Navier–Stokes' Equation with the Generalized Impermeability Boundary Conditions and Initial Data in Domains of Powers of the Stokes Operator

Petr Kučera[1], Jiří Neustupa[2], Patrick Penel[3]

Abstract

We deal with the Navier–Stokes equation with the generalized impermeability boundary conditions. We give basic information on these boundary conditions and we further study dynamical properties of solutions, like stability, fast decay and similar. The used norms are graph norms of powers of the Stokes operator S.

MSC 2000: Primary 76D05; Secondary 35Q30

Keywords: Navier–Stokes equation, generalized impermeability boundary conditions, Stokes operator

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with the boundary $\partial\Omega$ of the Hölder class $C^{2+\beta}$ for some $\beta > 0$. Suppose that $0 < T \leq +\infty$. Put $Q_T = \Omega \times (0,T)$. We deal with the initial–boundary value Navier–Stokes problem

$$\partial_t \boldsymbol{u} + \nu\, \mathbf{curl}^2 \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla q \;=\; \boldsymbol{f} \qquad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} \;=\; 0 \qquad \text{in } Q_T, \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} \;=\; 0, \quad \mathbf{curl}\, \boldsymbol{u}\cdot\boldsymbol{n} \;=\; 0, \quad \mathbf{curl}^2 \boldsymbol{u}\cdot\boldsymbol{n} \;=\; 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$\boldsymbol{u}(0) \;=\; \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{1.4}$$

We denote by $\boldsymbol{u}$ the velocity, by q the sum $p + \frac{1}{2}\,|\boldsymbol{u}|^2$ where p is the pressure, ν denotes the kinematic coefficient of viscosity, $\boldsymbol{f}$ is a specific body force and $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.

The boundary conditions (1.3) were introduced in [2] and we call them the *generalized impermeability boundary conditions.* We do not solve the question which boundary condition is more or less appropriate in which situation. Various physical considerations indicate that the answer is not simple and it depends on the actual smoothness of the wall which creates the boundary of the flow field, on mechanical and geometrical properties of particles of the

[1]Czech Technical University, Fac. of Civil Engineering, Department of Mathematics, Thákurova 7, 166 29 Praha 6, Czech Republic, e-mail: kucera@mat.fsv.cvut.cz

[2]Mathematical Institute of Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic, e-mail: neustupa@math.cas.cz

[3]Université du Sud Toulon–Var, Département de Mathématique, B.P. 20132, 83957 La Garde Cedex, France, e-mail: penel@univ-tln.fr

fluid and on the type of the considered flow. Our claim in this paper and in other papers where we deal with the generalized impermeability boundary conditions (1.3) is to present them as a logically correct alternative to the no–slip boundary condition

$$\boldsymbol{u} \,=\, \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T) \tag{1.5}$$

or to Navier's boundary condition

$$a\,\boldsymbol{u} + b\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} \,=\, \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T) \tag{1.6}$$

and to show that the conditions (1.3) enable us to obtain at least the same qualitative results as the conditions (1.5) or (1.6).

We define the weak problem corresponding to (1.1)–(1.4) in Section 2 and we explain that the third condition in (1.3), although not explicitly involved in the weak formulation, naturally follows from the weak formulation and the first two conditions in (1.3) if a weak solution is "smooth". The physical meaning of the generalized impermeability boundary conditions (1.3) is explained in subsection 2.2. In subsection 2.3, we show that the conditions (1.3) naturally induce boundary conditions of the same type for vorticity and a Neumann–type boundary condition for pressure which is simpler than the same condition obtained in the case of the no–slip boundary condition (1.5) for velocity.

Section 3 is devoted to some properties of powers of the Stokes operator S (defined on a set of functions satisfying the boundary conditions (1.3)). A theorem on stability of a solution with respect to small perturbations of initial data and the acting body force is derived in Section 4. The perturbations of the initial velocity are measured in the graph norm of operator $S^{1/4}$. Finally, in Section 5, we prove a theorem which provides the existence of a solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) whose norm $\|S^{\alpha}\boldsymbol{v}(0)\|_2$ (with $\frac{1}{4} < \alpha \leq \frac{1}{2}$) is arbitrarily large and whose values $\boldsymbol{v}(t)$ (for t in a time interval whose distance from zero is arbitrarily small) belong to an arbitrarily chosen open set U in the space $D(S^{\gamma})$ (with $\frac{3}{4} < \gamma < 1$).

We use the following notation:

- $L^2_{\sigma}(\Omega)$ is a closure of $\{\boldsymbol{v} \in C_0^{\infty}(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0\}$ in $L^2(\Omega)^3$. It is the Hilbert space of divergence–free (in the sense of distributions) vector functions $\boldsymbol{v}$ in $L^2(\Omega)^3$ such that $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces. (Here we use the existence of a continuous operator of traces from the space $L^2_{\mathrm{div}}(\Omega) := \{\boldsymbol{v} \in L^2(\Omega)^3;\ \operatorname{div}\boldsymbol{v} \in L^2(\Omega)\}$ to $W^{-1/2,2}(\partial\Omega)$ which assigns to each smooth function $\boldsymbol{v}$ from $L^2_{\mathrm{div}}(\Omega)$ the normal component of $\boldsymbol{v}$ on $\partial\Omega$.)
- The scalar product in $L^2(\Omega)^3$ (and particularly also in $L^2_{\sigma}(\Omega)$) is denoted by $(\,.\,,\,.\,)_2$ and the associated norm is $\|\,.\,\|_2$.
- P_{σ} is the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_{\sigma}(\Omega)$.
- $L^2_{\sigma}(\Omega)^{\perp}$ is the orthogonal complement to $L^2_{\sigma}(\Omega)$ in $L^2(\Omega)^3$. It coincides with $\{\nabla\varphi;\ \varphi \in W^{1,2}(\Omega)\}$.
- $\|\,.\,\|_s$ denotes the norm in $L^s(\Omega)$ and $\|\,.\,\|_{k,s}$ is the norm in the Sobolev space $W^{k,s}(\Omega)$.
- The norms of vector–valued or tensor–valued functions are denoted in the same way as the norms of scalar functions.
- D^1 is the set of functions $\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_{\sigma}(\Omega)$ such that $(\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces. D^1 is a closed subspace of $W^{1,2}(\Omega)^3$.
- D^{-1} is the dual to D^1. The duality between the elements of D^{-1} and D^1 is denoted by $\langle\,.\,,\,.\,\rangle_{\Omega}$ and the norm in D^{-1} is denoted by $\|\,.\,\|_{-1,2}$.

- $A := \mathbf{curl}\,|_{D^1}$ (Thus the domain $D(A)$ of operator A coincides with space D^1.)
- D^2 denotes the domain of the operator A^2. It is proved in [2] that D^2 is a set of all divergence–free functions from $W^{2,2}(\Omega)^3$ that satisfy the boundary conditions (1.3). Note that $A^2 = \mathbf{curl}^2 = -\Delta$ on D^2.
- $S := A^2$ (S represents one of possible concrete realizations of the Stokes operator.)
- $\mathbb{Z}^*$ denotes the set of all integer numbers without zero: $\mathbb{Z}^* := \{\dots,\, -2,\, -1,\, 1,\, 2,\, \dots\}$.
- C denotes a generic constant, i.e. a constant whose value may change from line to line. C may depend on Ω, T or on other parameters, but it never depends on a concrete function. On the other hand, numbered constants have fixed values throughout the whole paper.

The next lemma brings some results from [2].

