta_\beta^\alpha \, [\omega \times \mathbf{x}] \right)$ equals zero for $w = \eta_\beta^\alpha$. The left hand side can be estimated from below:

$$2^{-1} \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, w \, d\mathbf{x} + 2^{-1} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \left(-\nu \, |\nabla w|^2 / w - k \partial_1 w \right) d\mathbf{x} \tag{1.5}$$

Because the term $-\nu \, |\nabla w|^2 / w - k \partial_1 w$ is known explicitly, we have the possibility to evaluate it from below by a small negative quantity in the form $-C \, \eta_{\beta-1}^{\alpha-1}$ without any constraint in $s(\cdot)$ (see Lemma 2.5).

An improved weighted Friedrichs-Poincaré type inequality in $\overset{\circ}{\mathbf{H}}{}^1_{\alpha, \beta}$ is necessary: it is the first main technical result of this paper. The obtained inequality allows us to compensate by the viscous Dirichlet integral the "small" negative contribution in the second integral of (1.5). We finally prove the existence of a weak solution (1.1) - (1.3) in $\mathbf{V}_{\alpha,\beta}$ by the Lax-Milgram theorem.

The main results can be summarized in the following theorems (parameters α, β, δ, ε are specified in Section 1.2):

Theorem 1.2 *Let $\beta > 0$. There are positive constants R_0, c_0, c_1 depending on α, β, δ, ε (explicit expressions of these constants are given by Lemma 2.3, essentially $c_0 = O(\varepsilon^{-2} + \delta^{-2})$ and $c_1 = O(\varepsilon^{-1} \delta^{-1})$ for δ and ε tending to zero) such that for all $\mathbf{v} \in \overset{\circ}{\mathbf{H}}{}^1_{\alpha, \beta}$*

$$\|\mathbf{v}\|^2_{2,\alpha-1,\beta-1} \ \le \ c_0 \int_{B_{R_0}} |\nabla \mathbf{v}|^2 \, \eta_\beta^\alpha \, d\mathbf{x} + c_1 \int_{B^{R_0}} |\nabla \mathbf{v}|^2 \, \eta_\beta^\alpha \, d\mathbf{x}. \tag{1.6}$$

Theorem 1.3 *(Existence and uniqueness) Let $0 < \beta \le 1$, $0 \le \alpha < y_1 \beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W_0^{1,2}$ with $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, and $\int_{\mathbb{R}^3} g \, d\mathbf{x} = 0$; y_1 will be precised in Lemma 4.3. Then there exists a unique weak solution $\{\mathbf{u}, \, p\}$ of the problem (1.1) - (1.3) such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C \, \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} \right).$$

2 Friedrichs-Poincaré inequality

In this section we derive an inequality of the Friedrichs-Poincaré type in weighted Sobolev spaces. We also recall some necessary technical assertions, for more details see [16].

Proposition 2.1 *For arbitrary* $\alpha, \beta \geq 0$ *and* $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq \mathbf{0}$ *:*

$$\Delta\eta_\beta^\alpha(\mathbf{x}) \geq 2\beta \min(1,\beta)\, \varepsilon\, \delta\, \eta_{\beta-1}^{\alpha-1}(\mathbf{x})$$

Proof. We introduce $\beta^* = \min(\beta,\, 1)$ in an explicit expression of $\Delta\eta_\beta^\alpha$:

$$\begin{aligned}\Delta\eta_\beta^\alpha &= \left\{\left(\alpha^2\delta^2\frac{1+\varepsilon s}{1+\delta r} - \alpha\delta^2\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\frac{s}{r}\right.\\ &\quad +2\beta(\beta-1)\frac{\varepsilon}{r}(1+\delta r)\frac{\varepsilon s}{1+\varepsilon s}\\ &\quad \left.+2\alpha\,\delta^2(1+\varepsilon s)\frac{1}{\delta r} + (1-\beta^*+\beta^*)\,2\beta\frac{\varepsilon}{r}(1+\delta r)\right\}\eta_{\beta-1}^{\alpha-1},\end{aligned}$$

for $r > 0$. We denote the five terms in $\{\ \}$ by $T_1, T_2, \ldots, T_5$, and overwrite the previous relation as $\Delta\eta_\beta^\alpha = \{[T_1 + T_4] + T_2 + [T_3 + (1-\beta^*)\,T_5] + \beta^* T_5\}\ \eta_{\beta-1}^{\alpha-1}$. Observing that $T_5 \geq 2\beta\varepsilon\delta$, the proposition is trivial. □

Proposition 2.2 *Let* $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ *and* $\kappa > 1$. *Then for* $\mathbf{x} \in \mathbb{R}^3$, $|\mathbf{x}| \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right|(\kappa-1)^{-1}$*:*

$$\left|\nabla\eta_\beta^\alpha(\mathbf{x})\right|^2 \leq 2\,\kappa\,\delta\,\varepsilon\ (\alpha+\beta)^2\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2 \tag{2.7}$$

Let $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ *and* $(\beta-\alpha)(2\varepsilon-\delta) \geq 0$. *Then for* $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq 0$*:*

$$\left|\nabla\eta_\beta^\alpha(\mathbf{x})\right|^2 \leq (\alpha\delta + 2\beta\varepsilon)^2\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2 \tag{2.8}$$

Proof. If $\beta = 0$ and $\alpha = 0$ then both inequalities (2.7) and (2.8) are valid. Let us concentrate on the nontrivial cases:

For $r > 0$, $s \in [0, 2r]$, we have that $\partial g/\partial s > 0$, where g is a function defined by relations:

$$\begin{aligned}\left|\nabla\eta_\beta^\alpha(\mathbf{x})\right|^2 &= g(s(\mathbf{x}), r(\mathbf{x}))\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2,\\ g(s,r) &\equiv \alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\frac{s}{r} + 2\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r}.\end{aligned}$$

So, $g(s,r)$ is increasing as a function of s and

$$\begin{aligned}G(r) &\equiv \max_{s\in[0,2r]} g(s,r) = g(2r, r) \\ &= \alpha^2\delta^2\frac{1+2\varepsilon r}{1+\delta r} + 4\alpha\beta\delta\varepsilon + 4\beta^2\varepsilon^2\frac{1+\delta r}{1+2\varepsilon r} \ \leq\ 2\kappa(\alpha+\beta)^2\delta\varepsilon\end{aligned} \tag{2.9}$$

for $\kappa > 1$ and $r \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right|(\kappa-1)^{-1}$. So, inequality (2.7) is proved.

To justify second inequality (2.8), we observe that for the given values of α, β, δ, ε and for $r > 0$, $G(r) \leq G(0)$. $\square$

Next we derive an inequality of the Friedrichs-Poincaré type in the space $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$. It is necessary for our aim to get expressions of constants in this inequality. It follows from Proposition 2.1

Lemma 2.3 *Let $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$, $\kappa > 1$. Let δ and ε be arbitrary positive constants, such that $(\beta - \alpha)(2\varepsilon - \delta) \geq 0$. Then for all $\mathbf{u} \in \overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$*

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta-1} \leq c_0 \,\| \nabla\mathbf{u}|\, B_{R_0}\|^2_{2,\alpha,\beta} + c_1 \left\| \nabla\mathbf{u}|\, B^{R_0}\right\|^2_{2,\alpha,\beta}, \tag{2.10}$$

where $c_0 = \left[(\alpha\delta + 2\beta\varepsilon)\,/\,(\beta\beta^*\delta\varepsilon)\right]^2$, $c_1 = \left[(2\kappa)\,/\,(\delta\varepsilon)\right] \cdot \left[(\alpha+\beta)\,/\,(\beta\beta^*)\right]^2$ *and* $R_0 \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right|(\kappa - 1)^{-1}$.

