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A direct approach to vorticity transport & diffusion

Summary

The evolution in time of the vorticity w of incompressible viscous flow in bounded 3-dimensional domains is governed by the initial boundary value problem of the vorticity transport & diffusion equation, which imposes a nonlocal boundary condition on w . In suitable solution spaces defined below this boundary condition holds true. We will prove the uniqueness of generalized solutions to the problem above as well as the local in time existence of a unique strong solution which even exists globally in case of sufficiently small initial data.

0. Introduction. Notations.

In a bounded open set $\Omega \subset \mathbb{R}^3$, Ω having the smooth boundary $\partial\Omega \in C^4$, the evolution of the vorticity $w = w(t, x)$ of an incompressible viscous flow in time $t \in [0, T)$ at $x \in \bar{\Omega} = \Omega \cup \partial\Omega$ is governed by the initial value problem

$$(0.1) \quad \begin{cases} \frac{\partial}{\partial t} w - \Delta w &= w \cdot \nabla v - v \cdot \nabla w, \\ \operatorname{div} w &= 0, \\ w(0, \cdot) &= w_0, \\ v(t, x) &= \operatorname{rot}^{-1} w(t, x), \end{cases}$$

the function $v(t, x)$ denoting the flow velocity. The usual condition of adherence

$$(0.2) \quad v(t, x)|_{\partial\Omega} = 0$$

for the flow velocity at the boundary constitutes a nonlocal boundary condition in terms of w .

In case of a sufficiently smooth solution $v(t, x)$ with pressure function $p(t, x)$ to the initial boundary value problem of the Navier-Stokes equations

$$(0.3) \quad \begin{cases} \frac{\partial}{\partial t} v - \Delta v + \nabla p + v \cdot \nabla v &= 0, \\ \operatorname{div} v &= 0, \\ v|_{\partial\Omega} &= 0, \\ v(0, \cdot) &= v_0, \end{cases}$$

the equations (0.1) follow for $w = \text{rot } v$ by formally applying the operator rot on the equations (0.3).

Evidently system (0.1), (0.2) in itself does not imply the compatibility condition due to the pressure gradient ∇p in (0.3) which there severely restricts the solution's initial regularity [9, 19, 22, 8, 15, 23, 12].

In any flow the vorticity being an important feature, the direct solution of the initial boundary problem (0.1), (0.2) would be of special interest. In the rich literature concerned with flow vorticity, functional properties of operator rot as well as its use in numerical approximations mainly have been studied in the frame of $H^1(\Omega)$ and related trace spaces, cp. [21, 5, 26] and the citations there. A potentialtheoretic representation of rot^{-1} with zero boundary condition is established in [24]. In general Sobolev spaces the construction of rot^{-1} is presented in [3] likewise for vanishing boundary values, and recently in [17] for the case of Hölder-continuous functions with vanishing normal components at the boundary. In [2] boundary conditions for solutions of the Navier-Stokes equations have been formulated in terms of the vorticity. Many fundamental aspects of vorticity, mainly for flows in the whole \mathbb{R}^2 or \mathbb{R}^3 , are presented in [11]. However, the difficulty which stems from the nonlocal boundary condition (0.2) in the initial boundary value problem (0.1) seemingly has not been overcome until now.

Below in Section 1, suitable solution spaces, which with a view of (0.2) we introduce in terms of eigenfunctions of the Stokes operator, will be characterized independently of these eigenfunctions. In Section 2 having defined generalized as well as strong solutions to (0.1), (0.2) we will prove the uniqueness of generalized solutions. In Section 3 by a Galerkin Ansatz based again on Stokes eigenfunctions we will show the existence of a unique strong solution to (0.1), (0.2) locally in time, which even exists globally in case of sufficiently small initial data. To the last section I have found inspiring devices in [7].

Besides the Lebesgue spaces $L^q = L^q(\Omega)$ of vector valued functions $f : \Omega \rightarrow \mathbb{R}^n$ with norm $\|\cdot\|_{L^q}$ we will need the Hilbert spaces $H^m = H^m(\Omega)$ of vector valued functions $f : \bar{\Omega} \rightarrow \mathbb{R}^3$ which have all partial derivatives $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} f = D^\alpha f \in L^2(\Omega)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq m$ up to the order $m \in \mathbb{N}$ in the distributional sense, $\mathbb{N} = \{0, 1, 2, \dots\}$ denoting the set of natural numbers, $\mathbb{N}_1 = \{1, 2, \dots\}$. As usual, the norm $\|\cdot\| = \|\cdot\|_{H^0}$ in $H^0 = L^2(\Omega)$ is given by the inner

product

$$\langle f, g \rangle = \int_{\Omega} f(x) \cdot g(x) dx, \quad \|f\|^2 = \langle f, f \rangle,$$

and the inner product

$$\langle f, g \rangle_{H^m} = \sum_{|\alpha| \leq m} \langle D^\alpha f, D^\alpha g \rangle$$

in H^m defines there the norm $\|\cdot\|_{H^m}$.

By C_c^∞ we will denote the space of functions $f : \bar{\Omega} \rightarrow \mathbb{R}^3$ which possess partial derivatives of all orders, f having compact support in Ω , and $C_{c,\sigma}^\infty$ stands for the subspace of divergence-free functions in C_c^∞ . We will write

$$L_\sigma^2 = \text{closure}(C_{c,\sigma}^\infty) \subset L^2(\Omega), \text{ or } \overset{\circ}{H}_\sigma^1 = \text{closure}(C_{c,\sigma}^\infty) \subset H^1(\Omega), \text{ or}$$

$\overset{\circ}{H}^1 = \text{closure}(C_c^\infty) \subset H^1(\Omega)$, for the closure of $C_{c,\sigma}^\infty$ in L^2 , or in H^1 , or for the closure of C_c^∞ in H^1 , respectively. For definition of the space $L^\infty = L^\infty(\Omega)$ with the usual norm $\|\cdot\|_{L^\infty}$, and of the Sobolev spaces $W^{m,q}(\Omega)$, $q \geq 1$, and finally of the fractional order spaces $H^s = H^s(\Omega)$, $H^s(\partial\Omega)$ with general $s \in \mathbb{R}$ cp. [1]. In case of any given interval $J \subset \mathbb{R}$ and Banach space B with norm $\|\cdot\|_B$, we will denote by $C^0(J, B)$ the Banach space of continuous maps $f : J \rightarrow B$, $C^0(J, B)$ being equipped with the supremum norm $\sup_{t \in J} \|f(t)\|_B$. In addition $C^m(\Omega)$ or

$C^m(\bar{\Omega})$ stands for the space of all continuous functions $f : \Omega \rightarrow \mathbb{R}^3$ or $f : \bar{\Omega} \rightarrow \mathbb{R}^3$, f possessing in Ω continuous or in Ω uniformly continuous partial derivatives $D^\alpha f$ of all orders $|\alpha| \leq m$, respectively. Again $C_\sigma^m(\bar{\Omega})$ means the subspace of all $f \in C^m(\bar{\Omega})$ with $\text{div } f = 0$.

Finally we write H. Weyl's projection $P : L^2(\Omega) \rightarrow L_\sigma^2$ in the form

$$(0.4) \quad Pf = f - \nabla q,$$

∇q being the gradient of a (possibly weak) solution of the boundary value problem

$$\Delta q = \text{div } f, \quad N \cdot \nabla q|_{\partial\Omega} = N \cdot f|_{\partial\Omega},$$

N denoting the field of outer normals on $\partial\Omega$. In case $\partial\Omega \in C^{m+1}$, $m \in \mathbb{N}_1$, $P : H^m \rightarrow H^m$ represents a bounded linear map, [21, p. 18].