Lemma 1.1 *a) Space D^1 can be characterized by the identities*

$$\begin{aligned} D^1 &= P_\sigma\, W_0^{1,2}(\Omega)^3 \\ &= \{\boldsymbol{v} = \boldsymbol{v}_0 + \nabla\varphi;\ \boldsymbol{v}_0 \in W_0^{1,2}(\Omega)^3,\ \Delta\varphi = -\mathrm{div}\,\boldsymbol{v}_0 \text{ in } \Omega \text{ and } \partial\varphi/\partial\boldsymbol{n}\,|_{\partial\Omega} = 0\}. \end{aligned}$$

b) Operator A is selfadjoint and has a compact resolvent in $L^2_\sigma(\Omega)$.

c) The spectrum of A consists of a countable set of eigenvalues $\dots \le \lambda_{-2} \le \lambda_{-1} < \lambda_1 \le \lambda_2 \le \dots$ which cluster both at $-\infty$ and $+\infty$ and $\lambda_i < 0$ for $i < 0$, $\lambda_i > 0$ for $i > 0$. Each of the eigenvalues has a finite algebraic (=geometric) multiplicity. Corresponding eigenfunctions $\dots,\, \boldsymbol{e}_{-2},\, \boldsymbol{e}_{-1},\, \boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \dots$ can be chosen so that they form a complete orthonormal system in space $L^2_\sigma(\Omega)$ and a complete orthogonal system in D^1 and in D^2.

d) The spaces D^k ($k = 1,\, 2$) satisfy the identities

$$D^k = \Big\{\boldsymbol{v} = \sum_{i\in\mathbb{Z}^*} \alpha_i\, \boldsymbol{e}_i;\ \ \sum_{i\in\mathbb{Z}^*} \alpha_i^2 \lambda_i^{2k} < +\infty\Big\}$$

e) The norm $\|\,.\,\|_{k,2}$ is equivalent with the norm $\|A^k\,.\,\|_2$ in D^k for $k = 1,\, 2$.

The self–adjointness of operator A was already earlier proved by Z. Yosida, Y. Giga [18] and R. Picard [10]. The fact that 0 is not an eigenvalue of operator A is a consequence of the assumption on the simple connectedness of domain Ω. A series of further properties of operator **curl** follows from articles of O. A. Ladyzhenskaya, V. A. Solonnikov and their co–workers; let us cite e.g. [7].

2 Navier–Stokes equation with the generalized impermeability boundary conditions (1.3)

2.1 The weak Navier–Stokes problem with boundary conditions (1.3)

Definition 2.1 Let $T > 0$, $\boldsymbol{f} \in L^2(0, T;\, D^{-1})$ and $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)$. We call a function $\boldsymbol{u} \in L^\infty(0, T;\, L^2_\sigma(\Omega)) \cap L^2(0, T;\, D^1)$ a *weak solution* of the problem (1.1)–(1.4) if

$$\begin{aligned} &\int_0^T \int_\Omega \big[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nu\, A\boldsymbol{u}\cdot A\boldsymbol{\phi} + (A\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(0)\,\mathrm{d}\boldsymbol{x} \\ &\quad = \int_0^T \big\langle \boldsymbol{f},\, \boldsymbol{\phi}\big\rangle_\Omega\,\mathrm{d}t \end{aligned} \tag{2.1}$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\, D^1)$ such that $\boldsymbol{\phi}(T) = \mathbf{0}$.

Obviously, a weak solution $\boldsymbol{u}$ satisfies the first two boundary conditions in (1.3) in the sense of traces for a.a. $t \in (0,T)$. However, it is not apparent at the first sight that the weak problem formulated in Definition 2.1 also in a certain sense involves the third boundary condition in (1.3), i.e. the condition $\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n} = 0$. Let us therefore explain it in a greater detail. If $\boldsymbol{u}$ is a weak solution which, moreover, belongs to $L^2(0,T;\, W^{2,2}(\Omega)^3)$ and $\partial_t \boldsymbol{u} \in L^2(0,T;\, L^2_\sigma(\Omega))$ then we can easily verify, using a standard procedure, that there exists a scalar function q such that $\nabla q \in L^2(Q_T)^3$ and the pair $(\boldsymbol{u}, q)$ is a strong solution of equation (1.1). Using this information and integrating by parts in the terms containing $\boldsymbol{u} \cdot \partial_t \boldsymbol{\phi}$ and $A\boldsymbol{u} \cdot A\boldsymbol{\phi}$ in (2.1), we obtain:

$$\int_0^T \int_{\partial\Omega} A\boldsymbol{u} \cdot (\boldsymbol{\phi} \times \boldsymbol{n})\, \mathrm{d}S\, \mathrm{d}t = 0. \tag{2.2}$$

One can deduce from the characterization of D^1, see Lemma 1.1, that the test function $\boldsymbol{\phi}$ can be expressed in the form $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ where $\boldsymbol{\phi}_0 \in C^\infty([0,T];\, W_0^{1,2}(\Omega)^3)$ and $\varphi \in C^\infty([0,T];\, W^{2,2}(\Omega))$. Hence (2.2) implies that

$$\begin{aligned} 0 \;&=\; \int_0^T \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{u} \cdot (\nabla\varphi \times \boldsymbol{n})\, \mathrm{d}S\, \mathrm{d}t \;=\; -\int_0^T \int_\Omega \mathrm{div}\,(\nabla\varphi \times \mathbf{curl}\,\boldsymbol{u})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\ &=\; \int_0^T \int_\Omega \nabla\varphi \cdot \mathbf{curl}^2\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \;=\; \int_0^T \big\langle (\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n}),\, \varphi \big\rangle_{\partial\Omega}\, \mathrm{d}t \end{aligned}$$

where $\langle\,.\,,\,.\,\rangle_{\partial\Omega}$ denotes the duality between elements of $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$. For each $t \in (0,T)$, the set of traces on $\partial\Omega$ of all considerable functions φ is dense in $W^{1/2,2}(\partial\Omega)$. Thus, for a.a. $t \in (0,T)$, the condition $\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n} = 0$ is satisfied in the sense of equality in $W^{-1/2,2}(\partial\Omega)$. This shows that each "smooth" weak solution satisfies the third condition in (1.3) as a boundary condition which naturally follows from the weak formulation.

The existence of the weak solution of the problem (1.1)–(1.4) can be proved e.g. by the Galerkin method in the same way as in the case of the no–slip boundary condition (1.5). The Galerkin approximations can be constructed as linear combinations of the eigenfunctions of operator A. Further qualitative results on the Navier–Stokes problem (1.1)–(1.4) can be found in [2], [8] and [9].

2.2 A note to the physical sense of boundary conditions (1.3)

Although the boundary conditions (1.3) seem to be substantially different from the "traditional" no–slip boundary condition (1.5) at the first sight, the difference is in fact only subtle. It is known, and it is also explained in detail in [8], that the no–slip boundary condition (1.5) is equivalent with

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{2.3}$$

for divergence–free vector fields $\boldsymbol{u} \in W^{1,2}(\Omega)^3$. The first two conditions in (1.3) and in (2.3) are identical and they express the zero flux of $\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ through the boundary of Ω. In the incompressible Newtonian fluid, the rate of deformation tensor $\mathbb{D}$ (which equals the symmetrized gradient of velocity) and the dynamic stress tensor $\mathbb{T}_\mathrm{d}$ are related through the formula $2\nu\mathbb{D} = \mathbb{T}_{\rm d}$. Thus, the third condition in (2.3) can be written in the form $\boldsymbol{n}\cdot\mathbb{T}_{\rm d}\cdot\boldsymbol{n} = 0$ which means that the normal component of the viscous stress acting on $\partial\Omega$ equals zero.

On the other hand, since $\nu\,\mathbf{curl}^2\boldsymbol{v} = -\mathrm{Div}\,\mathbb{T}_{\rm d}$, the third condition in (1.3) says that the normal component of the intensity of production of the viscous stress on $\partial\Omega$ equals zero. Moreover, the term $\mathrm{Div}\,\mathbb{T}_{\rm d}$ represents the viscous force per unit volume in the general equation of balance of momentum. Thus, the condition $\mathrm{Div}\,\mathbb{T}_{\rm d}\cdot\boldsymbol{n} = 0$ can also be interpreted as a requirement that the normal component of this force equals zero on $\partial\Omega$.