Remark 2.4 Let us observe that if additionally $\delta < 2\varepsilon$ and $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$ then $c_0 \geq c_1$.

Proof of Lemma 2.3 Due to the density of $\mathbf{C}_0^\infty$ in $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$ it is sufficient to prove the inequality for all $\mathbf{u} \in \mathbf{C}_0^\infty$. From Proposition 2.1 it follows that for $\mathbf{v} \in \mathbf{C}_0^\infty$

$$\begin{aligned}
& 2\beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} \leq \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2 \Delta\eta^\alpha_\beta \, d\mathbf{x} \\
= \; & -2\int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}\nabla\, \mathbf{v}\cdot\nabla\eta^\alpha_\beta\, d\mathbf{x} + \int_{\partial B_\rho} \mathbf{v}^2 \nabla\eta^\alpha_\beta\cdot\mathbf{n}\, dS \\
\leq \; & \beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} + \frac{1}{\beta\beta^*\delta\varepsilon}\int_{\mathbb{R}^3\setminus B_\rho} |\nabla\mathbf{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\mathbf{x} \\
& + \int_{\partial B_\rho} \mathbf{v}^2\, \nabla\eta^\alpha_\beta\cdot\mathbf{n}\, dS.
\end{aligned}$$

Hence, because the surface integral is a value of the order $O(\rho^2)$, we have:

$$\beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3} \mathbf{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} \leq \frac{1}{\beta\beta^*\delta\varepsilon}\int_{\mathbb{R}^3} |\nabla\mathbf{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\mathbf{x} \tag{2.11}$$

By means of the Cauchy-Schwarz inequality and from Proposition 2.2 with $\mathbb{R}^3 = B_{R_0} \cup B^{R_0}$, $R_0 \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right| / (\kappa - 1)$ we finally get (2.10). $\square$

We will need some technical lemmas. Let us define $F_{\alpha,\beta}(s, r;\, \nu)$ by the relation:

$$F_{\alpha,\beta}(s, r;\, \nu) \cdot \eta^{\alpha-1}_{\beta-1} \equiv -\nu \left|\nabla\eta^\alpha_\beta\right|^2 / \eta^\alpha_\beta - k\, \partial_1\, \eta^\alpha_\beta \tag{2.12}$$

The following lemma gives the evaluation of $F_{\alpha,\beta}(s, r;\, \nu)$ from below

Lemma 2.5 *Let $0 \leq \alpha < \beta$, $\kappa > 1$, $0 < \varepsilon \leq (1/(2\kappa))\cdot(k/\nu) \cdot ((\beta-\alpha)/\beta^2)$ and δ, ν, $k > 0$. Then*

$$F_{\alpha,\beta}(s, r; \nu) - \left(1 - \kappa^{-1}\right) k\delta\varepsilon\, (\beta - \alpha)\, s \geq -\alpha\delta k \left(1 + \nu k^{-1}\alpha\delta\right) \tag{2.13}$$

for all $r > 0$ and $s \in [0, 2r]$.

Proof. Expressing the function $F_{\alpha,\beta}(s,r;\nu)$ explicitly we get:

$$\begin{aligned} F_{\alpha,\beta}(s,r;\nu) &= -\nu\alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right)-2\nu\alpha\beta\delta\varepsilon\frac{s}{r}-2\nu\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r} \\ &\quad -k\alpha\delta(1+\varepsilon s)\frac{r-s}{r}+k\beta\varepsilon(1+\delta r)\frac{s}{r} \end{aligned}$$

For convenient use we subtract $(1-\kappa^{-1})\,k\delta\varepsilon(\beta-\alpha)\,s$ from $F_{\alpha,\beta}(s,r;\nu)$. We observe (see Appendix A) that, for the given α, β, ε, κ, for all δ, ν, $k>0$ and for $r>0$, $F_{\alpha,\beta}(s,r;\nu)-(1-\kappa^{-1})\,k\delta\varepsilon(\beta-\alpha)\,s \geq F_{\alpha,\beta}(0,r;\nu)$, which immediately gives inequality (2.13). □

The following technical proposition about the existence of a solution of an ordinary differential equation in a space of periodical functions we need in the proof of uniqueness of a solution of problem (1.1)–(1.3), see the proof of Theorem 3.1, and also in the proof of existence of a solution of the problem for checking solenoidality of a constructed solution, see the proof of Theorem 4.4.

Proposition 2.6 *Let $a\in\mathbb{C}$, $Re\ a>0$. Let $f\in C^\infty(\mathbb{R})$ be a 2π-periodical complex function. Then there is unique 2π-periodical solution $g\in C^\infty(\mathbb{R})$ of the equation*

$$g'+a\,g=f$$

and the solution g can be expressed in the following form:

$$g(\varphi)=\left(e^{2\pi a}-1\right)^{-1}\int_0^{2\pi}e^{at}f(\varphi+t)\,dt=e^{-a\varphi}\int_{-\infty}^{\varphi}e^{at}f(t)\,dt$$

Proof of the proposition follows from standard computations.

3 Uniqueness in $\mathbb{R}^3$

In this section we will prove two theorems about uniqueness of a weak solution of problem (1.1)–(1.3). The first method gives the uniqueness in "larger" function spaces. On the other hand the second method can be used without any change also in the case of an exterior domain. In the present paper we need only one of these two uniqueness results. But, the both theorems (the second formulated in an exterior domain) are necessary for extension the results of present paper onto the case of an exterior domain, see [13].

Theorem 3.1 *(Uniqueness in $\mathbb{R}^3$) Let $\{\mathbf{u},p\}$ be a distributional solution of the problem (1.1)–(1.3) with $\mathbf{f}=\mathbf{0}$, $g=0$ such that $\mathbf{u}\in\mathbf{D}_0^{1,2}$ and $p\in L^2_{loc}$. Then $\mathbf{u}=\mathbf{0}$ and $p=const$.*

Proof. From the condition $\mathbf{u}\in\mathbf{D}_0^{1,2}$ we get $\nabla\mathbf{u}\in\mathbf{L}^2$, $\mathbf{u}\in\mathbf{L}^6$, $\mathbf{u}\in\mathcal{S}'$. Because $\operatorname{div}((\omega\times\mathbf{x})\cdot\nabla\mathbf{u}-\omega\times\mathbf{u})=(\omega\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u}=0$, we have $\triangle p=0$. Hence, applying Laplacian and the Fourier transform we get

$$\triangle\left(-\nu\,\Delta\mathbf{u}+k\,\partial_1\mathbf{u}-(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}+\omega\times\mathbf{u}\right)=\mathbf{0},$$

$$|\xi|^2\left(\nu\,|\xi|^2\,\widehat{\mathbf{u}}+i\,k\,\xi_1\widehat{\mathbf{u}}-(\omega\times\xi)\cdot\nabla_\xi\widehat{\mathbf{u}}+\omega\times\widehat{\mathbf{u}}\right)=\mathbf{0}\qquad\text{in }\mathcal{S}'.$$

Assuming the equation in cylindrical coordinates (ξ_1,ρ,φ), and denoting $T(\varphi)\,\widehat{\mathbf{v}}=\widehat{\mathbf{u}}\,(\xi_1,\rho,\varphi)$, where

$$T\,(\varphi)=\begin{bmatrix}1, & 0, & 0\\ 0, & \cos(\varphi)\,, & -\sin(\varphi)\\ 0, & \sin(\varphi)\,, & \cos(\varphi)\end{bmatrix},$$

we get

$$|\xi|^2\left\{-\partial_\varphi\widehat{\mathbf{v}}+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\,(k/\widetilde{\omega})\,\xi_1\right]\widehat{\mathbf{v}}\right\}=\mathbf{0}\ \text{ in }\mathcal{S}'.\tag{3.14}$$