We will need the Hölder inequalities

$$(0.5) \quad \|\varphi \cdot \psi\|_{L^p} \leq \|\varphi\|_{L^q} \cdot \|\psi\|_{L^{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p}, \quad 1 \leq p \leq q,$$

$$\left| \int_{\Omega} |f| \cdot |g| \cdot |h| dx \right| \leq \|f\|_{L^6} \cdot \|g\|_{L^3} \cdot \|h\|,$$

for all $\varphi \in L^q$, $\psi \in L^{q'}$, $f \in L^6$, $g \in L^3$, $h \in L^2$, furthermore Young's inequality

$$(0.6) \quad |a| \cdot |b| \leq \mu \cdot |a|^q + c_\mu \cdot |b|^{q'} \text{ with} \\ \mu = \epsilon^q/q, \quad c_\mu = \epsilon^{-q}/q', \epsilon > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad q > 1, [10],$$

finally the embedding

$$(0.7) \quad \begin{aligned} H^{j+m} &\hookrightarrow W^{j,q} && \text{if } 2 \leq q \leq \frac{2}{1-\frac{2}{3}m}, j, m \in \mathbb{N}, \text{ and} \\ H^{j+m} &\hookrightarrow C^j(\bar{\Omega}) && \text{if } m-1 < \frac{3}{2} < m, [1]. \end{aligned}$$

By c, c_0, c_1, \dots we mean constants which may have different values even in the same chain of inequalities.

1. The solution spaces and basic estimates

A key to fundamental results in the theory of the Navier-Stokes equations is the quadratic form

$$(1.1) \quad a(v, \varphi) = \langle -P\Delta v, \varphi \rangle = \langle \nabla v, \nabla \varphi \rangle,$$

the representation (1.1) being valid with $\varphi \in \mathring{H}_\sigma^1$ for all

$$(1.2) \quad v \in \mathring{H}_\sigma^1 \cap H^2$$

which in particular are divergence-free and fulfill the condition of adherence $v|_{\partial\Omega} = 0$. We will check whether a similarly useful quadratic form can be found for the rotation $w = \text{rot } v$, w only fulfilling the boundary condition

$$(1.3) \quad N \cdot w|_{\partial\Omega} = 0$$

in case $v \in \mathring{H}_\sigma^1$, [18, p. 241]. Approximating w in H^2 by divergence-free functions $W \in C_\sigma^2(\bar{\Omega})$, and φ in H^1 by functions $\Phi \in C^1(\bar{\Omega})$, we find

$$\begin{aligned} \langle -\Delta W, \Phi \rangle &= \langle \text{rot } P \text{ rot } W, \Phi \rangle = \\ &= \langle P \text{ rot } W, P \text{ rot } \Phi \rangle + \int_{\partial\Omega} N \cdot [(P \text{ rot } W) \times \Phi] dS, \end{aligned}$$

the latter due to the Gauss theorem. Thus for the limits w, φ we get

Proposition 1.1: *For each $w = \text{rot } v$, $v \in \mathring{H}_\sigma^1 \cap H^3$, and all $\varphi \in H^1$, the representation*

$$(1.4) \quad b(w, \varphi) = \langle -\Delta w, \varphi \rangle = \langle P \text{ rot } w, P \text{ rot } \varphi \rangle$$

is valid, if v fulfils the additional boundary condition

$$(1.5) \quad P \operatorname{rot} w|_{\partial\Omega} = P \operatorname{rot}^2 v|_{\partial\Omega} = -P\Delta v|_{\partial\Omega} = 0. \square$$

Equation (1.5) e.g. holds true for each eigenfunction $e_j = v \in \overset{\circ}{H}_\sigma^1 \cap H^4$, [21, p. 39], of the Stokes operator

$$A = -P\Delta : D_A = \overset{\circ}{H}_\sigma^1 \cap H^2 \rightarrow L_\sigma^2,$$

the function e_j being related to the j^{th} eigenvalue λ_j :

$$(1.6) \quad \left\{ \begin{array}{l} Ae_j = \lambda_j \cdot e_j, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty \text{ with } k \rightarrow \infty, \\ \operatorname{div} e_j = 0, \\ e_j|_{\partial\Omega} = 0, \\ \text{the sequence } (e_j) \text{ being orthonormalized:} \\ \langle e_j, e_k \rangle = \delta_{jk} = 1 \text{ if } j = k, \delta_{jk} = 0 \text{ if } j \neq k. \end{array} \right.$$

Therefore from Proposition 1.1 we find

Corollary 1.1: *The representation (1.4) holds for each $w = \operatorname{rot} v$,*

$$(1.7) \quad v \in E^3 := \operatorname{closure} \left\{ \sum_{j=1}^k a_j \cdot e_j \mid k \in \mathbb{N}_1, a_j \in \mathbb{R} \right\} \subset H^3(\Omega),$$

and all $\varphi \in H^1(\Omega)$.

Proof: H^3 -convergence of the sums $S_k = \sum_{j=1}^k a_j \cdot e_j$ includes the

H^1 -convergence of the sequence $(P\Delta S_k)$, thus the $H^{1/2}(\partial\Omega)$ -convergence of the boundary values $P\Delta S_k|_{\partial\Omega} = 0$. \square

For the rotations $w = \operatorname{rot} v$ of functions $v \in E^3$, we find a variant of the Cattabriga-Solonnikov estimate

$$(1.8) \quad \|f\|_{H^m} \leq c \cdot \|P\Delta f\|_{H^{m-2}} \text{ for all } f \in \overset{\circ}{H}_\sigma^1 \cap H^m, m \geq 1,$$

if $\partial\Omega \in C^n$, $n = \max\{2, m\}$, [4, 9] :

Proposition 1.2.: *The rotation $w = \operatorname{rot} v$ of any function $v \in E^3$ obeys the estimate*

$$(1.9) \quad \|w\|_{H^2} \leq c \cdot \|P\Delta w\| \leq c \cdot \|\Delta w\|$$

with some constant c being independent of w and v .

Proof: From (1.8) we find

$$\begin{aligned} \|w\|_{H^2} &\leq c_1 \cdot \|v\|_{H^3} \leq c_2 \cdot \|P\Delta v\|_{H^1} \leq c_3 \cdot \|\nabla P\Delta v\| = \\ &\leq c_3 \|P \operatorname{rot} P\Delta v\| = c_3 \cdot \|P \operatorname{rot}^2 w\| = c_3 \cdot \|P\Delta w\|, \end{aligned}$$

since for $v \in E^3$ we have $P\Delta v \in \mathring{H}_\sigma^1$, and the equation

$$(1.10) \quad \|\nabla f\| = \|\operatorname{rot} Pf\| = \|P \operatorname{rot} f\|$$

holds true for all $f \in \mathring{H}_\sigma^1$. The latter equation can easily be verified by the usual approximation procedure, noting $P \operatorname{rot} f = \operatorname{rot} f$ because of (1.3). \square

For rotations $w = \operatorname{rot} v$ of functions $v \in D_A$, the following variant of Poincaré's inequality

$$(1.11) \quad \|f\| \leq c \cdot \|\nabla f\| \text{ for all } f \in \mathring{H}^1$$

holds:

Proposition 1.3: *The rotation $w = \operatorname{rot} v$ of any $v \in D_A$ satisfies the inequality*

$$(1.12) \quad \|w\|_{H^1} \leq c \cdot \|P \operatorname{rot} w\|$$

with some constant c being independent of w and v .