2.3 Boundary conditions for vorticity and pressure

Applying the operator **curl** to equation (1.1) and denoting $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$, we obtain the equation

$$\partial_t\boldsymbol{\omega} + \nu\,\mathbf{curl}^2\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} \ =\ \mathbf{curl}\,\boldsymbol{f} \tag{2.4}$$

in Q_T. Suppose that $\boldsymbol{u}$ is a "smooth" solution of the problem (1.1)–(1.4) and equation (2.4) is valid up to the boundary of Ω. If $\boldsymbol{u}$ is supposed to satisfy the no–slip boundary condition (1.5) then we can only derive that $(\boldsymbol{\omega}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$, but neither the equation (2.4) nor condition (1.5) enable us to obtain more information on the behavior of $\boldsymbol{\omega}$ on the boundary and to formulate a well–posed problem for $\boldsymbol{\omega}$.

On the other hand, if $\boldsymbol{u}$ satisfies boundary conditions (1.3) then we can derive that

$$\boldsymbol{\omega}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n} = \frac{1}{\nu}\,\mathbf{curl}\,\boldsymbol{f}\cdot\boldsymbol{n} \tag{2.5}$$

on $\partial\Omega\times(0,T)$. Indeed, the first two conditions in (2.5) coincide with the second and the third condition in (1.3). The third condition in (2.5) follows from the equation (2.4): multiplying this equation by $\boldsymbol{n}$ on $\partial\Omega\times(0,T)$ and using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = \mathbf{curl}\,(\boldsymbol{\omega}\times\boldsymbol{u})$, we obtain

$$\mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n} = \frac{1}{\nu}\big[\mathbf{curl}\,\boldsymbol{f}\cdot\boldsymbol{n} - \mathbf{curl}\,(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\boldsymbol{n}\big]. \tag{2.6}$$

Since $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are tangent to $\partial\Omega$, their cross product is normal and its curl is again tangent. Hence $\mathbf{curl}\,(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T)$. The identity (2.6) now provides the third condition in (2.5). One can observe that (2.5) are the boundary conditions of the same type as (1.3), however not all the right hand sides are equal to zero. We do not discuss properties of solutions with such boundary conditions in this text; a paper on this theme is being prepared.

In order to derive a well posed problem for pressure p from the Navier–Stokes problem (1.1)–(1.4), it is better to write the Navier–Stokes equation (1.1) in the form

$$\partial_t\boldsymbol{u} + \nu\,\mathbf{curl}^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p \ =\ \boldsymbol{f}. \tag{2.7}$$

Applying operator divergence to equation (2.7), one can derive the well known Poisson equation for the pressure: $\Delta p = -(\partial_i u_j)(\partial_j u_i) + \mathrm{div}\,\boldsymbol{f}$. Multiplying formally equation (2.7) by the normal vector $\boldsymbol{n}$ and considering it on the boundary of Ω, we obtain

$$\begin{aligned}\frac{\partial p}{\partial\boldsymbol{n}} \ &=\ \boldsymbol{f}\cdot\boldsymbol{n} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{n} \ =\ \boldsymbol{f}\cdot\boldsymbol{n} - u_j\,(\partial_j u_i)\,n_i \\ &=\ \boldsymbol{f}\cdot\boldsymbol{n} - u_j\,\partial_j(u_i n_i) + u_j u_i\,(\partial_j n_i) \ =\ \boldsymbol{f}\cdot\boldsymbol{n} + u_j u_i\,(\partial_j n_i).\end{aligned} \tag{2.8}$$

(The term $u_j\, \partial_j(u_i n_i)$ expresses the derivative of the product $\boldsymbol{u}\cdot\boldsymbol{n}$, which is equal to zero on the boundary, in the direction tangential to the boundary, hence it equals zero.) (2.8) represents the Neumann boundary condition for p. This condition is simpler than the condition which we obtain if the velocity is supposed to satisfy the no–slip boundary condition (1.5). It is remarkable that the right hand side of (2.8) depends on the curvature of the boundary: the term $u_j u_i\, (\partial_j n_i)$ equals zero on those parts of $\partial\Omega$ where $\partial\Omega$ coincides with a plane.

3 Preliminary results on powers of the Stokes operator

We recall that our Stokes operator is $S = A^2$. It follows from Lemma 1.1 that S is a positive selfadjoint operator with a compact resolvent in $L^2_\sigma(\Omega)$. The eigenvalues of S coincide with λ_i^2 $(i \in \mathbb{Z}^*)$ where λ_i are the eigenvalues of operator A.

We denote by $D(S^\alpha)$ (for $1 < \alpha < +\infty$) the domain of the operator S^α, equipped with the norm $\|S^\alpha .\|_2$. The space $D(S^\alpha)$ can be characterized as

$$D(S^\alpha) \;=\; \Big\{\boldsymbol{v} = \sum_{i\in\mathbb{Z}^*} \beta_i\, \boldsymbol{e}_i;\;\; \sum_{i\in\mathbb{Z}^*} \beta_i^2 (\lambda_i^2)^{2\alpha} < +\infty\Big\} \tag{3.1}$$

and if $\boldsymbol{v}$ has the form from this identity then the power $S^\alpha \boldsymbol{v}$ can be expressed:

$$S^\alpha \boldsymbol{v} \;=\; \sum_{i\in\mathbb{Z}^*} \beta_i\, (\lambda_i^2)^\alpha\, \boldsymbol{e}_i. \tag{3.2}$$

By analogy with D. Henry [6], Exercise 5 in Sec. 1.4, one can prove the interpolation inequality

$$\|S^\alpha \boldsymbol{v}\|_2 \;\leq\; \big\|S^{\alpha_1}\boldsymbol{v}\big\|_2^{\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1}}\; \big\|S^{\alpha_2}\boldsymbol{v}\big\|_2^{\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}}. \tag{3.3}$$

which holds for $0 < \alpha_1 < \alpha < \alpha_2 < \infty$ and $\boldsymbol{v} \in D(S^{\alpha_2})$. Furthermore, using Theorem 1.4.8 in [6], we obtain the compact imbedding $D(S^{\alpha_2}) \hookrightarrow\hookrightarrow D(S^{\alpha_1})$.

The "traditional" Stokes operator $\widetilde{S} := -P_\sigma\Delta$ with the domain $D(\widetilde{S}) := W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)$ (see for instance the book by H. Sohr [16]) is different from our operator S. Nevertheless, the following Lemma 3.1 and 3.2 (for S) can basically be proved in a similar way as Lemma 2.4.2 and 2.4.3 (for $\widetilde{S}$) in [16]. So, we do not repeat the whole procedure from [16], we only present and comment the main steps.