We will show that from this equation follows that $\operatorname{supp}\widehat{\mathbf{v}}\subset\{0\}$, and due to the definition of $\widehat{\mathbf{v}}$ we will have also $\operatorname{supp}\widehat{\mathbf{u}}\subset\{\mathbf{0}\}$. This means that $\mathbf{u}$ is a polynomial of x_1, x_2, x_3. Because $\mathbf{u}\in\mathbf{L}^6$ we get $\mathbf{u}=0$. Substituting into (1.1) we get $\nabla p=0$ and $p=const.$

So, we have to prove that for an arbitrary real vector function $\Psi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ defined for $[\xi_1,\xi_2,\xi_3]\in\mathbb{R}^3$ we have $\langle\widehat{\mathbf{v}},\Psi\rangle=0$. If for each $\Psi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ there is a function $\Phi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ such that

$$\partial_\varphi\left(|\xi|^2\,\Phi\right)+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\ (k/\widetilde{\omega})\,\xi_1\right]\left(|\xi|^2\,\Phi\right)=\Psi\tag{3.15}$$

then from (3.14) follows:

$$\begin{aligned}0&=\left\langle|\xi|^2\left\{-\partial_\varphi\widehat{\mathbf{v}}+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\ (k/\widetilde{\omega})\,\xi_1\right]\widehat{\mathbf{v}}\right\},\ \Phi\right\rangle\\&=\left\langle\widehat{\mathbf{v}},\ \partial_\varphi\left(|\xi|^2\,\Phi\right)+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\ (k/\widetilde{\omega})\,\xi_1\right]\left(|\xi|^2\,\Phi\right)\right\rangle=\langle\widehat{\mathbf{v}},\ \Psi\rangle\end{aligned}$$

Hence, the proof of $\operatorname{supp}\widehat{\mathbf{v}}\subset\{0\}$ is reduced to the solvability of (3.15). First we note that it is sufficient to solve the equation

$$\partial_\varphi\zeta+\left((\nu/\widetilde{\omega})\ |\xi|^2+i\ (k/\widetilde{\omega})\ \xi_1\right)\zeta=\Psi\tag{3.16}$$

because the division on the expression $|\xi|^2$ defines the one-to-one correspondence of the space $\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ onto $\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$.

To analyze the equation (3.16) we assume this equation in cylindrical coordinates $[\xi_1,\rho,\varphi]$, $\rho=(\xi_2^2+\xi_3^2)^{1/2}$. For an arbitrary real vector function $\Psi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ defined for $[\xi_1,\xi_2,\xi_3]\in\mathbb{R}^3$ we define $f\,(t):=\Psi\,(\xi_1,\ \rho\,\cos t,\ \rho\,\sin t)$, $a:=(\nu/\widetilde{\omega})\,|\xi|^2+i\,(k/\widetilde{\omega})\,\xi_1$, assuming $\widetilde{\omega}>0$. Using the Proposition 2.6 we get the solution of (3.16) in the form

$$\begin{aligned}\zeta\,(\xi_1,\rho,\varphi)&=\left\{\exp\left[2\pi\left(\frac{\nu}{\widetilde{\omega}}\,|\xi|^2+i\,\frac{k}{\widetilde{\omega}}\right)\right]-1\right\}^{-1}\\&\cdot\int_0^{2\pi}\exp\left[\left(\frac{\nu}{\widetilde{\omega}}\,|\xi|^2+i\,\frac{k}{\widetilde{\omega}}\xi_1\right)t\right]\ \Psi\,(\xi_1,\ \rho\,\cos(t+\varphi)\,,\ \rho\,\sin(t+\varphi))\ dt.\end{aligned}$$

It is easy to see that function ζ as the function of $[\xi_1,\xi_2,\xi_3]$ is infinitely differentiable with respect to these variables and $\zeta\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$. Finally we put $\Phi=\zeta/\,|\xi|^2$.

□

Theorem 3.2 *Let $\{\mathbf{u},p\}$ be a distributional solution of the problem (1.1)–(1.3) with $\mathbf{f}=\mathbf{0}$ and $g=0$ such that $\mathbf{u}\in\mathbf{V}_{0,0}$ and $p\in L^2_{-1,0}$. Then $\mathbf{u}=\mathbf{0}$ and $p=0$.*

Proof. Let $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ be a non-increasing cut-off function such that $\Phi(z) \equiv 1$ for $z < 1/2$ and $\Phi(z) \equiv 0$ for $z > 1$. Let $|\Phi'| \leq 3$. Let $\Phi_R \equiv \Phi_R(\mathbf{x}) \equiv \Phi(|\mathbf{x}|/R)$. We have $|\nabla \Phi_R| \leq 3/R$ and $|\partial_1 \Phi_R| \leq 3/R$ for $\mathbf{x} \in \mathbb{R}^3$, $R/2 \leq |\mathbf{x}| \leq R$. Let $\{R_j\} \in \mathbb{R}$ be an increasing sequence of radii with the limit $+\infty$. So we have that $\mathbf{u}_j \equiv \mathbf{u} \cdot \Phi_{R_j} \in \overset{\circ}{\mathbf{H}}{}^1$, and $\{\mathbf{u}_j\}$ is a sequence of functions with limit $\mathbf{u}$ in the space $\mathbf{V}_{0,0}$. Using the (non-solenoidal) test functions $\varphi = \mathbf{u}\, \Phi_{R_j}^2 = \mathbf{u}_j\, \Phi_{R_j} \in \overset{\circ}{\mathbf{H}}{}^1$ for equation (1.1) we get:

$$\begin{aligned} &\nu \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla \left(\mathbf{u}\, \Phi_{R_j}^2 \right) d\mathbf{x} + k \int_{\mathbb{R}^3} \partial_1 \mathbf{u} \cdot \mathbf{u}\, \Phi_{R_j}^2\, d\mathbf{x} \\ &+ \int_{\mathbb{R}^3} (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} \cdot \mathbf{u}\, \Phi_{R_j}^2\, d\mathbf{x} + \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u}\, \Phi_{R_j}^2\, d\mathbf{x} = 0 \end{aligned} \tag{3.17}$$

Using in (3.17) relation $\nabla \mathbf{u} \cdot \nabla \left(\mathbf{u}\, \Phi_{R_j}^2 \right) = |\nabla \mathbf{u}_j|^2 - \nabla \Phi_{R_j} \cdot \nabla \Phi_{R_j}\, \mathbf{u}^2$, integrating by parts, we get after some evident rearrangements

$$\begin{aligned} &\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}_j|^2\, d\mathbf{x} - \frac{1}{2} \int_{\mathbb{R}^3} \operatorname{div}(\omega \times \mathbf{x})\, \mathbf{u}_j^2\, d\mathbf{x} \\ &- \frac{k}{2} \int_{\mathbb{R}^3} \mathbf{u}^2\, \partial_1 \Phi_{R_j}^2 d\mathbf{x} - \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{u}^2\, (\omega \times \mathbf{x}) \cdot \nabla \Phi_{R_j}^2 d\mathbf{x} \\ &- \nu \int_{\mathbb{R}^3} \left| \nabla \Phi_{R_j} \right|^2\, \mathbf{u}^2\, d\mathbf{x} - \int_{\mathbb{R}^3} p \mathbf{u} \cdot \nabla \left(\Phi_{R_j}^2 \right) d\mathbf{x} = 0. \end{aligned}$$

$$\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}_j|^2\, d\mathbf{x} \leq C \left(\int_{B_{R_j}^{R_j/2}} \mathbf{u}^2 r^{-1}\, d\mathbf{x} + \int_{B_{R_j}^{R_j/2}} |p|\, |\mathbf{u}|\, r^{-1}\, d\mathbf{x} \right).$$