Proof: From (1.8) we see

$$\|w\|_{H^1} \leq c_1 \cdot \|v\|_{H^2} \leq c_2 \cdot \|P\Delta v\| = c_2 \cdot \|P \operatorname{rot}^2 v\| = c_2 \cdot \|P \operatorname{rot} w\|. \square$$

In addition we prove a slightly more general result as in [21, p. 163]:

Proposition 1.4: *For each $f \in \mathring{H}_\sigma^1$, the trilinearform*

$$c(f, g, h) = \int_{\Omega} (f \cdot \nabla g) \cdot h \, dx$$

is skew symmetric in g, h for all $g, h \in H^1$: There holds

$$(1.13) \quad c(f, g, h) = -c(f, h, g).$$

Proof: We approximate f in H^1 by functions $F \in C_{c,\sigma}^\infty$, g and h in H^1 , by functions $G \in C^1(\bar{\Omega})$ and $H \in C^1(\bar{\Omega})$, respectively. Using the Gauss theorem and $\operatorname{div} F = 0, F_{\partial\Omega} = 0$ we find (1.13) first of all for F, G, H , from which this equation follows for the limits f, g, h , too. \square

We will fix our solution spaces in

Definition 1.1: (i) For $s \in \mathbb{R}_+ = \{\sigma \in \mathbb{R} \mid 0 \leq \sigma\}$ we denote by

$$E^s := \text{closure} \left\{ \sum_{j=1}^k a_j \cdot e_j \mid k \in \mathbb{N}_1, a_j \in \mathbb{R} \right\} \subset H^s(\Omega), \text{ or}$$

$$F^s := \text{closure} \left\{ \sum_{j=1}^k b_j \cdot \tilde{e}_j \mid k \in \mathbb{N}_1, b_j \in \mathbb{R} \right\} \subset H^s(\Omega)$$

the closed linear subspace of the Hilbert space $H^s(\Omega)$, generated by the Stokes eigenfunctions e_j , or by the functions

$$(1.14) \quad \tilde{e}_j := \frac{\text{rot } e_j}{\lambda_j^{1/2}}, \quad \text{respectively.}$$

(ii) For $s \in \mathbb{N}$, by the inner product

$$(1.15) \quad \langle f, g \rangle_s = \int_{\Omega} (P \text{rot}^s f) \cdot (P \text{rot}^s g) dx$$

we define

$$(1.16) \quad \|f\|_s := \langle f, f \rangle_s^{1/2} \text{ for all } f, g \in H^s(\Omega).$$

Remark 1.1: Evidently, the equalities $E^0 = L^2_{\sigma}$, $E^1 = \overset{\circ}{H}^1_{\sigma}$, $E^2 = \overset{\circ}{H}^2_{\sigma} \cap H^2$ result from the completeness properties of the system (e_j) . The representation $E^3 = A^{-1} \overset{\circ}{H}^1_{\sigma}$ will be established in the proof of the next Proposition 1.5.

Remark 1.2: The orthonormality of the \tilde{e}_j results from

$$\langle \tilde{e}_j, \tilde{e}_l \rangle \cdot \lambda_j^{1/2} \cdot \lambda_l^{1/2} = \langle e_j, \text{rot}^2 e_l \rangle + \int_{\partial\Omega} N \cdot [e_j \times \text{rot } e_l] dS = \langle e_j, P \text{rot}^2 e_l \rangle = \delta_{jl} \cdot \lambda_l \text{ because of (1.6).}$$

Proposition 1.5: (i) On each Hilbert space E^s , $s = 0, 1, 2, 3$, as well as on each Hilbert space F^s , $s = 0, 1, 2$ the functional $\|\cdot\|_s$ defines a norm which is equivalent to $\|\cdot\|_{H^s}$. Thus there holds

$$(1.17) \quad c_1 \cdot \|f\|_s \leq \|f\|_{H^s} \leq c_2 \cdot \|f\|_s \quad \text{for all } f \in E^s, s = 0, 1, 2, 3,$$

and for all $f \in F^s$, $s = 0, 1, 2$, with positive constants c_j being independent of f .

(ii) The operator $\text{rot} : E^{s+1} \rightarrow F^s$ maps E^{s+1} isomorphically onto F^s , rot being isometrical with respect to the norm $\|\cdot\|_s$.

Proof:

(1.) The functional $\|\cdot\|_s$ on E^s :

Evidently each closed subspace E^s and F^s in H^s forms again a Hilbert space being equipped with the norm $\|\cdot\|_{H^s}$ of $H^s(\Omega)$, and the inequality

$$(1.18) \quad \|f\|_s \leq c \cdot \|f\|_{H^s}$$

holds by Definition (1.16) for each $f \in H^s, s \in \mathbb{N}$, with some constant $c > 0, c$ being independent of f . In case $s = 0$, (1.18) holds true even with equality sign and $c = 1$.

For any $f \in E^s$, the additional inequality

$$(1.19) \quad \|f\|_{H^s} \leq c \cdot \|f\|_s$$

follows in case $s = 1$, thus $f \in E^1 = \overset{\circ}{H}_\sigma^1$, from (1.10) and Poincaré's inequality (1.11). In case $s = 2$, thus $f \in E^2 = \overset{\circ}{H}_\sigma^1 \cap H^2$, we see

$$(1.20) \quad \|f\|_{H^2} \leq c \cdot \|P\Delta f\| = c \cdot \|P \operatorname{rot}^2 f\| = c \cdot \|f\|_2$$

from Cattabriga-Solonnikov's estimate (1.8). In case $s = 3$, for each $f \in E^3$ we conclude

$$(1.21) \quad P\Delta f \in \overset{\circ}{H}_\sigma^1$$

from Definition 1.1., $P\Delta f$ being limit in $H^1(\Omega)$, thus $P\Delta f|_{\partial\Omega}$ in $H^{1/2}(\partial\Omega)$, of sums

$$\sum_{j=1}^k a_j \cdot P\Delta e_j = \sum_{j=1}^k a_j \cdot \lambda_j \cdot e_j, \quad k \longrightarrow \infty. \quad \text{Therefore we get (1.19) from}$$

$$\|f\|_{H^3} \leq c \cdot \|P\Delta f\|_{H^1} \leq c \cdot \|\nabla P\Delta f\| = c \cdot \|P \operatorname{rot} P\Delta f\| = c \cdot \|P \operatorname{rot}^3 f\| = c \cdot \|f\|_3,$$

using (1.8), (1.11), and (1.10) again. \square

In addition, (1.21) being equivalent to $f \in A^{-1} \overset{\circ}{H}_\sigma^1$, we have proved the last statement in Remark 1.1. \square

(2.) The equality $\mathbf{F}^s = \operatorname{rot} \mathbf{E}^{s+1}, s = 0, 1, 2$:

We conclude

$$(1.22) \quad \operatorname{rot} E^{s+1} \subset F^s$$

from the fact that each first order distributional partial derivative of any H^{s+1} -convergent sequence commutes in H^s with the limiting process. The additional inclusion

$$(1.23) \quad F^s \subset \operatorname{rot} E^{s+1}$$

results in case $s = 0$, if to each $w \in F^s$,

$$w = \lim_{k \rightarrow \infty} w_k = \lim_{k \rightarrow \infty} \left(\sum_{j=1}^k b_j \cdot \lambda_j^{-1/2} \cdot \text{rot } e_j \right) \quad \text{in } H^s,$$

we consider the related sequence

$$(1.24) \quad v_k := \text{rot}^{-1} w_k = \sum_{j=1}^k b_j \cdot \lambda_j^{-1/2} \cdot e_j \in \overset{\circ}{H}_\sigma^1 \cap H^3, k \in \mathbb{N}_1.$$

The convergence of (v_k) in E^{s+1} follows from (1.10) and (1.11) in case $s = 0$. In case $s = 1$, now with (1.8), we see the E^2 -convergence of $(v_k) \rightarrow v \in E^2$, and $w = \text{rot } v$, from

$$(1.25) \quad \|v_k\|_{H^2} \leq c \cdot \|P \text{rot}^2 v_k\| = c \|P \text{rot } w_k\| \leq c \|w_k\|_{H^1}$$

and the analogous estimate $\|v_k - v_l\|_{H^2} \leq c \|w_k - w_l\|_{H^1}$, for all $k, l \in \mathbb{N}_1$, (w_k) being F^1 -convergent to the limit $w \in F^1$.