Lemma 3.1 *Suppose that $\alpha \in [0, \frac{1}{2}]$ and $q \in [2, 6]$ are two real numbers which satisfy $2\alpha + 3/q = \frac{3}{2}$. Then there exists a constant $c_1 > 0$ (depending only on Ω, q and α) such that for all $\boldsymbol{v} \in D(S^\alpha)$, we have*

$$\|\boldsymbol{v}\|_q \;\leq\; c_1 \|S^\alpha \boldsymbol{v}\|_2. \tag{3.4}$$

Lemma 3.2 *Suppose that $\alpha \in [\frac{1}{2}, 1]$ and $q \in [2, 6]$ are two real numbers which satisfy $2\alpha + 3/q = \frac{5}{2}$. Then there exists a constant $c_2 > 0$ (depending only on Ω, q and α) such that for all $\boldsymbol{v} \in D(S^\alpha)$, we have*

$$\|\boldsymbol{v}\|_{1,q} \;\leq\; c_2\, \|S^\alpha \boldsymbol{v}\|_2. \tag{3.5}$$

Proof of Lemma 3.1 and 3.2. We proceed as in [16], with a slight modification due to the different boundary conditions. The proofs of Lemma 2.4.2 and 2.4.3 in [16] are based on this lemma:

Lemma 3.3 (E. Heinz [5]) *Suppose that $\mathcal{H}'$, $\mathcal{H}''$ are two Hilbert spaces with the norms $\|\,.\,\|'$, $\|\,.\,\|''$ and $\mathcal{A}'$ (respectively $\mathcal{A}''$) are two positive selfadjoint injective operators in $\mathcal{H}'$ (respectively in $\mathcal{H}''$). Suppose further that $\mathcal{B}$ is a bounded linear operator from $\mathcal{H}'$ into $\mathcal{H}''$ that maps $D(\mathcal{A}')$ into $D(\mathcal{A}'')$ and*

$$\|\mathcal{A}''\mathcal{B}\boldsymbol{v}\|'' \le c_3\,\|\mathcal{A}'\boldsymbol{v}\|' \qquad \text{for all } \boldsymbol{v} \in D(\mathcal{A}').$$

Then for $0 \le \beta \le 1$, $\mathcal{B}$ maps $D((\mathcal{A}')^\beta)$ into $D((\mathcal{A}'')^\beta)$ and

$$\|(\mathcal{A}'')^\beta\mathcal{B}\boldsymbol{v}\|'' \le c_3^\beta\,\|\mathcal{B}\|_{\mathcal{H}'\to\mathcal{H}''}^{1-\beta}\,\|(\mathcal{A}')^\beta\boldsymbol{v}\|' \qquad \textit{for all } \boldsymbol{v} \in D((\mathcal{A}')^\beta).$$

In the no–slip case, the operator $\mathcal{B}$ can be chosen as the extension by zero from Ω onto $\mathbb{R}^3 - \Omega$. Such an operator $\mathcal{B}$ is a bounded linear operator from $L^2_\sigma(\Omega)$ into $L^2_\sigma(\mathbb{R}^3)$ and it maps the domain of $\widetilde{S}^{1/2}$ into the domain of $S_{\mathbb{R}^3}^{1/2}$ where $S_{\mathbb{R}^3}$ denotes the Stokes operator in $L^2_\sigma(\mathbb{R}^3)$. One can arrive at the identities

$$\|S_{\mathbb{R}^3}^{1/2}B\boldsymbol{v}\|_{2;\,\mathbb{R}^3} = \|\nabla B\boldsymbol{v}\|_{2;\,\mathbb{R}^3} = \|\nabla\boldsymbol{v}\|_2 = \|\widetilde{S}^{1/2}\boldsymbol{v}\|_2 \tag{3.6}$$

which hold for all $v \in D(\widetilde{S}^{1/2})$, with $\|\,.\,\|_{2;\,\mathbb{R}^3}$ being the norm in $L^2(\mathbb{R}^3)^3$. Then the Heinz lemma is applied with $\mathcal{H}' = L^2_\sigma(\Omega)$, $\mathcal{A}' = \widetilde{S}^{1/2}$, $\mathcal{H}'' = L^2_\sigma(\mathbb{R}^3)$, $\mathcal{A}'' = S_{\mathbb{R}^3}^{1/2}$, $\mathcal{B} = B$ and $\beta = 2\alpha$. One finally obtains the desired estimate.

With our boundary conditions (1.3), however, it is not generally true that if $\boldsymbol{v} \in D(S^{1/2})$ then the extension of $\boldsymbol{v}$ by zero belongs to $D(S_{\mathbb{R}^3}^{1/2})$ and also the square root $S^{1/2}$ does not satisfy the last equality in (3.6), as $\widetilde{S}^{1/2}$. This is why we use a general extension operator E which is a bounded linear operator from $L^2(\Omega)^3$ into $L^2(\mathbb{R}^3)^3$ and from $W^{2,2}(\Omega)^3$ into $W^{2,2}(\mathbb{R}^3)^3$ and which satisfies $E\boldsymbol{v}\,|_\Omega = \boldsymbol{v}$ for all $\boldsymbol{v} \in L^2(\Omega)^3$. (The existence of such an operator is proved in [1], part IV, 4.29.) Using Lemma 1.1, part e) with $k = 2$, we get

$$\|(-\Delta)E\boldsymbol{v}\|_{2;\,\mathbb{R}^3} \le C\,\|E\boldsymbol{v}\|_{2,2;\,\mathbb{R}^3} \le C\,\|\boldsymbol{v}\|_{2,2} \le C\,\|S\boldsymbol{v}\|_2 \qquad \text{for all } \boldsymbol{v} \in D(S)$$

where $\|\,.\,\|_{2,2;\,\mathbb{R}^3}$ denotes the norm in $W^{2,2}(\mathbb{R}^3)^3$. Therefore we may put $\mathcal{B} = E$ and $\mathcal{H}' = L^2_\sigma(\Omega)$, $\mathcal{A}' = S$, $\mathcal{H}'' = L^2(\mathbb{R}^3)^3$, $\mathcal{A}'' = -\Delta$, $\beta = 2\alpha$ and to apply the Heinz lemma. It yields

$$\|(-\Delta)^\alpha E\boldsymbol{v}\|_{2;\,\mathbb{R}^3} \le C\,\|S^\alpha\boldsymbol{v}\|_2 \qquad \text{for all } \boldsymbol{v} \in D(S^\alpha).$$

The proof of Lemma 3.1 can now be completed by means of this estimate, the estimate $\|E\boldsymbol{v}\|_q \le C\,\|(-\Delta)^\alpha E\boldsymbol{v}\|_{2;\,\mathbb{R}^3}$ (following from Lemma 3.3.1 in [16], p. 102) and the boundedness of operator E from $L^q(\Omega)^3$ into $L^q(\mathbb{R}^3)^3$.

Lemma 3.2 can be proved in a similar way. Using Lemma 1.1, item e), and the same extension operator E, we can obtain

$$\|(I-\Delta)E\boldsymbol{v}\|_{2;\,\mathbb{R}^3} \le C\,\|E\boldsymbol{v}\|_{2,2;\,\mathbb{R}^3} \le C\,\|\boldsymbol{v}\|_{2,2} \le C\,\|S\boldsymbol{v}\|_2.$$

Then, applying again the Heinz lemma, we get

$$\|(I-\Delta)^\alpha E\boldsymbol{v}\|_{2;\,\mathbb{R}^3} \le C\,\|S^\alpha\boldsymbol{v}\|_2$$

for $0 \le \alpha \le 1$ and $\boldsymbol{v} \in D(S^\alpha)$. The estimate (3.5) now follows from this inequality and from the inequalities

$$\|\boldsymbol{v}\|_{1,q} \le C\,\|E\boldsymbol{v}\|_{1,q;\,\mathbb{R}^3} \le C\,\|(I-\Delta)^\alpha E\boldsymbol{v}\|_{2;\,\mathbb{R}^3}.$$

(The last one is proved in [16], pp. 103–104.) □

4 Small perturbations of the initial velocity and of the body force

Theorem 4.1 *Suppose that $\boldsymbol{u}$ is a weak solution of the problem (1.1)–(1.4) with the input data $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in D(S^{1/4})$ and $P_\sigma \boldsymbol{f} \in L^2(0,T;\, L^2_\sigma(\Omega))$ which satisfies*

$$\int_0^T \Big(\|S^{3/4}\boldsymbol{u}(t)\|_2^2 + \|S^{1/2}\boldsymbol{u}(t)\|_2^4 \Big)\, \mathrm{d}t \, < \, \infty. \tag{4.1}$$