$\mathbf{u} \in \mathbf{L}^2_{-1,0}$, $p \in L^2_{-1,0}$, $p\mathbf{u} \in \mathbf{L}^1_{-1,0}$. So, for $j \to \infty$ we get $\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, d\mathbf{x} \leq 0$. Hence, the function $\nabla \mathbf{u} = 0$ a.e. in $\mathbb{R}^3$, and this means $\mathbf{u}$ is a constant a.e. in $\mathbb{R}^3$. From $\mathbf{u} \in \mathbf{L}^2_{-1,0}$ follows that $\mathbf{u} = \mathbf{0}$ a.e. in $\mathbb{R}^3$. Using now an arbitrary test function ϕ for equation (1.1), we get $\int_{\mathbb{R}^3} \nabla p\, \phi\, d\mathbf{x} = 0$. So, the function $\nabla p = 0$ a.e. in $\mathbb{R}^3$, and this means p is a constant a.e. in $\mathbb{R}^3$. From $p \in L^2_{-1,0}$ follows that $p = 0$ a.e. in $\mathbb{R}^3$, and the uniqueness is proved. □

4 Existence of a solenoidal solution

In this section we will construct a weak solution of the problem assuming that $g = 0$.

4.1 Existence of the pressure in $\mathbb{R}^3$

If there exist distributions $\mathbf{u}, p$ satisfying

$$\begin{aligned} -\nu\, \Delta \mathbf{u} + k\, \partial_1 \mathbf{u} - (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \mathbb{R}^3 \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

then pressure p satisfies the equation

$$\triangle p = \operatorname{div} \mathbf{f} \tag{4.18}$$

because div $((\omega\times\mathbf{x})\cdot\nabla\mathbf{u}-\omega\times\mathbf{u})=(\omega\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u}=0$, and $\operatorname{div}(\Delta\mathbf{u}+k\,\partial_1\mathbf{u})=0$ provided $\operatorname{div}\mathbf{u}=0$.

Let $\mathcal{E}$ be the fundamental solution of the Laplace equation, i.e. $\mathcal{E}=-1/(4\pi r)$. Assuming firstly $\mathbf{f}\in\mathbf{C}_0^\infty$ we have $p=\mathcal{E}\star\operatorname{div}\mathbf{f}$ and $\nabla p=\nabla\mathcal{E}\star\operatorname{div}\mathbf{f}$ and so, $p=\nabla\mathcal{E}\star\mathbf{f}$ and $\nabla p=\nabla^2\mathcal{E}\star\mathbf{f}$. It is well known that both formulas can be extended for $\mathbf{f}\in\mathbf{L}^2_{\alpha+1,\,\beta}$ with $0<\beta<1$ and $-2<\alpha+\beta<2$ (the last convolution $\nabla p=\nabla^2\mathcal{E}\star\mathbf{f}$ due to the fact that $\nabla^2\mathcal{E}$ is the singular kernel of the Calderon-Zygmund type and that $\eta_\beta^{\alpha+1}$ belongs to the Muckenhoupt class of weights A_2), see [2, Thm. 3.2, Thm 5.5], [15, Thm. 4.4, Thm 5.4], where the theorems are formulated for the pressure part $\mathcal{P}$ of the fundamental solution of the classical Oseen problem, so $\mathcal{P}=\nabla\mathcal{E}$ and $\nabla\mathcal{P}=\nabla^2\mathcal{E}$. For $\mathbf{f}\in\mathbf{L}^2_{\alpha+1,\,\beta}$ we get $p\in L^2_{\alpha,\,\beta-1}$ and $\nabla p\in\mathbf{L}^2_{\alpha+1,\,\beta}$, and there are positive constants C_1, C_2 such that the following estimates are satisfied:

$$\|p\|^2_{2,\alpha,\beta-1}\le C_1\,\|\mathbf{f}\|^2_{2,\alpha+1,\beta}\ ,\quad \|\nabla p\|^2_{2,\alpha+1,\beta}\le C_2\,\|\mathbf{f}\|^2_{2,\alpha+1,\beta} \tag{4.19}$$

4.2 The problem in B_R.

We will study in this section the existence of a weak solution of the following problem in a bounded domain B_R, pressure p is assumed here to be known, the right hand side $\mathbf{f}-\nabla p=\widetilde{\mathbf{f}}\in\mathbf{L}^2_{\alpha+1,\,\beta}$:

$$\begin{aligned} -\nu\Delta\mathbf{u}+k\,\partial_1\mathbf{u}-(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}+\omega\times\mathbf{u} &= \widetilde{\mathbf{f}}\quad\text{in } B_R && (4.20)\\ \mathbf{u} &= 0\quad\text{on } \partial B_R && (4.21)\end{aligned}$$

We show the existence of a weak solution $\mathbf{u}_R\in\overset{\circ}{\mathbf{H}}(B_R)$. Following (1.4), (1.5) again with $w=\eta^0_{\beta_0}$, $\beta_0\in(0,1]$, using notation (2.12), let us introduce a continuous bilinear form $\widetilde{Q}\,(\cdot,\cdot)$ on $\overset{\circ}{\mathbf{H}}(B_R)\times\overset{\circ}{\mathbf{H}}(B_R)$:

$$\begin{aligned}\widetilde{Q}\,(\mathbf{u},\mathbf{v}) &= \int_{B_R}\nu\nabla\mathbf{u}\cdot\nabla\left(\mathbf{v}\cdot\eta^0_{\beta_0}\right)\,d\mathbf{x}+k\int_{B_R}\partial_1\mathbf{u}\cdot\left(\mathbf{v}\eta^0_{\beta_0}\right)\,d\mathbf{x}\\ &\quad+\int_{B_R}(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}\,\left(\mathbf{v}\eta^0_{\beta_0}\right)\,d\mathbf{x},\end{aligned}$$

$$\widetilde{Q}\,(\mathbf{v},\mathbf{v})\ge 2^{-1}\nu\int_{B_R}|\nabla\mathbf{v}|^2\,\eta^0_{\beta_0}d\mathbf{x}+2^{-1}\int_{B_R}\mathbf{v}^2F_{0,\beta_0}\,(s,r;\,\nu)\,\eta^{-1}_{\beta_0-1}d\mathbf{x}. \tag{4.22}$$

Lemma 4.1 *Let $0<\beta_0\le 1$. Then, for all $\widetilde{\mathbf{f}}\in\mathbf{L}^2_{1,\beta_0}\,(B_R)$, $\varepsilon_0<(1/2)\cdot(k/\nu)\cdot(1/\beta_0)$, $\eta^\alpha_{\beta_0}\equiv\eta^{\alpha,\varepsilon_0}_{\beta_0,\varepsilon_0}$, there exists unique $\mathbf{u}_R\in\overset{\circ}{\mathbf{H}}(B_R)$ such that for all $\mathbf{v}\in\overset{\circ}{\mathbf{H}}(B_R)$*

$$\widetilde{Q}\,(\mathbf{u}_R,\mathbf{v})=\int_{B_R}\widetilde{\mathbf{f}}\cdot\mathbf{v}\eta^0_{\beta_0}\,d\mathbf{x}. \tag{4.23}$$