Finally in case $s = 2$, using (1.8), (1.10), (1.11), we get the E^3 -convergence $(v_k) \rightarrow v \in E^3$, and the equality $w = \text{rot } v$, from

$$(1.26) \quad \begin{aligned} \|v_k\|_{H^3} \leq c \cdot \|\nabla P \text{rot}^2 v_k\| &= c \cdot \|P \text{rot}^3 v_k\| \\ &= c \cdot \|P \text{rot}^2 w_k\| \leq c \|P w_k\|_{H^2} \end{aligned}$$

and from the analogous estimate

$$\|v_k - v_l\|_{H^3} \leq c \cdot \|w_k - w_l\|_{H^2} \text{ for all } k, l \in \mathbb{N}_1,$$

the sequence (w_k) being F^2 -convergent to the limit $w \in F^2$. Consequently, the linear map $\text{rot} : E^{s+1} \rightarrow F^s$ is surjective, $s = 0, 1, 2$. \square

(3.) The functional $\|\cdot\|_s$ on F^s , $s = 0, 1, 2$:

As we have seen in the last section (2.), each $w \in F^s$ has the representation $w = \text{rot } v$ with some $v \in E^{s+1}$, $s = 0, 1, 2$. The functional $\|\cdot\|_0$ being just the L^2 -norm, we have to prove the additional inequality (1.19) only in the cases $s = 1, 2$. Firstly in case $s = 1$, from (1.8) we find

$$(1.27) \quad \|w\|_{H^1} \leq c \cdot \|v\|_{H^2} \leq c \cdot \|P \Delta v\| = c \cdot \|P \text{rot } w\| = c \cdot \|w\|_1,$$

and in case $s = 2$ similarly we get

$$(1.28) \quad \begin{aligned} \|w\|_{H^2} &\leq c \cdot \|v\|_{H^3} \leq c \|P \Delta v\|_{H^1} \leq c \cdot \|\nabla P \Delta v\| = \\ &= c \cdot \|P \text{rot } P \Delta v\| = c \cdot \|P \text{rot}^2 w\| = c \cdot \|w\|_2 \end{aligned}$$

because of $P \Delta v|_{\partial\Omega} = 0$ for each $v \in E^3$ and (1.10), (1.11).

Thus the functional $\|\cdot\|_s$ representing the Hilbert space norm on E^s as well as on F^s , from the equality

$$(1.29) \quad \|v\|_{s+1} = \|\operatorname{rot} v\|_s = \|w\|_s,$$

which immediately results from Definition (1.15), we conclude that the surjective linear map $\operatorname{rot} : E^{s+1} \rightarrow F^s$ is isometrical, therefore isomorphism, too, having the bounded inverse $\operatorname{rot}^{-1} : F^s \rightarrow E^{s+1}$, $s = 0, 1, 2$. \square

In an analogous way like John Heywood in [7], we will use the following well known Lemma from the theory of ordinary differential equations, cp. [25]:

Lemma 1.1: *Let $g(t, x) \geq 0, f(t) \geq 0$, and $\psi(t) \geq 0$ denote continuous functions defined for positive arguments $t \geq 0, x \geq 0$, $g(t, x)$ being locally Lipschitz continuous in x .*

(a) *Then each solution $\varphi = \varphi(t, \varphi_0)$ of the differential inequality*

$$(1.30) \quad \begin{aligned} \frac{d}{dt}\varphi + \psi(t) &\leq g(t, \varphi) + f(t) && \text{for } t > t_0 \geq 0, \\ \varphi(t_0) &\leq \varphi_0 && \text{for some } \varphi_0 \geq 0 \end{aligned}$$

is bounded from above by the solution $\Phi(t, \varphi_0)$ to the initial value problem

$$(1.31) \quad \begin{aligned} \frac{d}{dt}\Phi &= g(t, \Phi) + f(t) && \text{for } t > t_0 \geq 0, \\ \Phi(t_0) &= \varphi_0, \end{aligned}$$

on the right hand maximal interval $[t_0, T)$ of existence of Φ ,

the function $\Phi(t, \varphi_0)$ being monotone increasing in φ_0 .

In case $g(t, \varphi)$ being monotone increasing in φ , additionally there holds

$$(1.32) \quad \int_{t_0}^t \psi(\tau) d\tau \leq \tilde{\Phi}(t, \varphi_0) := \varphi_0 + \int_{t_0}^t [g(\tau, \Phi(\tau, \varphi_0)) + f(\tau)] d\tau,$$

$\tilde{\Phi}(t, \varphi_0)$ being monotone increasing in $t \in [t_0, T)$.

(b) *In the special case $g(t, \varphi) = h(t) \cdot \varphi$ we get the solution Ψ of (1.31) explicitly from*

$$(1.33) \quad \Phi(t, \varphi_0) = \left(\varphi_0 + \int_{t_0}^t f(\tau) e^{-\int_{t_0}^{\tau} h(\sigma) d\sigma} d\tau \right) \cdot e^{\int_{t_0}^t h(\tau) d\tau} := \Psi(t, \varphi_0, h, f),$$

and the bound $\tilde{\Phi}$ is given by

$$(1.34) \quad \tilde{\Phi}(t, \varphi_0) = \varphi_0 + \int_{t_0}^t [h(\tau) \cdot \Phi(\tau, \varphi_0) + f(\tau)] d\tau := \tilde{\Psi}(t, \varphi_0, h, f).$$

If we take φ_0 from any sequence converging to 0, and the functions $h(t), f(t)$ each from some sequences converging to zero uniformly on any fixed interval $[t_0, t_1] \subset [t_0, T)$, then the related bounds $\Psi(t, \varphi_0, h, f)$ and $\tilde{\Psi}(t, \varphi_0, h, f)$ converge to zero uniformly in $t \in [t_0, t_1]$.

2. Uniqueness of generalized solutions

Definition 2.1: We will call generalized solution to the initial value problem (0.1), (0.2) each function $w \in C^0([0, T), F^1)$, $\frac{\partial}{\partial t} w \in C^0((0, T), F^0)$, which

- (i) has the representation $w(t) = \text{rot } v(t)$ with some function $v \in C^0([0, T), E^2)$ and
- (ii) fullfils the equations

$$(2.1) \quad \begin{cases} \langle \frac{\partial}{\partial t} w, \varphi \rangle + \langle P \text{rot } w, P \text{rot } \varphi \rangle &= \langle w \cdot \nabla v - v \cdot \nabla w, \varphi \rangle, \\ t \in J_0 = (0, T), \\ \langle w(0), \varphi \rangle &= \langle w_0, \varphi \rangle, \end{cases}$$

for all $\varphi \in F^1$.

Each solution $w = \text{rot } v \in C^0([0, T), F^2)$, $\frac{\partial}{\partial t} w \in C^0([0, T), F^0)$ with $v \in C^0([0, T), E^3)$ to (0.1), (0.2) on $[0, T)$ will be called strong solution of the initial value problem.

Evidently, because of (1.4) and $F^2 \subset F^1, E^3 \subset E^2$, each strong solution of (0.1), (0.2) represents a generalized solution, too.

Theorem 2.1: The initial value problem (0.1), (0.2) of the vorticity transport & diffusion equation admits at most one generalized solution.

Proof: Let $w_m = \text{rot } v_m, m = 1, 2$, denote two solution of (2.1) having the same initial value $w_0 \in F^1$. Then by a short calculation we see that their differences

$$\eta := w_2 - w_1, \quad \zeta := v_2 - v_1, \quad \text{thus } \eta = \text{rot } \zeta \in F^1,$$

fullfil

$$(2.2) \quad \langle \frac{\partial}{\partial t} \eta, \varphi \rangle + \langle P \text{rot } \eta, P \text{rot } \varphi \rangle = \langle f, \varphi \rangle, t \in J_0, \langle \eta(0), \varphi \rangle = 0,$$

where

$$(2.3) \quad f = \langle \eta \cdot \nabla v_2, \varphi \rangle + \langle w_1 \cdot \nabla \zeta, \varphi \rangle - \langle \zeta \cdot \nabla w_2, \varphi \rangle - \langle v_1 \cdot \nabla \eta, \varphi \rangle.$$

If we take $\varphi := \eta$, the last term in (2.3) vanishes because of the skew symmetry of this inner product in η and φ . For the first term in (2.3) we find

$$(2.4) \quad |\langle \eta \cdot \nabla v_2, \eta \rangle| \leq \|\eta\|_{L^6} \cdot \|\nabla v_2\| \cdot \|\eta\|_{L^3} \leq c \|\eta\|_{H^1}^{3/2} \cdot \|w_2\| \cdot \|\eta\|^{1/2} \leq \mu \|P \operatorname{rot} \eta\|^2 + c_\mu \cdot \|w_2\|^4 \cdot \|\eta\|^2$$

using (0.5), the embedding (0.7), (1.10), (1.12), and finally Young's inequality (0.6).