Then to given $\epsilon > 0$ there exists $\delta > 0$ such that if $\boldsymbol{v}_0 \in D(S^{1/4})$ and $P_\sigma \boldsymbol{g} \in L^2(0,T;\, L^2_\sigma(\Omega))$ are functions satisfying

$$\|S^{1/4}\boldsymbol{u}_0 - S^{1/4}\boldsymbol{v}_0\|_2 + \int_0^T \|P_\sigma \boldsymbol{f}(t) - P_\sigma \boldsymbol{g}(t)\|_2^2 \, < \, \delta \tag{4.2}$$

then there exists a unique weak solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) with the data $\boldsymbol{v}_0$ and $\boldsymbol{g}$ (instead of $\boldsymbol{u}_0$ and $\boldsymbol{f}$) on the time interval $(0,T)$, such that

$$\|S^{1/4}\boldsymbol{v}(t) - S^{1/4}\boldsymbol{u}(t)\|_2^2 \, < \, \epsilon \qquad \text{for all } t \in (0,T), \tag{4.3}$$

$$\int_0^T \|S^{3/4}\boldsymbol{v}(s) - S^{3/4}\boldsymbol{u}(s)\|_2^2 \, \mathrm{d}s \, < \, \epsilon. \tag{4.4}$$

Remark 4.1 A similar result for the Navier–Stokes problem with the no–slip boundary condition (1.5) was proved by G. Ponce et. al. in [11]. However, our assumption (4.2) is weaker because we measure the difference between the initial velocities $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$ in the norm $\|S^{1/4}\,.\,\|_2$ while the authors of [11] have used the norm $\|\,.\,\|_{1,2}$, equivalent with $\|S^{1/2}.\,\|_2$.

Remark 4.2 (on condition (4.1)) A strong solution of the problem (1.1)–(1.4) on the interval $(0,T)$ is usually supposed to belong to $L^\infty(0,T;\, W^{1,2}(\Omega)^3) \cap L^2(0,T; W^{2,2}(\Omega)^3)$ and it automatically satisfies (4.1). Thus, the condition (4.1) can be replaced by the assumption that $\boldsymbol{u}$ is a strong solution.

In fact, inequality (4.1) follows from a weaker assumption, i.e. from the assumption that $\boldsymbol{u} \in L^\infty(0,T;\, D(S^{1/4})) \cap L^2(0,T;\, D(S^{3/4}))$ because then, using the interpolation inequality (3.3) with $\alpha = \frac{1}{2}$, $\alpha_1 = \frac{1}{4}$ and $\alpha_2 = \frac{3}{4}$, we get

$$\int_0^T \Big(\|S^{3/4}\boldsymbol{u}(t)\|_2^2 + \|S^{1/2}\boldsymbol{u}(t)\|_2^4 \Big)\, \mathrm{d}t \, \leq \, \int_0^T \|S^{3/4}\boldsymbol{u}(t)\|_2^2 \Big(1 + \|S^{1/4}\boldsymbol{u}(t)\|_2^2 \Big)\, \mathrm{d}t$$

$$\leq \, \Big[1 + \operatorname*{ess\,sup}_{0<t<T} \|S^{1/4}\boldsymbol{u}(t)\|_2^2 \Big] \int_0^T \|S^{3/4}\boldsymbol{u}(t)\|_2^2\, \mathrm{d}t \, < \, \infty.$$

Condition (4.1) and Lemma 3.1 imply that solution $\boldsymbol{u}$ belongs to the anisotropic Lebesgue space $L^r(0,T;\, L^s(\Omega)^3)$ (with $r = 4$ and $s = 6$) where the exponents r, s satisfy Serrin's condition $2/r + 3/s \leq 1$. It can be deduced from known results on regularity of solutions to the Navier–Stokes equation, see e.g. Y. Giga [4] and W. von Wahl [17], that such a solution is regular, i.e. it has no singular points. The exact rate of regularity depends on regularity of the body force $\boldsymbol{f}$. Nevertheless, since $P_\sigma \boldsymbol{f} \in L^2(0,T;\, L^2_\sigma(\Omega))$, the condition (4.1) enables us to deduce that a) if $\boldsymbol{u}_0 \in D(S^{1/2})$ then $\boldsymbol{u}$ is a strong solution of the problem (1.1)–(1.4) on the time interval $(0,T)$ or b) if $\boldsymbol{u}_0 \in D(S^{1/4})$ then $\boldsymbol{u}$ is a strong solution of the problem (1.1)–(1.4) on each time interval of the type (τ,T) where $0 < \tau < T$.

Proof of Theorem 4.1. We seek for solution $\boldsymbol{v}$ in the form $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{w}$ is a new unknown function. This function should be a weak solution of the problem

$$\begin{aligned} \partial_t \boldsymbol{w} + \nu\, \mathbf{curl}^2 \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u} & \\ + \mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{w} + \nabla q \ &= \ \boldsymbol{g} - \boldsymbol{f} \qquad \text{in } Q_T, \qquad (4.5)\\ \operatorname{div} \boldsymbol{w} \ &= \ 0 \qquad \text{in } Q_T, \qquad (4.6)\\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}^2 \boldsymbol{w} \cdot \boldsymbol{n} \ &= \ 0 \qquad \text{on } \partial\Omega \times (0,T), \qquad (4.7)\\ \boldsymbol{w}(0) \ &= \ \boldsymbol{v}_0 - \boldsymbol{u}_0 \qquad \text{in } \Omega. \qquad (4.8) \end{aligned}$$

Applying formally the projection P_σ to equation (4.5), writing A instead of $\mathbf{curl}$ and using the notation $S = A^2$, we can write the problem (4.5)–(4.8) in the form of one operator equation

$$\partial_t \boldsymbol{w} + \nu S \boldsymbol{w} + P_\sigma [A\boldsymbol{u} \times \boldsymbol{w}] + P_\sigma [A\boldsymbol{w} \times \boldsymbol{u}] + P_\sigma [A\boldsymbol{w} \times \boldsymbol{w}] \ = \ P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}. \qquad (4.9)$$

The equation of continuity (4.6) and the first two boundary conditions (4.7) are now replaced by the requirement that $\boldsymbol{w}(t)$ belongs to the domain of A for a.a. $t \in (0,T)$ and the third boundary condition in (4.7) is a natural boundary condition which is satisfied by $\boldsymbol{w}$ if $\boldsymbol{w}$ is "smooth". (See the explanation in subsection 2.1.)

The weak solution of equation (4.9) can be constructed in a standard way which basically copies the proof of the existence of a weak solution of the Navier–Stokes initial–boundary value problem with the no–slip boundary condition (1.5). The proof is described in detail e.g. in the survey article by G. P. Galdi [3]. The Galerkin approximations $\boldsymbol{w}^n$ can be constructed as linear combinations of the eigenfunctions $\boldsymbol{e}_i$ ($i \in \mathbb{Z}^*$) of operator A. The crucial step is the derivation of their estimates. The estimates then enable us to deduce that the sequence of approximations contains a subsequence whose limit $\boldsymbol{w}$ (= the strong limit in $L^2(0,T;\, L^2_\sigma(\Omega))$, the weak limit in $L^2(0,T;\, D^1)$ and the weak–$*$ limit in $L^\infty(0,T;\, L^2_\sigma(\Omega))$) is a weak solution of the problem (4.5)–(4.8) or of the operator equation (4.9). In order not to complicate the proof, we shall formally derive the estimates directly from equation (4.9); such estimates are usually called *a priori estimates.*

Multiplying equation (4.9) by $S^{1/2}\boldsymbol{w}$ and integrating on Ω, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|S^{1/4}\boldsymbol{w}\|_2^2 + \nu \|S^{3/4}\boldsymbol{w}\|_2^2 \ \leq\ & \int_\Omega |\nabla \boldsymbol{u}|\, |\boldsymbol{w}|\, |S^{1/2}(\boldsymbol{w})|\, \mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla \boldsymbol{w}|\, |\boldsymbol{u}|\, |S^{1/2}\boldsymbol{w}|\, \mathrm{d}\boldsymbol{x} \\ & + \int_\Omega |\nabla \boldsymbol{w}|\, |\boldsymbol{w}|\, |S^{1/2}\boldsymbol{w}|\, \mathrm{d}\boldsymbol{x} + \int_\Omega |P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}|\, |S^{1/2}\boldsymbol{w}|\, \mathrm{d}\boldsymbol{x}. \qquad (4.10) \end{aligned}$$