Proof. The bilinear form $\widetilde{Q}$ is coercive, i.e. there exists a constant $C_R>0$ such that $\widetilde{Q}\,(\mathbf{v},\mathbf{v})\ge C_R\,\|\mathbf{v}\|^2$, where $\|\cdot\|$ is here the norm in the space $\overset{\circ}{\mathbf{H}}(B_R)$. Indeed, we get

$$\widetilde{Q}\,(\mathbf{v},\mathbf{v})\ge\frac{\nu}{2}\int_{B_R}|\nabla\mathbf{v}|^2\,\eta^0_{\beta_0}d\mathbf{x}+\frac{1}{2}\int_{B_R}\mathbf{v}^2F_{0,\beta_0}(s,r;\,\nu)\,\eta^{-1}_{\beta_0-1}d\mathbf{x}$$

Because $\varepsilon_0 < (1/2)\cdot(k/\nu)\cdot(1/\beta_0)$ there is a constant κ satisfying all previous conditions and additionally $\varepsilon_0 \leq (1/2\kappa)\cdot(k/\nu)\cdot(1/\beta_0)$. Because $\alpha = 0$ we get from Lemma 2.5

$$\int_{B_R} \mathbf{v}^2 F_{0,\beta_0}(s,r;\,\nu)\,\eta_{\beta_0-1}^{-1} d\mathbf{x} \geq \left(1-\kappa^{-1}\right) k\varepsilon_0^2\beta_0 \int_{B_R} \mathbf{v}^2\eta_{\beta_0-1}^{-1} s\, d\mathbf{x},$$

$$\widetilde{Q}\,(\mathbf{v},\mathbf{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla\mathbf{v}|^2\,\eta_{\beta_0}^0 d\mathbf{x} + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} \mathbf{v}^2\eta_{\beta_0-1}^{-1}\,(\varepsilon_0 s)\; d\mathbf{x}.$$

Using Lemma 2.3 and Remark 2.4 we derive:

$$\begin{aligned}
\widetilde{Q}\,(\mathbf{v},\mathbf{v}) \;&\geq\; \frac{\nu}{4}\int_{B_R} |\nabla\mathbf{v}|^2\,\eta_{\beta_0}^0\, d\mathbf{x} + \frac{\nu}{16}\varepsilon_0^2\beta_0^2\int_{B_R} \mathbf{v}^2\eta_{\beta_0-1}^{-1} d\mathbf{x} \\
&\quad + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0\int_{B_R} \mathbf{v}^2\eta_{\beta_0-1}^{-1}\,(\varepsilon_0 s)\; d\mathbf{x} \\
&\geq\; \left(1-\frac{1}{\kappa}\right)\frac{\nu}{4}\min\left\{1,\,\frac{1}{4}\varepsilon_0^2\beta_0^2,\,2\frac{k}{\nu}\beta_0\varepsilon_0\right\} \qquad (4.24)\\
&\qquad\cdot\left(\int_{B_R} |\nabla v|^2\,\eta_{\beta_0}^0 d\mathbf{x} + \int_{B_R} \mathbf{v}^2\eta_{\beta_0}^{-1} d\mathbf{x}\right)
\end{aligned}$$

$$\widetilde{Q}\,(\mathbf{v},\mathbf{v}) \geq C_R\left(\int_{B_R} |\nabla\mathbf{v}|^2\, d\mathbf{x} + \int_{B_R} \mathbf{v}^2 d\mathbf{x}\right) = C_R\,\|\mathbf{v}\|^2, \qquad (4.25)$$

where $C_R = (\nu/4)\cdot(1-\kappa^{-1})\cdot\min\{1,\,\varepsilon_0^2\beta_0^2/4,\,2\,(k/\nu)\,\beta\varepsilon_0\}\cdot(1+\varepsilon_0\,R)$. Using Lax-Milgram theorem we get that there is $\mathbf{u}_R \in \overset{\circ}{\mathbf{H}}(B_R)$ such that (4.23) is satisfied. $\square$

Remark 4.2 An arbitrary function $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}(B_R)$ *can be expressed in the form* $\mathbf{\Phi} = \phi\,\eta_{\beta_0}^0$, *where* ϕ *is a function from* $\overset{\circ}{\mathbf{H}}(B_R)$. *Therefore we have for* $\mathbf{u}_R$

$$Q\,(\mathbf{u}_R,\mathbf{\Phi}) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{\Phi}\, d\mathbf{x}, \qquad (4.26)$$

for an all $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}(B_R)$ *where by the definition* $Q\,(\mathbf{u}_R,\mathbf{\Phi}) \equiv Q\left(\mathbf{u}_R,\phi\cdot\eta_{\beta_0}^0\right) \equiv \widetilde{Q}\,(\mathbf{u}_R,\phi)$.

4.3 Uniform estimates of $\mathbf{u}_R$

Our next aim is to prove that the weak solutions $\mathbf{u}_R$ of (4.23) are uniformly bounded in $\mathbf{V}_{\alpha,\beta}$ as $R \to +\infty$.

Let y_1 be the unique real solution of the algebraic equation $4y^3+8y^2+5y-1=0$. It is easy to verify that $y_1 \in (0,1)$. We will explain later, why the control of α/β by y_1 is necessary.

Lemma 4.3 *Let* $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$, $\widetilde{\mathbf{f}} \in \mathbf{L}^2_{\alpha+1,\beta}$. *Then, as* $R \to +\infty$, *the weak solutions* $\mathbf{u}_R$ *of (4.23) given by Lemma 4.1 are uniformly bounded in* $\mathbf{V}_{\alpha,\beta}$. *There is a constant* $c > 0$, *which does not depend on* R *such that*

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_R^2\eta_\beta^{\alpha-1} d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\tilde{\mathbf{u}}_R|^2\,\eta_\beta^\alpha d\mathbf{x} \leq c\int_{\mathbb{R}^3} \left|\widetilde{\mathbf{f}}\right|^2\,\eta_\beta^{\alpha+1} d\mathbf{x} \qquad (4.27)$$

for all R *greater than some* $R_0 > 0$, $\tilde{\mathbf{u}}_R$ *being extension by zero of* $\mathbf{u}_R$ *on* $\mathbb{R}^3 \setminus B_R$.

Proof. First, we derive estimate of $\mathbf{u}_R$ on a bounded subdomain $B_{R_0} \subset B_R$; the choice of R_0 will be given in the next part of the proof. Our aim is to get an estimate with a constant not depending on R. Let us substitute $\phi = \mathbf{u}_R$ into (4.23). Hence, we get from (4.24):

$$\widetilde{Q}\left(\mathbf{u}_R, \mathbf{u}_R\right) = \int_{B_R} \widetilde{\mathbf{f}}\, \mathbf{u}_R \eta^0_{\beta_0} d\mathbf{x} \geq C_1 \left(\int_{B_R} \left|\nabla \mathbf{u}_R\right|^2 \eta^0_{\beta_0} d\mathbf{x} + \int_{B_R} \mathbf{u}_R^2 \eta^{-1}_{\beta_0} d\mathbf{x} \right),$$

with the constant $C_1 > 0$ stated in (4.24). Let R_0 be some fixed positive number such that $0 < R_0 < R$. We get

$$\int_{B_R} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} + \int_{B_R} \mathbf{u}_R^2 \eta^{\alpha-1}_\beta d\mathbf{x} \leq C_2 \int_{B_R} \left|\widetilde{\mathbf{f}}\right| \left|\mathbf{u}_R\right| \eta^\alpha_\beta d\mathbf{x}, \tag{4.28}$$

where the constant $C_2 = C_1^{-1} \left(1 + \varepsilon_0\, R_0\right)^\alpha \left(1 + \varepsilon_0\, 2\, R_0\right)^{|\beta - \beta_0|}$ depend on k, ν, α, β, β_0, ε_0, R_0, κ, but does not depend on R.