Similarly we get

$$(2.5) \quad \begin{aligned} |\langle w_1 \cdot \nabla \zeta, \eta \rangle| &\leq \|w_1\| \cdot \|\nabla \zeta\|_{L^6} \cdot \|\eta\|_{L^3} \\ &\leq c \|w_1\| \cdot \|P \operatorname{rot} \eta\| \cdot \|\eta\|_{H^1}^{1/2} \cdot \|\eta\|^{1/2} \leq \\ &\leq \mu \cdot \|P \operatorname{rot} \eta\|^2 + c_\mu \cdot \|w_1\|^4 \cdot \|\eta\|^2, \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} |\langle \zeta \cdot \nabla w_2, \eta \rangle| &\leq \|\zeta\|_{L^6} \cdot \|\nabla w_2\| \cdot \|\eta\|_{L^3} \\ &\leq c \|\eta\|_{L^2}^{3/2} \cdot \|P \operatorname{rot} w_2\| \cdot \|P \operatorname{rot} \eta\|^{1/2} \leq \\ &\leq \mu \cdot \|P \operatorname{rot} \eta\|^2 + c_\mu \cdot \|P \operatorname{rot} w_2\|^4 \cdot \|\eta\|^2. \end{aligned}$$

Summing up the last three inequalities, from (2.2) we see

$$(2.7) \quad \begin{aligned} \frac{d}{dt} \|\eta\|^2 + 2(1 - 3\mu) \cdot \|P \operatorname{rot} \eta\|^2 &\leq c_\mu \cdot \|\eta\|^2 \cdot \{\|w_1\|^4 + \\ &+ \|w_2\|^4 + \|P \operatorname{rot} w_2\|^4\}, t \in J_0, \\ \|\eta(0)\|^2 &= 0. \end{aligned}$$

Since, by our assumption, $\|w_j(t)\|$ and $\|P \operatorname{rot} w_j(t)\|$ are given continuous functions on $[0, T)$, from Lemma 1.1 we conclude $\eta(t) = w_2(t) - w_1(t) = 0$ in F^1 on their interval of existence. \square

3. Existence of a unique strong solution

In this section we will prove the existence of a unique F^2 -continuous solution $w(t) = \operatorname{rot} v(t)$ to the initial value problem (0.1), (0.2) by means of Galerkin approximations on the basis of the complete orthonormal system (\tilde{e}_j) in F^0 from (1.14). The k^{th} Galerkin approximation

$$(3.1) \quad w_k(t, x) := \sum_{j=1}^k b_{kj}(t) \cdot \tilde{e}_j(x) \in F^\circ, \quad b_{kj}(t) = \langle w_k(t, \cdot), \tilde{e}_j \rangle$$

defines the related velocity $v_k(t, \cdot) := \text{rot}^{-1} w_k(t, \cdot) \in E^1$,

$$(3.2) \quad v_k(t, x) := \sum_{j=1}^k a_{kj}(t) \cdot e_j(x), \quad a_{kj}(t) = \langle v_k(t, \cdot), e_j \rangle = \lambda_j^{-1/2} b_{kj}(t)$$

in a unique way, $\|\text{rot} \cdot\|$ being norm in E^1 . Let $P_k : L^2(\Omega) \rightarrow E_k^0$, $Q_k : L^2(\Omega) \rightarrow F_k^0$ denote the projections of $L^2(\Omega)$ onto the span E_k^0 or F_k^0 of the first k functions e_j or $\tilde{e}_j = \lambda_j^{-1/2} \text{rot} e_j$, $j = 1, \dots, k$, respectively. Thus we have

$$(3.3) \quad P_k f = \sum_{j=1}^k \langle f, e_j \rangle e_j, \quad Q_k f = \sum_{j=1}^k \langle f, \tilde{e}_j \rangle \cdot \tilde{e}_j$$

for $f \in L^2(\Omega)$.

Lemma 3.1: *The equation*

$$(3.4) \quad P \text{rot} Q_k f = P_k \text{rot} f$$

holds for all $f \in H^1$, and we have

$$(3.5) \quad Q_k \text{rot} f = \text{rot} P_k f$$

for all $f \in \overset{\circ}{H}_\sigma^1$.

Proof: Approximating f in H^1 by functions $\phi_m \in C^1(\bar{\Omega})$, and recalling $P \text{rot}^2 e_j = \lambda_j e_j$, we see

$$(3.6) \quad \langle \phi_m, \tilde{e}_j \rangle = \langle \text{rot} \phi_m, \lambda_j^{-1/2} e_j \rangle$$

because of $\int_{\partial\Omega} N \cdot [e_j \times \phi_m] dS = 0$. From (3.6) with $\phi_m \rightarrow f$ in H^1 , we get (3.4).

Similarly approximating $f \in \overset{\circ}{H}_\sigma^1$ in H^1 by functions $\phi_m \in C_{c,\sigma}^\infty$, from the equality

$$(3.7) \quad \langle \text{rot} \phi_m, \tilde{e}_j \rangle = \langle \phi_m, \lambda_j^{-1/2} \cdot P \text{rot}^2 e_j \rangle$$

we find (3.5). \square

We will calculate w_k from the initial value problem

$$(3.8) \quad \begin{cases} \frac{\partial}{\partial t} w_k - \Delta w_k &= Q_k [w_k \cdot \nabla v_k - v_k \cdot \nabla w_k], \quad t \geq 0, \\ w_k(0) &= Q_k w(0), \quad \text{with} \\ v_k &= \text{rot}^{-1} w_k \text{ from (3.2),} \end{cases}$$

for all $k \in \mathbb{N}_1$.

Theorem 3.1: *Assume $w(0) = \text{rot } v(0) \in F^2, v(0) \in E^3$. Then all Galerkin approximations $w_k(t) = \text{rot } v_k(t) \in F^2, v_k(t) \in E^3$ calculated from (3.8), exist on a sufficiently small time interval $J = [0, T)$.*

The whole sequences $(w_k(t))$, or $(v_k(t))$, or $(w_{kt}(t))$ converge in F^2 , or E^3 , or F^0 , respectively, uniformly with respect to $t \in J' = [0, T']$ for each $T' \in (0, T)$. Their limit functions $w = (\text{rot } v) \in C^0(J, F^2), v \in C^0(J, E^3)$, and $w_t \in C^0(J, F^0)$ represent altogether the unique strong solution of (0.1), (0.2).

The proof is given in the following Sections 3.1 - 3.7.

3.1. Existence of the w_k, v_k :

The initial value problem (3.8) is equivalent to the following initial value problem of k ordinary differential equations for the k coefficients $b_{kj}(t)$: Taking the inner product of (3.8) with \tilde{e}_j we find

$$(3.9) \quad \begin{aligned} \frac{d}{dt} b_{kj} + \lambda_j b_{kj} &= \sum_{m,n=1}^k \lambda_n^{-1/2} \cdot b_{km} \cdot c_{mnj} \cdot b_{kn}, \\ b_{kj}(0) &= \langle w(0), \tilde{e}_j \rangle, \quad j = 1, \dots, k, \end{aligned}$$

because of (1.4). The coefficients

$$(3.10) \quad c_{mnj} = \langle \tilde{e}_m \cdot \nabla e_n - e_n \cdot \nabla \tilde{e}_m, \tilde{e}_j \rangle$$

of the quadratic form in (3.9) are determined by the \tilde{e}_m, e_n . It is well known from the theory of ordinary differential equations that for each initial value $(b_{kj}(0)) \in \mathbb{R}^k$ the system (3.9) has a unique solution $(b_{kj}) \in C^\infty(J, \mathbb{R}^k)$ on each (possibly small) time interval $J = [0, T), J$ having the property that the $|b_{kj}|$ remain uniformly bounded on each compact subinterval $[0, T']$ with any fixed $T' \in [0, T)$, [6].