We will now estimate the integrals on the right hand side of (4.10). We shall use Lemma 3.1 and Lemma 3.3. We obtain

$$\begin{aligned} & \int_\Omega |\nabla \boldsymbol{u}|\, |\boldsymbol{w}|\, |S^{1/2}\boldsymbol{w}|\, \mathrm{d}\boldsymbol{x} \ \leq\ \|\nabla \boldsymbol{u}\|_3\, \|\boldsymbol{w}\|_6\, \|S^{1/2}\boldsymbol{w}\|_2 \\ & \quad \leq\ C\, \|S^{3/4}\boldsymbol{u}\|_2\, \|S^{1/2}\boldsymbol{w}\|_2^2 \ \leq\ C\, \|S^{3/4}\boldsymbol{u}\|_2\, \|S^{3/4}\boldsymbol{w}\|_2\, \|S^{1/4}\boldsymbol{w}\|_2 \\ & \quad \leq\ \frac{\nu}{6}\, \|S^{3/4}\boldsymbol{w}\|_2^2 + C\, \|S^{1/4}\boldsymbol{w}\|_2^2\, \|S^{3/4}\boldsymbol{u}\|_2^2, \qquad (4.11) \end{aligned}$$

$$
\begin{aligned}
&\int_\Omega |\nabla \boldsymbol{w}|\, |\boldsymbol{u}|\, |S^{1/2}\boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \;\le\; \|\nabla\boldsymbol{w}\|_3\, \|\boldsymbol{u}\|_6\, \|S^{1/2}\boldsymbol{w}\|_2 \\
&\quad \le\; C\, \|S^{3/4}\boldsymbol{w}\|_2\, \|S^{1/2}\boldsymbol{u}\|_2\, \|S^{1/2}\boldsymbol{w}\|_2 \;\le\; C\, \|S^{3/4}\boldsymbol{w}\|_2^{3/2}\, \|S^{1/4}\boldsymbol{w}\|_2^{1/2}\, \|S^{1/2}\boldsymbol{u}\|_2 \\
&\quad \le\; \frac{\nu}{6}\, \|S^{3/4}\boldsymbol{w}\|_2^2 + C\, \|S^{1/4}\boldsymbol{w}\|_2^2\, \|S^{1/2}\boldsymbol{u}\|_2^4, \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega |\nabla \boldsymbol{w}|\, |\boldsymbol{w}|\, |S^{1/2}\boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \;\le\; \|\nabla\boldsymbol{w}\|_3\, \|\boldsymbol{w}\|_6\, \|S^{1/2}\boldsymbol{w}\|_2 \\
&\quad \le\; C\, \|S^{3/4}\boldsymbol{w}\|_2\, \|S^{1/2}\boldsymbol{w}\|_2^2 \;\le\; C\, \|S^{3/4}\boldsymbol{w}\|_2^2\, \|S^{1/4}\boldsymbol{w}\|_2, \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega |P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}|\, |S^{1/2}\boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \;\le\; \|P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}\|_2\, \|S^{1/2}\boldsymbol{w}\|_2 \\
&\quad \le\; \frac{\nu}{6}\, \|S^{3/4}\boldsymbol{w}\|_2^2 + C\, \|P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}\|_2^2. \qquad (4.14)
\end{aligned}
$$

We have also used the estimate $\|S^{1/2}(\boldsymbol{w})\|_2 \le C\, \|S^{3/4}(\boldsymbol{w})\|_2$ in (4.14). Substituting now (4.11)–(4.14) into (4.10), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2}\, \|S^{1/4}\boldsymbol{w}\|_2^2 + \Big(\frac{\nu}{2} - C\, \|S^{1/4}\boldsymbol{w}\|_2 \Big)\, \|S^{3/4}\boldsymbol{w}\|_2^2 \\
&\quad \le\; C\, \Big(\|S^{3/4}\boldsymbol{u}\|_2^2 + \|S^{1/2}\boldsymbol{u}\|_2^4 \Big)\, \|S^{1/4}\boldsymbol{w}\|_2^2 + C\, \|P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}\|_2^2. \qquad (4.15)
\end{aligned}
$$

If we denote

$$
\begin{aligned}
\zeta(t) &= \|S^{3/4}\boldsymbol{u}\|_2^2 + \|S^{1/2}\boldsymbol{u}\|_2^4, \\
\vartheta(t) &= \|P_\sigma \boldsymbol{g} - P_\sigma \boldsymbol{f}\|_2^2
\end{aligned}
$$

then (4.15) can be written in the form

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\, \|S^{1/4}\boldsymbol{w}\|_2^2 + \Big(\nu - c_4\, \|S^{1/4}\boldsymbol{w}\|_2 \Big)\, \|S^{3/4}\boldsymbol{w}\|_2^2 \\
&\quad \le\; c_5\, \zeta(t)\, \|S^{1/4}\boldsymbol{w}\|_2^2 + c_6\, \vartheta(t) \qquad (4.16)
\end{aligned}
$$

where c_4–c_6 are appropriate constants which depend only on Ω and ν. The integral of ζ on the time interval $(0,T)$ is less than or equal to c_7 where c_7 denotes the left hand side of (4.1). Let us further compare function $\|S^{1/4}\boldsymbol{w}(t)\|_2^2$ with function $z(t)$ such that $z(0) = \|S^{1/4}(\boldsymbol{w})(0)\|_2^2$ and z satisfies the equation

$$
z'(t) \;=\; c_5\, \zeta(t)\, z(t) + c_6\, \vartheta(t). \qquad (4.17)
$$

Integrating (4.17), we obtain that

$$
\begin{aligned}
z(t) &= \exp\Big[\int_0^t c_5\, \zeta(\tau)\, \mathrm{d}\tau \Big]\, z(0) + \int_0^t \exp\Big[\int_s^t c_5\, \zeta(\tau)\, \mathrm{d}\tau \Big]\, c_6\, \vartheta(s)\, \mathrm{d}s \\
&\le \mathrm{e}^{c_5 c_7}\, z(0) + c_6\, \mathrm{e}^{c_5 c_7} \int_0^t \vartheta(s)\, \mathrm{d}s \qquad (4.18)
\end{aligned}
$$

for all $t \in (0,T)$. Obviously, the comparison of (4.16) with (4.17) yields the inequality

$$
\|S^{1/4}\boldsymbol{w}(t)\|_2^2 \;\le\; z(t) \qquad (4.19)
$$

on each interval $(0,T')$ such that

$$\nu - c_4\, \|S^{1/4}\boldsymbol{w}(t)\|_2^2 \,\geq\, \frac{\nu}{2} \tag{4.20}$$

also holds for all $t \in (0,T')$. However, (4.18) and (4.19) imply that provided

$$\mathrm{e}^{c_5 c_7}\, \|S^{1/4}\boldsymbol{w}(0)\|_2^2 + c_6\, \mathrm{e}^{c_5 c_7} \int_0^T \vartheta(s)\; \mathrm{d}s \,\leq\, \frac{\nu}{2c_4}, \tag{4.21}$$