Now, we are going to derive an estimate of $\mathbf{u}_R$ on domain B_R. Using the test function $\mathbf{\Phi} = \mathbf{u}_R \eta^\alpha_\beta = \mathbf{u}_R (1 + \delta r)^\alpha (1 + \varepsilon s)^\beta \in \overset{\circ}{\mathbf{H}}(B_R)$ in (4.26) we get after integration by parts:

$$\nu \int_{B_R} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} + \nu \int_{B_R} \mathbf{u}_R \nabla \mathbf{u}_R \cdot \nabla \eta^\alpha_\beta \, d\mathbf{x} - \frac{k}{2} \int_{B_R} \mathbf{u}_R^2\, \partial_1 \eta^\alpha_\beta d\mathbf{x}$$
$$= \int_{B_R} \widetilde{\mathbf{f}} \mathbf{u}_R \eta^\alpha_\beta d\mathbf{x}$$

So, we get for some $\kappa > 1$:

$$\frac{\nu}{2} \int_{B_R} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} + \frac{1}{2} \int_{B_R} \mathbf{u}_R^2 F_{\alpha,\beta}(s, r;\, \nu) \eta^{\alpha-1}_{\beta-1} d\mathbf{x} \leq \int_{B_R} \left|\widetilde{\mathbf{f}}\right| \left|\mathbf{u}_R\right| \eta^\alpha_\beta d\mathbf{x}$$

Let $R_0 \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right| (\kappa - 1)^{-1}$. Using Lemma 2.5 (with $0 \leq \alpha < \beta$, $\varepsilon \leq \left(1/\left(2\kappa\right)\right) \left(k/\nu\right) \left(\left(\beta - \alpha\right)/\beta^2\right)$) and Lemma 2.3 (with $\delta < 2\varepsilon$), the second term in the previous estimate can be evaluated from below:

$$\begin{aligned} & \int_{B_R} \mathbf{u}_R^2\, F_{\alpha,\beta}(s, r;\, \nu)) \eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} \\ \geq \;& -\alpha\delta k \left(1 + \frac{\nu\kappa}{k}\alpha\delta\right) \frac{2\kappa}{\delta\varepsilon} \left(\frac{\alpha+\beta}{\beta\beta^*}\right)^2 \int\limits_{B_R^{R_0}} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} \\ & + \left(1 - \kappa^{-1}\right) k\delta\varepsilon \left(\beta - \alpha\right) \int_{B_R^{R_0}} \mathbf{u}_R^2 \eta^{\alpha-1}_{\beta-1} s\, d\mathbf{x} - 2C_4 \int_{B_{R_0}} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} \end{aligned}$$

Denote $C_5 = \alpha\delta k \left(1 + \kappa \left(\nu/\kappa\right) \alpha\delta\right) \left(\kappa/\left(\delta\, \varepsilon\right)\right) \left(\left(\alpha+\beta\right)/\left(\beta\beta^*\right)\right)^2$. It is clear that $C_5 \leq \nu/\left(2\, \kappa^2\right) < \nu/\left(2\, \kappa\right)$ if $1 + \nu\kappa\alpha\delta/k \leq \kappa$ (i.e. $\delta \leq \left(k/\nu\right) \cdot \left(\left(\kappa - 1\right)\right)/\left(\kappa\beta\right)$) and $\alpha \leq \left(1/\left(2\kappa^4\right)\right) \cdot \left(\nu/k\right) \cdot \left(\left(\beta\, \beta^*\right)/\left(\alpha+\beta\right)\right)^2 \varepsilon$. We have

$$\frac{\nu}{2\kappa} \int_{B_R} \left|\nabla \mathbf{u}_R\right|^2\; \eta^\alpha_\beta\, d\mathbf{x} + \frac{1}{2}\left(1 - \frac{1}{\kappa}\right) k\delta\varepsilon \left(\beta - \alpha\right) \int_{B_R} \mathbf{u}_R^2 \eta^{\alpha-1}_{\beta-1}\, s\, d\mathbf{x}$$
$$-C_6 \int_{B_{R_0}} \mathbf{u}_R^2 \eta^{\alpha-1}_{\beta-1} d\mathbf{x} - C_7 \int_{B_{R_0}} \left|\nabla \mathbf{u}_R\right|^2 \eta^\alpha_\beta d\mathbf{x} \leq \int_{B_R} \left|\widetilde{\mathbf{f}}\right| \left|\mathbf{u}_R\right| \eta^\alpha_\beta d\mathbf{x}.$$

We use now relation (4.28) in order to estimate the integrals computed on the domain B_{R_0}. Before using the mentioned inequality we should re-scale it with respect to new values ε, δ, see Remark 1.1. The new constant in (4.28) after rescaling we denote C_2'.

$$\frac{\nu}{\kappa}\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha d\mathbf{x}+k\delta\varepsilon\,(\beta-\alpha)\int_{B_R}\mathbf{u}_R^2\eta_{\beta-1}^{\alpha-1}s d\mathbf{x}\le C_8\int_{B_R}\left|\widetilde{\mathbf{f}}\right||\mathbf{u}_R|\,\eta_\beta^\alpha d\mathbf{x},$$

where $C_8=\{1+C_2'\max(C_6,\,C_7)\}\cdot 2\cdot(1-\kappa^{-1})^{-1}$. We use Lemma 2.3 and Remark 2.4. So, if $\delta<2\varepsilon$ and $1<\kappa\le 2\varepsilon/\delta+\delta/(2\varepsilon)-1$ we get

$$\frac{\nu}{2\,\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta+2\beta\varepsilon}\right)^2\int_{B_R}\mathbf{u}_R^2\,\eta_{\beta-1}^{\alpha-1}\,d\mathbf{x}\ \le\ \frac{\nu}{2\kappa}\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha\,d\mathbf{x},$$

$$\begin{aligned}&\frac{\nu}{2\kappa}\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha dx+\frac{\nu}{2\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta+2\beta\varepsilon}\right)^2\int_{B_R}\mathbf{u}_R^2\eta_{\beta-1}^{\alpha-1}\,d\mathbf{x}\\&+k\delta\varepsilon\,(\beta-\alpha)\int_{B_R}\mathbf{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\mathbf{x}\le C_8\int_{B_R}\left|\widetilde{\mathbf{f}}\right||\mathbf{u}_R|\,\eta_\beta^\alpha d\mathbf{x}.\end{aligned}$$

So we get

$$\begin{aligned}&\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha d\mathbf{x}+2\int_{B_R}\mathbf{u}_R^2\eta_{\beta-1}^{\alpha-1}\,d\mathbf{x}+2\varepsilon\int_{B_R}\mathbf{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\mathbf{x}\\&=\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha d\mathbf{x}+2\int_{B_R}\mathbf{u}_R^2\eta_\beta^{\alpha-1}\,d\mathbf{x}\le C_{10}\int_{B_R}\left|\widetilde{\mathbf{f}}\right||\mathbf{u}_R|\,\eta_\beta^\alpha d\mathbf{x},\end{aligned}$$

$C_9=\min\left(\nu/(2\kappa),(\nu/(2\kappa))\ (\beta\beta^*\delta\varepsilon/(\alpha\delta+2\beta\varepsilon))^2,k\delta\,(\beta-\alpha)\ /2\right)$ and $C_{10}=C_8/C_9$. We have also:

$$\int_{B_R}\left|\widetilde{\mathbf{f}}\right||\mathbf{u}_R|\,\eta_\beta^\alpha dx\le\frac{t}{2}\int_{B_R}\mathbf{u}_R^2\eta_\beta^{\alpha-1}d\mathbf{x}+\frac{1}{2t}\int_{B_R}\left|\widetilde{\mathbf{f}}\right|^2\eta_\beta^{\alpha+1}dx$$

So, if we choose $t=2\cdot C_{10}^{-1}$ then we get :

$$\int_{B_R}|\nabla \mathbf{u}_R|^2\,\eta_\beta^\alpha d\mathbf{x}+\int_{B_R}\mathbf{u}_R^2\,\eta_\beta^{\alpha-1}d\mathbf{x}\le c\int_{\mathbb{R}^3}\left|\widetilde{\mathbf{f}}\right|^2\eta_\beta^{\alpha+1}d\mathbf{x},$$

It can be easily shown that the all conditions on α, β, δ, ε, κ used in the proof are compatible if $0\le\alpha<y_1\beta$, see Appendix B. □

4.4 The problem in $\mathbb{R}^3$ with zero divergence

Let y_1 be the same as in Lemma 4.3.