3.2. Bounds in F^1, E^2 :

For the projection $Q_k f_k$ of the function

$$(3.11) \quad f_k = w_k \cdot \nabla v_k - v_k \nabla w_k, \quad w_k = \text{rot } v_k,$$

from (0.5), the imbedding theorems and (1.10), (1.12) we find the estimate

$$(3.12) \quad \|Q_k f_k\| \leq \|f_k\| \leq \|w_k\|_{L^3} \cdot \|\nabla v_k\|_{L^6} + \|v_k\|_{L^\infty} \cdot \|\nabla w_k\| \leq c \|P \text{rot } w_k\|^2.$$

Taking the inner product in $L^2(\Omega)$ of equation (3.8) with $-\Delta w_k$ gives

$$\frac{d}{dt} \|P \operatorname{rot} w_k\|^2 + 2 \|\Delta w_k\|^2 = 2 \langle Q_k f_k, -\Delta w_k \rangle \leq c \|P \operatorname{rot} w_k\|^2 \cdot \|\Delta w_k\|$$

because of (1.4) and $w_k, w_{kt} \in F^2$. Thus applying Young's inequality (0.6) we get

$$(3.13) \quad \begin{aligned} \frac{d}{dt} \|P \operatorname{rot} w_k\|^2 + 2(1 - \mu) \|\Delta w_k\|^2 &\leq c_\mu \|P \operatorname{rot} w_k\|^4, \\ \|P \operatorname{rot} w_k(0)\|^2 &\leq \|P \operatorname{rot} w(0)\|^2 = \varphi_1, \end{aligned}$$

with some $\mu \in (0, 1)$, c_μ from (0.6). The estimate of the initial values results from Bessel's inequality because of the orthogonality (1.6).

Recalling Lemma 1.1 we find the estimate

$$(3.14) \quad \begin{aligned} \|P \operatorname{rot} w_k(t)\|^2 &\leq \Phi_1(t, \varphi_1) = \\ &= \frac{\varphi_1}{1 - ct \cdot \varphi_1} \text{ on } \mathbf{J} = [\mathbf{0}, \mathbf{T}), \quad \mathbf{T} = \frac{\mathbf{1}}{c \cdot \varphi_1} \end{aligned}$$

for each solution to (3.13). In addition we have

$$(3.15) \quad \|w_k(t)\|_{H^1}^2 \leq c \Phi_1(t, \varphi_1) \text{ and } \|v_k(t)\|_{H^2}^2 \leq c \cdot \Phi_1(t, \varphi_1)$$

because of (1.8), (1.12). Thus looking at the definitions (3.1), (3.2) with the orthonormal system (\tilde{e}_j) we see that the $|b_{kj}(t)|$ are uniformly bounded on each $J' = [0, T']$, $T' < T$.

Therefore all solutions $w_k(t) \in F^2$, $v_k(t) \in E^3$ exist for each $t \in [0, T)$.

Moreover, Lemma 1.1 with (1.32) gives the estimate

$$(3.16), \quad \int_0^t \|\Delta w_k(\tau)\|^2 d\tau \leq \tilde{\Phi}_1(t, \varphi_1)$$

the continuous function $\tilde{\Phi}_1(t, \varphi_1)$ being monotone increasing in $t \in [0, T)$.

3.3. Bounds in $\mathbf{F}^2, \mathbf{E}^3$:

We will write $h_{(j)} = \frac{\partial}{\partial x_j} h$ for the (generalized) first order derivative of any function $h \in H^1(\Omega)$. Then using (0.5), for

$$(3.17) \quad f_{k(j)} = w_{k(j)} \nabla v_k - v_k \cdot \nabla w_{k(j)} + w_k \cdot \nabla v_{k(j)} - v_{k(j)} \cdot \nabla w_k$$

we get the estimate

$$(3.18) \quad \begin{aligned} \|f_{k(j)}\| &\leq \|w_{k(j)}\|_{L^6} \cdot \|\nabla v_k\|_{L^3} + \|v_k\|_{L^\infty} \cdot \|\nabla w_{k(j)}\| \\ &+ \|w_k\|_{L^\infty} \cdot \|\nabla v_{k(j)}\| + \|v_{k(j)}\|_{L^3} \|\nabla w_k\|_{L^6}. \end{aligned}$$

From this with (3.4) we see

$$(3.19) \quad \|P \operatorname{rot} Q_k f_k\| \leq c \cdot \|\operatorname{rot} f_k\| \leq c \cdot \|P \Delta w_k\| \cdot \|P \operatorname{rot} w_k\|,$$

since P_k is projection and each single term of the sum in (3.18) is bounded by the right hand side in (3.19).

Due to our regularity assumptions concerning the Stokes eigenfunctions e_j , and because of Lemma 3.1, we can apply the operator $P \operatorname{rot}$ on both sides of equation (3.8), finding

$$(3.20) \quad \begin{aligned} \frac{\partial}{\partial t} P \operatorname{rot} w_k - P \Delta P \operatorname{rot} w_k &= P \operatorname{rot} Q_k f_k \text{ for } t \in [0, T), \\ P \operatorname{rot} w_k(0) &= P_k \operatorname{rot} w(0). \end{aligned}$$

Taking the inner product of both sides in (3.20) with $-\Delta \operatorname{rot} w_k$ and using (3.19), (1.4), (0.6) leads to

$$(3.21) \quad \begin{aligned} \frac{d}{dt} \|(\Delta w_k)\|^2 + 2(1 - \mu) \|P \Delta (P \operatorname{rot} w_k)\|^2 \\ \leq c_\mu \|(\Delta w_k)\|^2 \cdot \|P \operatorname{rot} w_k\|^2, \end{aligned}$$

$$(3.22) \quad \|(\Delta w_k(0))\|^2 \leq \|\Delta(w(0))\|^2 = \varphi_2.$$

The estimate (3.22) expresses Bessel's inequality which holds due to the orthogonality (1.6). Since for all $t \in J = [0, T)$, the norm values $\|P \operatorname{rot} w_k(t)\|^2$ are bounded by Φ_1 in (3.14), Lemma 1.1 applies on (3.21), (3.22), yielding

$$(3.23) \quad \|\Delta w_k(t)\|^2 \leq \Phi_2(t, \varphi_2).$$

Thus from (1.8), (1.9) we see

$$(3.24) \quad \|w_k(t)\|_{H^2}^2 \leq c \cdot \Phi_2(t, \varphi_2) \quad \text{and}$$

$$(3.25) \quad \|v_k(t)\|_{H^3}^2 \leq c \cdot \Phi_2(t, \varphi_2) \text{ for all } t \in J.$$

3.4. Bounds for w_{kt}, v_{kt} :

Writing $\frac{\partial}{\partial t} f_k = f_{kt} = f_{k(0)}$, from (3.17) with $j = 0$ using (0.5) we find

$$(3.26) \quad \|f_{kt}\| \leq \left\{ \begin{aligned} &\|w_{kt}\|_{L^6} \cdot \|\nabla v_k\|_{L^3} + \|v_k\|_{L^\infty} \cdot \|\nabla w_{kt}\| + \\ &+ \|w_k\|_{L^6} \cdot \|\nabla v_{kt}\|_{L^3} + \|v_{kt}\|_{L^\infty} \cdot \|\nabla w_k\|. \end{aligned} \right.$$

From this recalling (0.7), (1.8), (1.12) we get

$$(3.27) \quad \|Q_k f_{kt}\| \leq \|f_{kt}\| \leq c \|P \operatorname{rot} w_{kt}\| \cdot \|P \operatorname{rot} w_k\|,$$

since Q_k is a projection in F^0 and each single term in the sum of (3.26) is bounded by the right hand side in (3.27).