(4.20) holds on the interval $(0,T)$ and consequently,

$$\|S^{1/4}\boldsymbol{w}(t)\|_2^2 \,\leq\, z(t) \,\leq\, \mathrm{e}^{c_5 c_7}\, z(0) + c_6\, \mathrm{e}^{c_5 c_7} \int_0^T \vartheta(s)\; \mathrm{d}s \tag{4.22}$$

also holds on the interval $(0,T)$. Finally, integrating (4.16) on the time interval $(0,T)$ and using (4.22), we obtain

$$\begin{aligned} \frac{\nu}{2} \int_0^T \|S^{3/4}\boldsymbol{w}\|_2^2\, \mathrm{d}t \,&\leq\, c_5 \left(\int_0^T \zeta(t)\, \mathrm{d}t \right) \left(\mathrm{e}^{c_5 c_7}\, z(0) + c_6\, \mathrm{e}^{c_5 c_7} \int_0^T \vartheta(s)\; \mathrm{d}s \right) \\ &\quad + c_6 \int_0^T \vartheta(t)\; \mathrm{d}t. \end{aligned} \tag{4.23}$$

It can now be observed that if number δ in (4.2) is so small that (4.21) holds then (4.22) and (4.23) give the a priori estimates which are in fact satisfied by the Galerkin approximations. However, as we have already mentioned, using an appropriate and standard limit procedure, we can obtain the existence of a solution $\boldsymbol{w}$ which satisfies the same estimates. Moreover, given $\epsilon > 0$, both the right hand sides of (4.22) and (4.23) can be achieved to be less than ϵ if we choose $\delta > 0$ sufficiently small. Hence, writing $\boldsymbol{w}$ in the form $\boldsymbol{v} - \boldsymbol{u}$, we observe that the difference $\boldsymbol{v} - \boldsymbol{u}$ satisfies (4.3) and (4.4).

Using (4.22), (4.23) and the interpolation inequality (3.3), we obtain:

$$\int_0^T \|S^{1/2}\boldsymbol{w}\|_2^4\, \mathrm{d}t \,\leq\, \operatorname*{ess\,sup}_{0<t<T}\, \|S^{1/4}\boldsymbol{w}(t)\|_2^2 \int_0^T \|S^{3/4}\boldsymbol{w}(t)\|_2^2\, \mathrm{d}t \,<\, \infty.$$

From this inequality, we can deduce that solution $\boldsymbol{v}$ also satisfies the inequality (4.1), which was originally supposed to be satisfies by $\boldsymbol{u}$.

As to the uniqueness of solution $\boldsymbol{v}$, it is obviously guaranteed in the class of functions $\boldsymbol{v}$ that satisfy (4.1) (with $\boldsymbol{v}$ instead of $\boldsymbol{u}$), (4.3) and (4.4). (The opposite can easily be denied by contradiction.) Using known results on uniqueness of solutions, we can even say that $\boldsymbol{v}$ is unique in the class of functions which satisfy the energy inequality. However, the uniqueness in the class of all possible weak solutions is an open problem. □

Theorem 4.1 provides information on stability of solution $\boldsymbol{u}$ with respect to small perturbations of the initial velocity in the norm $\|S^{1/4}\,.\,\|_2$ and with respect to small perturbations of the right hand side (projected onto $L^2_\sigma(\Omega)$ by projection P_σ) in the norm of the space $L^2(0,T;\, L^2_\sigma(\Omega))$. We usually speak on stability of a solution if we have information on behavior of "near" solutions on an unbounded time interval, but this does not contradict with our result because Theorem 4.1 is also true in the particular case when $T = +\infty$.

5 Large perturbations of the initial velocity

In this section, we apply Theorem 4.1 and we derive results on solutions of the Navier–Stokes problem (1.1)–(1.4) which represent a modification and generalization of theorems from Scarpellini's papers [12] and [13]. The modification consists in the fact that our Stokes operator S is defined on a set of functions satisfying the boundary conditions (1.3), while B. Scarpellini worked with the Stokes operator $\widetilde{S} = -P_\sigma \Delta$ defined on $D(\widetilde{S}) = W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)$ which we have already mentioned in Section 3. The generalization concerns the values of exponents of the Stokes operator in our propositions and it will be further explained.

In [12] and [13], assuming that $\boldsymbol{f} = \boldsymbol{0}$, B. Scarpellini constructed a regular solution of the Navier–Stokes equation on the time interval $(0, +\infty)$ with an arbitrarily large initial velocity (in the norm $\|\widetilde{S}^{1/2} \,.\, \|_2$). Our Theorem 4.1 provides an opportunity to extend this result to the case of the initial velocity arbitrarily large in the norm $\|S^\alpha \,.\, \|_2$ for some α less than $\frac{1}{2}$: Suppose that $T = +\infty$, $\boldsymbol{f} \equiv \boldsymbol{0}$ and $\boldsymbol{u} \equiv \boldsymbol{0}$ is a zero solution of (1.1)–(1.4). Let $\delta > 0$ be the number given by Theorem 4.1, corresponding e.g. to $\epsilon = 1$. Suppose further that $\frac{1}{4} < \alpha \leq \frac{1}{2}$ and $R > 0$ is an arbitrarily large real number. Due to the density of $D(S^{1/2})$ and $D(S^\alpha)$ in $D(S^{1/4})$, there exists $\boldsymbol{v}_0 \in D(S^{1/2})$ such that $\|S^{1/4}\boldsymbol{v}_0\|_2 < \delta$ and $\|S^\alpha \boldsymbol{v}_0\|_2 > R$. Theorem 4.1 now implies that there exists a unique solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) with the initial velocity $\boldsymbol{v}_0$ and with the same body force $\boldsymbol{g} = \boldsymbol{f} \equiv \boldsymbol{0}$ which satisfies (4.3) and (4.4) on the time interval $(0, +\infty)$. Moreover, the inclusion $\boldsymbol{v}_0 \in D(S^{1/2})$ and (4.4) guarantee that $\boldsymbol{v}$ is a strong solution on $(0, +\infty)$.

Obviously, this result can further be generalized for the case of a non–zero (however "small") body force $\boldsymbol{g}$: If $\boldsymbol{v}_0$ and $P_\sigma \boldsymbol{g}$ satisfy (4.2) then due to Theorem 4.1, there exists a unique solution of the problem (1.1)–(1.4) (with the initial velocity $\boldsymbol{v}_0$ and the body force $\boldsymbol{g}$) which has the same properties as the solution $\boldsymbol{v}$ discussed above.

Our next goal in this section is to prove the following Theorem 5.1. The theorem shows that there exists a solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) such that the norm $\|S^\alpha \boldsymbol{v}(0)\|_2$ (with $\frac{1}{4} < \alpha \leq \frac{1}{2}$) can be arbitrarily large and the norm $\|S^\gamma \boldsymbol{v}(t)\|_2$ (with $\frac{3}{4} < \gamma < 1$) can be arbitrarily small for all t from a time interval which is arbitrarily close to zero. The theorem in fact says something more: the solution $\boldsymbol{v}$ can be constructed so that all its values $\boldsymbol{v}(t)$ in a certain time interval, whose distance from 0 is arbitrarily small, belong to an arbitrarily chosen open set U in $D(S^\gamma)$.

Theorem 5.1 *Suppose that $P_\sigma \boldsymbol{f} \in L^2(0, T_1;\, L^2_\sigma(\Omega))$, $\frac{1}{4} < \alpha \leq \frac{1}{2}$, $\frac{3}{4} < \gamma < 1$ and U is a nonempty open subset of $D(S^\gamma)$. Suppose that two real numbers $R > 0$ (arbitrarily large) and $\chi \in (0, T_1)$ (arbitrarily small) are given. Then there exists $\boldsymbol{v}_0 \in D(S^\alpha)$ and a weak solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) on the time interval $(0, T_1)$ such that*

$$\|S^\alpha \boldsymbol{v}_0\|_2 \geq R \tag{5.1}$$

and $\boldsymbol{v}(t) \in U$ at all instants of time $t \in (\frac{1}{2}\chi, \chi)$.

This theorem represents the main generalization in comparison with Scarpellini's result from [13] in the part which concerns the exponent α in (5.1). (B. Scarpellini worked with the fixed $\alpha = \frac{1}{2}$.) The second difference is that we consider the generalized impermeability boundary conditions (1.3) while Scarpellini worked with the boundary condition (1.5). As follows from the proof of Theorem 5.1, our approach is enabled by Theorem 4.1.