Theorem 4.4 *(Existence and uniqueness) Let $0<\beta\le 1$, $0\le\alpha<y_1\beta$, $\mathbf{f}\in\mathbf{L}^2_{\alpha+1,\beta}$. Then there exists a unique weak solution $\{\mathbf{u},\,p\}$ of the problem*

$$\begin{aligned}-\nu\,\Delta\mathbf{u}+k\partial_1\mathbf{u}-(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}+\omega\times\mathbf{u}+\nabla\mathbf{p}&=\mathbf{f}\ \ in\ \mathbb{R}^3, &(4.29)\\ div\,\mathbf{u}&=0\ \ in\ \mathbb{R}^3 &(4.30)\end{aligned}$$

such that $\mathbf{u}\in\mathbf{V}_{\alpha,\beta}$, $p\in L^2_{\alpha,\beta-1}$, $\nabla p\in\mathbf{L}^2_{\alpha+1,\beta}$ and

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta}+\|\nabla\mathbf{u}\|^2_{2,\alpha,\beta}+\|p\|^2_{2,\alpha,\beta-1}+\|\nabla p\|^2_{2,\alpha+1,\beta}\le C\,\|\mathbf{f}\|^2_{2,\alpha+1,\beta}\,. \qquad (4.31)$$

Proof. Existence. Let p be the same as in Subsection 4.1. Let $R_n \subset \mathbb{R}$, $R_n > 0$, $n \in \mathbb{N}$ be a sequence converging to $+\infty$. Let $\mathbf{u}_{R_n}$ be the weak solution of (4.20), (4.21) on B_{R_n}. Extending $\mathbf{u}_{R_n}$ by zero on $\mathbb{R}^3 \setminus B_{R_n}$ to a function $\tilde{\mathbf{u}}_\mathbf{n} \in \mathbf{V}_{\alpha,\beta}$ we get a bounded sequence $\{\tilde{\mathbf{u}}_\mathbf{n}\}$ in $\mathbf{V}_{\alpha,\beta}$. Thus, there is a subsequence $\tilde{\mathbf{u}}_{\mathbf{n_k}}$ of $\tilde{\mathbf{u}}_\mathbf{n}$ with a weak limit $\mathbf{u}$ in $\mathbf{V}_{\alpha,\beta}$. Obviously, $\mathbf{u}$ is a weak solution of (4.29) and

$$\begin{aligned} \|\mathbf{u}\|^2_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|^2_{2,\alpha,\beta} &\leq \liminf_{k\in\mathbb{N}} \left(\int_{\mathbb{R}^3} \tilde{\mathbf{u}}^2_{n_k} \eta_\beta^{\alpha-1}\, d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla \tilde{\mathbf{u}}_{n_k}|^2\, \eta^\alpha_\beta \, d\mathbf{x} \right) \\ &\leq c \left|\widetilde{\mathbf{f}}\right|^2 \eta_\beta^{\alpha+1}\, d\mathbf{x} = c \int_{\mathbb{R}^3} |\mathbf{f} - \nabla p|^2 \; \eta_\beta^{\alpha+1}\, d\mathbf{x}. \end{aligned}$$

Taking into account also relation (4.19) we get (4.31).

Let us also check that for $\mathbf{u}$ the equation (4.30) is satisfied. Let us mention that $\mathbf{u} \in \mathbf{H}^2_{loc}$ because $\mathbf{f} - \nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$. So, computing the divergence of (4.29) we get

$$-\nu\, \Delta\, (\operatorname{div} \mathbf{u}) + k\partial_1\, (\operatorname{div} \mathbf{u}) - (\omega \times \mathbf{x}) \cdot \nabla\, (\operatorname{div} \mathbf{u}) = \operatorname{div} \mathbf{f} - \triangle \mathbf{p} \tag{4.32}$$

in distributional sense. From (4.18) and (4.31) we have

$$-\nu\, \Delta \gamma + k\partial_1 \gamma - (\omega \times \mathbf{x}) \cdot \nabla \gamma = 0$$

for $\gamma = \operatorname{div} \mathbf{u} \in L^2_{\alpha,\beta} \subset L^2$. Using Fourier transform we get

$$\left(\nu\, |\xi|^2 + i\, k\, \xi_1\right) \widehat{\gamma} - (\omega \times \xi) \cdot \nabla_\xi \widehat{\gamma} = 0 \quad \text{in} \;\; \mathcal{S}'.$$

Assuming $\widehat{\gamma}$ in cylindrical coordinates $[\xi_1, \rho, \varphi]$, $\rho = \left(\xi_2^2 + \xi_3^2\right)^{1/2}$, we can overwrite the equation in the form:

$$-\partial_\varphi \widehat{\gamma} + \left[(\nu/\widetilde{\omega})\; |\xi|^2 + i\; (k/\widetilde{\omega})\; \xi_1\right] \widehat{\gamma} = 0.$$

Using the same approach as in the proof of the uniqueness Theorem 3.1 we prove that $\operatorname{supp} \widehat{\gamma} \subset \{0\}$. The proof of this fact is reduced to the solvability of the equation (3.16) which was proved for arbitrary $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ in the proof of Theorem 3.1. So, by the same procedure we derive that γ is a polynomial in $\mathbb{R}^3$ and because $\gamma \in L^2$ we get $\gamma \equiv 0$, i.e. (4.30). The uniqueness of the solution follows from Theorem 3.1.

□

5 The problem with non-zero divergence

First of all let us formulate the lemma which will be used for the extension of our results to the case with nonzero divergence:

Lemma 5.1 *(M.E. Bogovski, G.P. Galdi, H. Sohr)*

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and $1 < q < \infty$, $n \in \mathbb{N}$. Then for each $g \in W_0^{k,q}(\Omega)$ with $\int_\Omega g\, dx = 0$, there exists $\mathbf{G} \in \left(W_0^{k+1,q}(\Omega)\right)^n$ satisfying

$$div\, \mathbf{G} = g\,, \qquad \|\mathbf{G}\|_{\left(W_0^{k+1,q}(\Omega)\right)^n} \leq C\, \|g\|_{W_0^{k,q}(\Omega)}$$

with some constant $C = C(q,\, k,\, \Omega) > 0$.