From (3.8) written with f_k from (3.11), by differentiation with respect to t we come to the initial value problem

$$(3.28) \quad \begin{cases} \frac{\partial}{\partial t} w_{kt} - \Delta w_{kt} &= Q_k f_{kt}, & t \geq 0, \\ w_{kt}(0) &= Q_k w_t(0), \\ w_{kt} &= \text{rot } v_{kt}. \end{cases}$$

In (3.28) we take the inner product in $L^2(\Omega)$ with w_{kt} or $w_{kt}(0)$, respectively. Recalling (1.4), (3.15), (3.22), (3.27) and (0.6) we see

$$(3.29) \quad \begin{cases} \frac{d}{dt} \|w_{kt}\|^2 + 2(1 - \mu) \|P \text{rot } w_{kt}\|^2 &\leq c_\mu \cdot \Phi_1 \cdot \|w_{kt}\|^2, \\ \|w_{kt}(0)\|^2 &\leq \|w_t(0)\|^2 \\ &\leq \varphi_2 + c\varphi_1 = \varphi_3, \end{cases}$$

the function $\Phi_1(t, \varphi_1)$ being continuous in $t \in [0, T)$. Thus Lemma 1.1 gives the estimate

$$(3.30) \quad \|w_{kt}\|^2 \leq \Phi_3(t, \varphi_3) \quad \text{for all } t \in [0, T)$$

with a continuous function $\Phi_3(t, \varphi_3)$ which is monotone increasing in φ_3 . In addition from (1.32) we see

$$(3.31) \quad \int_0^t \|P \text{rot } w_{kt}(\tau)\|^2 d\tau \leq \tilde{\Phi}_3(t, \varphi_3) \quad \text{for } t \in [0, T).$$

Moreover by (1.10), (1.11), inequality (3.30) implies

$$(3.32) \quad \|v_{kt}\|_{H^1}^2 \leq c \cdot \Phi_3(t, \varphi_3).$$

3.5. F^1 -convergence of (w_k) :

We will see that the convergence of the sequence (w_k) easily follows from the well known error estimates to Fourier expansions in terms of the complete orthogonal systems (\tilde{e}_j) or (e_j) in F^1 or E^1 , respectively.

Remark 3.1: For all vectorfunctions $\varphi \in \overset{\circ}{H}_\sigma^1 \cap H^3$, $\psi = \text{rot } \varphi$, the relation

$$(3.33) \quad g = (\psi \cdot \nabla \varphi - \varphi \cdot \nabla \psi) \in \overset{\circ}{H}_\sigma^1$$

results immediately from $\varphi|_{\partial\Omega} = 0$, the term $\psi \cdot \nabla \varphi$ representing some tangential derivative of φ along $\partial\Omega$ because of (1.3). Note that $g = \text{rot}(\varphi \cdot \nabla \varphi)$ holds.

Proposition 3.1: *The estimates*

$$(3.34) \quad \|(Q_m - Q_n)f\| \leq \lambda_{n+1}^{-1/2} \cdot \|P \text{rot } f\| \quad \text{for } f \in F^1,$$

$$(3.35) \quad \|(P_m - P_n)f\| \leq \lambda_{n+1}^{-1/2} \cdot \|P \text{rot } f\| \quad \text{for } f \in E^1,$$

hold true, $n, m \in \mathbb{N}_1$, $n < m$, λ_n from (1.6). Thus on each bounded subset M of F^1 or E^1 , the Fourier approximations $Q_n f$ or $P_n f$ converge with error $O(\lambda_{n+1}^{-1/2})$ to $f \in M$, respectively.

Proof: By (3.3) we have $(Q_m - Q_n)f = \sum_{j=n+1}^m \langle f, \tilde{e}_j \rangle \tilde{e}_j$. From the identity $\langle f, \tilde{e}_j \rangle = \lambda_j^{-1/2} \langle \text{rot } f, e_j \rangle$, which follows from $e_j|_{\partial\Omega} = 0$, we see

$$(3.36) \quad \langle f, \tilde{e}_j \rangle^2 = \lambda_j^{-1} \cdot \langle \text{rot } f, e_j \rangle^2 = \lambda_j^{-1} \cdot \langle P \text{rot } f, e_j \rangle^2,$$

which implies (3.34) by Bessel's inequality due to (1.6). Similarly for $f \in \overset{\circ}{H}_\sigma^1$, thus $Pf = f$, we have $(P_m - P_n)f = \sum_{j=n+1}^m \langle f, e_j \rangle e_j$, and the equation $\langle f, e_j \rangle = \langle f, \lambda_j^{-1} P \text{rot}^2 e_j \rangle = \lambda_j^{-1/2} \cdot \langle \text{rot } f, \lambda_j^{-1/2} \text{rot } e_j \rangle = \lambda_j^{-1/2} \langle P \text{rot } f, \tilde{e}_j \rangle$ implies (3.35) as above.

We will write $\eta = w_m - w_n$, $\zeta = v_m - v_n$, thus $\eta = \text{rot } \zeta$ for $m, n \in \mathbb{N}_1$. Due to (0.5), the difference

$$(3.37) \quad f_m - f_n = \eta \cdot \nabla v_m + w_n \cdot \nabla \zeta - (\zeta \cdot \nabla w_m + v_n \cdot \nabla \eta)$$

has the bound

$$(3.38) \quad \begin{aligned} \|f_m - f_n\| &\leq \|\eta\|_{L^3} \cdot \|\nabla v_m\|_{L^6} + \|w_n\|_{L^6} \cdot \|\nabla \zeta\|_{L^3} + \\ &+ \|\zeta\|_{L^\infty} \cdot \|\nabla w_m\|_{L^2} + \|v_n\|_{L^\infty} \cdot \|\nabla \eta\|. \end{aligned}$$

Thus there holds

$$(3.39) \quad \|f_m - f_n\| \leq c \cdot \|P \text{rot } \eta\| \cdot (\|P \text{rot } w_m\| + \|P \text{rot } w_n\|)$$

and

$$(3.40) \quad \begin{aligned} \|Q_m f_m - Q_n f_n\| &\leq \|(Q_m - Q_n)f_m\| + \|Q_n(f_m - f_n)\| \leq \\ &c \cdot (\lambda_{n+1}^{-1/2} \cdot \|P \Delta w_m\| + \|P \text{rot } \eta\|) \cdot \Phi_1^{1/2} \end{aligned}$$

because of (3.14), (3.19), (3.34), Q_n being projection, and since, due to (1.8), (1.12), each single term of the sum in (3.38) is bounded by the right hand side in (3.39).

Any two functions $w_m(t), w_n(t) \in F^2$ being solution of the initial value problem (3.8) for all $t \in [0, T) = J$, their difference $\eta = w_m(t) - w_n(t)$ must solve

$$(3.41) \quad \begin{cases} \frac{\partial}{\partial t} \eta - \Delta \eta &= (Q_m - Q_n)f_m + Q_n(f_m - f_n), & t \in J, \\ \eta(0) &= (Q_m - Q_n)w(0), \\ \eta &= \text{rot } \zeta, \quad \zeta = (v_m(t) - v_n(t)) \in E^3. \end{cases}$$

Multiplying through in (3.41) by $-\Delta\eta$ or $-\Delta\eta(0)$ respectively, integrating over Ω and applying Young's inequality (0.6) we obtain

$$(3.42) \quad \begin{aligned} \frac{d}{dt} \|P \operatorname{rot} \eta\|^2 + 2(1 - \mu) \|\Delta\eta\|^2 &\leq c_\mu \|(Q_m - Q_n)f_m\|^2 \\ &\quad + c_\mu \|(Q_n(f_n - f_m))\|^2, \quad t \in J, \\ \|P \operatorname{rot} \eta(0)\|^2 &\leq \|(P_m - P_n)P \operatorname{rot} w(0)\|^2 \\ &= \psi_1, \end{aligned}$$

because of (1.4). From (3.34) with (3.19), (3.14) we see

$$(3.43) \quad \|(Q_m - Q_n)f_m\|^2 \leq \lambda_{n+1}^{-1} \cdot c \cdot \Phi_1 \cdot \|P\Delta w_m\|^2,$$

and (3.39), (3.14) give

$$(3.44) \quad \|Q_n(f_m - f_n)\|^2 \leq c \cdot \Phi_1 \cdot \|P \operatorname{rot} \eta\|^2,$$

Q_n being projection in F^0 .