Proof of Theorem 5.1. Let U be an open set in $D(S^\gamma)$ and $\boldsymbol{u}_0 \in D(S) \cap U$. Due to Theorem 1 in [9], there exists $T^* \in (0, T_1)$ and a strong solution $\boldsymbol{u}$ of the problem (1.1)–(1.4) which is a continuous mapping from $[0, T^*)$ into $D(S)$. Thus, to every $\mu > 0$ there exists $T \in (0, T^*)$ such that the restriction of function $\boldsymbol{u}$ to the interval $[0, T]$ is a continuous mapping from $[0, T]$ into $D(S)$ and

$$B_\mu^\gamma(\boldsymbol{u}(t)) \subset U \tag{5.2}$$

for every $t \in [0, T]$. ($B_\mu^\gamma(\boldsymbol{u}(t))$ is the ball in $D(S^\gamma)$ with the center at $u(t)$ and radius μ.)

Let $\epsilon > 0$ be given. Due to Theorem 4.1, there exists $\delta > 0$ such that if

$$\|S^{1/4}\boldsymbol{v}_0 - S^{1/4}\boldsymbol{u}_0\|_2 \, < \, \delta$$

then there exists a weak solution $\boldsymbol{v}$ of the problem (1.1)–(1.4) on $(0, T_1)$ with the initial velocity $\boldsymbol{v}_0$ and with the same right hand side $\boldsymbol{f}$ such that $\boldsymbol{v}$ satisfies (4.3) and (4.4) on the "reduced" time interval $(0, T)$. Since $\alpha > \frac{1}{4}$, $\boldsymbol{v}_0$ can be chosen so that it satisfies (5.1). From (4.4) we can deduce that in each open time interval in $(0, T)$ whose length exceeds l there exists τ such that $\|S^{3/4}\boldsymbol{v}(\tau) - S^{3/4}\boldsymbol{u}(\tau)\|_2^2 < \epsilon/l$. Hence

$$\|S^{1/2}\boldsymbol{v}(\tau) - S^{1/2}\boldsymbol{u}(\tau)\|_2^4 \, \leq \, \|S^{1/4}\boldsymbol{v})(\tau) - S^{1/4}\boldsymbol{u}(\tau)\|_2^2 \, \|S^{3/4}\boldsymbol{v})(\tau) - S^{3/4}\boldsymbol{u}(\tau)\|_2^2 \, \leq \, \frac{\epsilon^2}{l}.$$

If the considered open time interval is $(0, \frac{1}{2}\chi)$ then $0 < \tau < \frac{1}{2}\chi$ and

$$\|S^{1/2}\boldsymbol{v}(\tau) - S^{1/2}\boldsymbol{u}(\tau)\|_2 \leq \left(\frac{2\epsilon^2}{\chi}\right)^{1/4}. \tag{5.3}$$

Slightly modifying the proof of Theorem 1 in [9], we can now show that if $\epsilon > 0$ is chosen sufficiently small then (5.3) implies that

$$\|S^\gamma\boldsymbol{v}(t) - S^\gamma\boldsymbol{u}(t)\|_2 \, < \, \mu \qquad \text{for all } \tfrac{1}{2}\chi < t < \chi. \tag{5.4}$$

Note that in the case of the no–slip boundary condition (1.5), the same implication, i.e. that (5.3) (for $\epsilon > 0$ small enough) implies (5.4), is a consequence of Proposition 3.4 in ([13].

The inclusion (5.2) (for all $t \in (\frac{1}{2}\chi, \chi)$) now follows from (5.4). $\square$

Remark 5.1 Choosing set U to be a sufficiently small neighborhood of zero (in the space $D(S^\gamma)$), Theorem 5.1 provides solution $\boldsymbol{v}$ which has the so called "big fall" at a very short instant of time $(0, \frac{1}{2}\chi)$. If, in addition, we assume that the specific body force $\boldsymbol{f}$ is "sufficiently small" on the time interval $(\chi, +\infty)$ then solution $\boldsymbol{v}$, due to its smallness at times $t \in (\frac{1}{2}\chi, \chi)$, can be prolonged as a strong solution onto the whole interval $(\chi, +\infty)$.

Further interesting theorems on global in time strong solutions which initially have "big falls" or on the other hand results restricting the "falls" of solutions of the Navier–Stokes equations can be found in the preprints [14] and [15] by Z. Skalák.

Acknowledgement. The research was supported by the Grant Agency of the Czech Academy of Sciences (grant No. IAA100190612), by the research plan of the Ministry of Education of the Czech Republic No. MSM 6840770010, by the Czech Academy of Sciences, Institutional Research Plan No. AV0Z10190503, and by the University of Sud Toulon–Var.

References

[1] R. S. Adams: *Sobolev Spaces.* Academic press, New York 1975.

[2] H. Bellout, J. Neustupa, P. Penel: On the Navier-Stokes equations with boundary conditions based on vorticity. *Math. Nachr.* 269–270, 2004, 59–72.

[3] G. P. Galdi: An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Math. Fluid Mechanics,* ed. G. P. Galdi, J. Heywood and R. Rannacher, series "Advances in Mathematical Fluid Mechanics", Birkhauser, Basel 2000, 1–98.

[4] Y. Giga: Solutions of semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes equations. *J. of Diff. Equations* 61, 1986, 186–212.

[5] E. Heinz: Beiträge zur Störungstheorie der Spektralzerlegung. *Math. Ann.* 123, 1951, 415–438.

[6] D. Henry: *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics 840, Springer–Verlag, Berlin–Heidelberg–New York 1981.

[7] O. A. Ladyzhenskaya and V. A. Solonnikov: Solution of some non–stationary problems of magnetohydrodynamics for viscous incompressible fluids. *Trudy Math. Inst. V. I. Steklova* 59, spec. vol. on "Mathematical Problems in Hydrodynamics and Magnetohydrodynamics", 1960, 115–173.

[8] J. Neustupa, P. Penel: Incompressible viscous fluid flows and the generalized impermeability boundary conditions. *IASME Transactions* 7, Vol. 2, 2005, 1254–1261.

[9] J. Neustupa, P. Penel: On regularity of a weak solution to the Navier–Stokes equation with generalized impermeability boundary conditions. To appear in *Nonlinear Analysis* 2006.

[10] R. Picard: On a selfadjoint realization of **curl** and some of its applications. *Ricerche di Matematica* XLVII, 1998, 153–180.

[11] G. Ponce, R. Racke, T. C. Sideris, E. S. Titi: Global stability of large solutions to the 3D Navier–Stokes equations. *Comm. Math. Phys.* 159, 1994, 329–341.

[12] B. Scarpellini: On a family of large solutions of the Navier–Stokes equations. *Houston J. of Math.* 28, No. 3, 2002, 641–648.

[13] B. Scarpellini: Fast decaying solutions of the Navier–Stokes equations and asymptotic properties. *J. of Math. Fluid Mech.* 6, 2004, 103–120.

[14] Z. Skalák: Fast decays of strong solutions of the Navier–Stokes equations. Preprint 2005. To appear in J. of Math. Fluid Mech.

[15] Z. Skalák: On asymptotic dynamics of solutions of the homogeneous Navier–Stokes equations. Preprint 2005. To appear in Nonlinear Analysis.

[16] H. Sohr: *The Navier–Stokes equations. An elementary functional analytic approach.* Birkhäuser advanced texts, Basel–Boston–Berlin 2001.

[17] W. von Wahl: Regularity of weak solutions of the Navier–Stokes equations. *Proc. Symp. Pure and Appl. Math.* 45, 1986, 497–503.

[18] Z. Yosida and Y. Giga: Remarks on spectra of operator **rot**. *Math. Zeitschrift* 204, 1990, 235–245.