For the proof and further references see e.g. [20, Lemma 2.3.1]. We will prove the following theorem:

Theorem 5.2 *(Existence and uniqueness) Let* $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W_0^{1,2}$ *with* $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, *and* $\int_{\mathbb{R}^3} g\,dx = 0$. *Then there exists a unique weak solution* $\{\mathbf{u}, p\}$ *of the problem*

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + k\partial_1\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla\mathbf{p} &= \mathbf{f} \quad in\ \mathbb{R}^3, \\ div\,\mathbf{u} &= g \quad in\ \mathbb{R}^3 \end{aligned}$$

such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ *and*

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|^2_{2,\alpha,\beta} + \|p\|^2_{2,\alpha,\beta-1} + \|\nabla p\|^2_{2,\alpha+1,\beta} \le C\left(\|\mathbf{f}\|^2_{2,\alpha+1,\beta} + \|g\|^2_{1,2}\right).$$

Proof. Using Lemma 5.1 we find $\mathbf{G} \in \mathbf{W}_0^{2,2}$, $\operatorname{supp}\mathbf{G} \subset \mathcal{K}$, where $\mathcal{K}$ is a bounded Lipschitz domain containing in $\varepsilon-$neighbourhood $\mathcal{K}_\varepsilon$ of compact set K for an arbitrary $\varepsilon > 0$, $\operatorname{div}\mathbf{G} = g$, $\|\mathbf{G}\|_{2,2} \le C\,\|g\|_{1,2}$. Let us assume the following problem

$$\begin{aligned} -\nu\,\Delta\mathbf{U} + k\,\partial_1\mathbf{U} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{U} + \omega\times\mathbf{U} + \nabla p &= \mathbf{F} \qquad \text{in } \mathbb{R}^3 \\ \operatorname{div}\mathbf{U} &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

where $\mathbf{U} = \mathbf{u} + \mathbf{G}$, $\mathbf{F} = \mathbf{f} - \nu\,\Delta\mathbf{G} + k\,\partial_1\mathbf{G} + (\omega\times\mathbf{x})\cdot\nabla\mathbf{G} - \omega\times\mathbf{G}$ with $\mathbf{G} \in \mathbf{W}_0^{2,2}$, function $\mathbf{G}$ has a compact support, and $\|\mathbf{G}\|_{2,2} \le C\,\|g\|_{1,2}$. The assertion of Theorem 5.2 follows from Theorem 4.4. □

Appendix A

Relation (2.13) follows from an estimate of the derivative of F_1 :

$$\begin{aligned} \frac{\partial}{\partial s}F_1(s,r) &\equiv \frac{\partial}{\partial s}\left\{F_{\alpha,\beta}(s,r;\,\nu) - \left(1-\kappa^{-1}\right)k\delta\varepsilon\,(\beta-\alpha)\,s\right\} \\ &= -\nu\alpha^2\delta^2\varepsilon\frac{1}{1+\delta r} - 2\nu\alpha\beta\delta\varepsilon\frac{1}{r} - 2\nu\beta^2\varepsilon^2\frac{1+\delta r}{r}\frac{1}{(1+\varepsilon s)^2} \\ &\quad -k\alpha\delta\varepsilon + k\alpha\delta\frac{1}{r}(1+2\varepsilon s) + k\beta\varepsilon\,(1+\delta r)\,\frac{1}{r} \\ &\quad -\left(1-\kappa^{-1}\right)k\delta\varepsilon\,(\beta-\alpha) \\ &\ge \delta\varepsilon\left\{r^{-1}\left[k\,(\alpha/\varepsilon+\beta/\delta) - \nu\alpha^2 - 2\nu\alpha\beta - 2\nu\beta^2\varepsilon/\delta\right]\right. \\ &\quad \left. + \left[-2\nu\beta^2\varepsilon + k\,(\beta-\alpha)/\kappa\right]\right\} \ \ge\ 0 \end{aligned}$$

The last inequality follows from the fact that we have $k\alpha/\varepsilon \ge \nu\,\alpha^2 + 2\,\nu\,\alpha\,\beta$, $k\beta/\delta \ge 2\,\nu\,\beta^2\varepsilon/\delta$, $k\,(\beta-\alpha)/\kappa \ge 2\nu\beta^2\varepsilon$ if $\varepsilon \le (1/(2\kappa))\,(k/\nu)\,((\beta-\alpha)/\beta^2)$. Hence, if the last inequality (which is included in the conditions of Lemma 2.5) is satisfied then $(\partial/\partial s)\,F_1(s,r) \ge 0$. So, we get immediately:

$$F_1(s,r) \ge F_1(0,r) \equiv -k\alpha\delta - \nu\alpha^2\delta^2\,(1+\delta r)^{-1} \ge -\alpha\delta k\left(1+\nu k^{-1}\alpha\delta\right)$$

Appendix B

Let us show that all conditions on α, β, δ, ε, κ used in the proof of Lemma 4.3 are compatible if $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$. Let us collect these assumptions: $0 < \delta < 2\varepsilon$, $1 < \kappa \le 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$, $0 \le \alpha < \beta$, $\varepsilon \le (1/(2\kappa^2)) \cdot (k/\nu) \cdot ((\beta-\alpha)/\beta^2)$, $\delta \le (k/\nu) \cdot (\kappa - 1)/(\kappa\beta)$, $\alpha \le (1/(2\kappa^4)) \cdot (k/\nu) \cdot (\beta\beta^*/(\alpha+\beta))^2 \varepsilon$.

From $\alpha \le (1/(2\kappa^4)) \cdot (k/\nu) \cdot (\beta\beta^*/(\alpha+\beta))^2 \varepsilon$, and $\varepsilon \le (1/(2\kappa^2)) \cdot (k/\nu) \cdot ((\beta-\alpha)/\beta^2)$ we get $\alpha \le (1/(4\kappa^6)) \cdot (\beta^*)^2 (\beta-\alpha)/(\alpha+\beta)^2$. So we get ($\kappa > 1$, $\beta \le 1$): $\alpha/\beta \le (1/(4\kappa^6))(1-\alpha/\beta)/(1+\alpha/\beta)^2$. By substitution $y = \alpha/\beta$ we get the inequality

$$4y^3 + 8y^2 + 4y + \kappa^{-6} \cdot (y-1) \le 0. \tag{5.33}$$

Taking into account the condition $0 \le \alpha < \beta$ we seek for solutions from $[0,1)$. It is clear that the equation $4y^3 + 8y^2 + y + \kappa^{-6}(y-1) = 0$ has a unique real solution $y_\kappa \in (0,1)$ for $\kappa > 1$. It is also clear that arbitrary $y \in [0, y_\kappa)$ solves (5.33). The value y_κ as a function of κ is decreasing. For $\kappa \to 1$ we get the inequality $4y^3 + 8y^2 + 5y - 1 \le 0$. This respective equation has a unique solution $y_1 = \left(\sqrt{13}/\left(6\sqrt{6}\right) + 53/216\right)^{1/3} + (1/30)\left(\sqrt{13}/\left(6\sqrt{6}\right) + 53/216\right)^{-1/3}$. Approximately, with an error less than 10^{-8} we have $y_1 \doteq 0.1582981$, ($y_1 > 1/7$). If $0 \le \alpha < y_1\beta$ then there is $\kappa > 1$ sufficiently close to number 1, such that $0 \le \alpha \le y_\kappa\beta$, so the relation $\alpha \le (1/(4\kappa^6)) \cdot (\beta^*)^2 (\beta-\alpha)/(\alpha+\beta)^2$ is satisfied. Then we can define $\varepsilon = 1/(2\kappa^2) \cdot (k/\nu) \cdot ((\beta-\alpha)/(\beta^2))$. The relation $\varepsilon \le (1/(2\kappa)) \cdot (k/\nu) \cdot (1/\beta)$ is satisfied. Then we take sufficiently small $\delta > 0$ such that $0 < \delta < 2\varepsilon$ and $1 < \kappa \le 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$. Hence, all conditions which we assume in the proof of Lemma 4.3 are satisfied.

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