Thus inequalities (3.42) imply

$$(3.45) \quad \begin{aligned} \frac{d}{dt} \|P \operatorname{rot} \eta\|^2 + 2(1 - \mu) \|\Delta\eta\|^2 &\leq c_\mu \cdot \Phi_1 \cdot \{\lambda_{n+1}^{-1} \cdot \|P\Delta w_m\|^2 \\ &\quad + \|P \operatorname{rot} \eta\|^2\}, \quad t \in J, \\ \|P \operatorname{rot} \eta(0)\|^2 &\leq \psi_1 \\ &\leq c \cdot \lambda_{n+1}^{-1} \cdot \|P \operatorname{rot}^2 w(0)\|^2 \\ &= \overline{\psi}_1, \end{aligned}$$

the latter because of (3.35). Therefore Lemma 1.1 (b) applies with $h(t) = c_\mu \cdot \Phi_1(t, \varphi_1)$, $f(t) = c_\mu \cdot \lambda_{n+1}^{-1} \cdot \Phi_1 \cdot \|P\Delta w_m\|^2$. In order to use the bound (3.16), in (1.33) under the first integral we introduce the bound $\overline{\Phi}_1 = \sup_{\tau \in [0, t]} \Phi_1(\tau, \varphi_1)$, getting

$$(3.46) \quad \begin{aligned} \|P \operatorname{rot} \eta\|^2 &\leq \{\psi_1 + c_\mu \cdot \lambda_{n+1}^{-1} \cdot \overline{\Phi}_1 \cdot \tilde{\Phi}_1(t, \varphi_1)\} \cdot e^{\int_0^t h(\tau) d\tau} \\ &= \Psi_1(t, \psi_1, h, f). \end{aligned}$$

Due to the completeness of the system (e_j) in L_σ^2 , $\psi_1 \rightarrow 0$ holds with $m, n \rightarrow \infty$ since $P \operatorname{rot} w(0) \in L_\sigma^2$. From (3.46) because of (1.8), (1.12) we get the estimates

$$(3.47) \quad \begin{aligned} \|\eta\|_{H^1} &= \|w_m(t) - w_n(t)\|_{H^1} \leq c \cdot \Psi_1 \\ \|\zeta\|_{H^2} &= \|v_m(t) - v_n(t)\|_{H^2} \leq c \cdot \Psi_1. \end{aligned}$$

From this inequality, recalling (3.14) on $[0, T]$, we see that even in case $w(0) \in F^1$, $v(0) \in E^2$ the sequences $(w_k(t))$ or $(v_k(t))$ are converging

in F^1 or E^2 uniformly on each compact time interval $[0, T'] \subset [0, T)$, respectively.

3.6 F^0 -convergence of the w_{kt} :

Due to the bound for $\|\Delta w_k(t)\|$ in (3.23) and the $H^1(\Omega)$ -convergence of the $w_k(t)$ established in (3.47), inequality (3.40) shows the $L^2(\Omega)$ -convergence of the sequence $(Q_k f_k(t))$ uniform in $t \in [0, T']$ for each $T' \in (0, T)$. Consequently from the differential equation in (3.8) and estimate (1.9) we see that the $H^2(\Omega)$ -convergence of the sequence $(w_k(t))$ (together with the $H^3(\Omega)$ -convergence of the related velocities $v_k(t)$) will result from the L^2 -convergence of the time derivatives $w_{kt}(t)$.

Differentiating (3.41) with respect to t , for the function $\frac{\partial}{\partial t}\eta = \eta_t$ we find the initial value problem

$$(3.48) \quad \begin{aligned} \frac{\partial}{\partial t}\eta_t - \Delta\eta_t &= (Q_m - Q_n)f_{mt} + Q_n(f_{mt} - f_{nt}), \quad t \in [0, T], \\ \eta_t(0) &= (Q_m - Q_n)w_t(0), \\ \eta_t &= \text{rot } \zeta_t. \end{aligned}$$

Multiplying through by η_t in $L^2(\Omega)$ and recalling (1.4) we obtain

$$(3.49) \quad \begin{aligned} \frac{d}{dt}\|\eta_t\|^2 + 2\|P \text{rot } \eta_t\|^2 &= \langle (Q_m - Q_n)f_{mt}, \eta_t \rangle \\ &\quad + \langle Q_n(f_{mt} - f_{nt}), \eta_t \rangle, \quad t \in [0, T], \\ \|\eta_t(0)\|^2 &= \|(Q_m - Q_n)w_t(0)\|^2 = \psi_3. \end{aligned}$$

Since the function $\text{rot}^{-1}(Q_m - Q_n)f_{mt}$ is vanishing on $\partial\Omega$, the identity

$$(3.50) \quad \langle \text{rot}^{-1}(Q_m - Q_n)f_{mt}, \text{rot } \eta_t \rangle = \langle (Q_m - Q_n)f_{mt}, \eta_t \rangle$$

follows by elementary calculus. Due to the statements in (1.6) and the Definition (3.3) of the projections P_k and Q_k we find

$$(3.51) \quad \begin{aligned} \text{rot}^{-1}(Q_m - Q_n)f_{mt} &= \sum_{j=n+1}^m \langle \text{rot } f_{mt}, \frac{e_j}{\lambda_j} \rangle e_j \\ &= \sum_{j=n+1}^m \langle A^{-1} \text{rot } f_{mt}, e_j \rangle e_j \\ &= (P_m - P_n)A^{-1} \text{rot } f_{mt}. \end{aligned}$$

Note that $\text{rot } f_{mt} \in L^2_\sigma$ hold true because of Remark 3.1.

Moreover our estimate (3.35) together with (1.8) implies the inequalities

$$\begin{aligned}
(3.52) \quad \|(P_m - P_n)A^{-1} \operatorname{rot} f_{mt}\| &\leq c \cdot \lambda_{n+1}^{-1/2} \cdot \|P \operatorname{rot} A^{-1} \operatorname{rot} f_{mt}\| \\
&\leq c \cdot \lambda_{n+1}^{-1/2} \cdot \|A^{-1} \operatorname{rot} f_{mt}\|_{H^1} \\
&\leq c \cdot \lambda_{n+1}^{-1/2} \cdot \|\operatorname{rot} f_{mt}\|_{H^{-1}} \\
&\leq c \cdot \lambda_{n+1}^{-1/2} \cdot \|f_{mt}\|.
\end{aligned}$$

The last inequality represents the usual $H^{-1}(\Omega)$ -bound for weak first order derivatives in Ω , cp. [1.p. 50].

From (3.50) - (3.52) together with (3.27), (1.12), (3.14) we find

$$(3.53) \quad |\langle (Q_m - Q_n)f_{mt}, \eta_t \rangle| \leq c \cdot \lambda_{n+1}^{-1/2} \cdot \Phi_1^{1/2} \cdot \|P \operatorname{rot} w_{mt}\|.$$

In order to estimate the term

$$(3.54) \quad \langle Q_n(f_{mt} - f_{nt}), \eta_t \rangle = \langle f_{mt} - f_{nt}, Q_n \eta_t \rangle,$$

too, writing $D = v_m \cdot \nabla v_m - v_n \cdot \nabla v_n$, we note

$$(3.55) \quad f_{mt} - f_{nt} = \frac{\partial}{\partial t} \operatorname{rot} D = \operatorname{rot} \frac{\partial}{\partial t} D,$$

recalling $a_{mj} = b_{mj} \cdot \lambda_j^{-1/2} \in C^\infty(J)$,

$$e_j \in \mathring{H}_\sigma^1 \cap H^4.$$

From (3.54), (3.55) we get

$$(3.56) \quad \langle Q_n(f_{mt} - f_{nt}), \eta_t \rangle = \langle \frac{\partial}{\partial t} D, \operatorname{rot} Q_n \eta_t \rangle$$

because of $v_m|_{\partial\Omega} = 0$. A short calculation shows

$$(3.57) \quad \frac{\partial}{\partial t} D = \zeta_t \cdot \nabla v_m + v_{nt} \cdot \nabla \zeta + \zeta \cdot \nabla v_{mt} + v_n \cdot \nabla \zeta_t,$$

consequently there holds

$$\begin{aligned}
(3.58) \quad \left\| \frac{\partial}{\partial t} D \right\| &\leq \|\zeta_t\| \cdot \|\nabla v_m\|_{L^\infty} + \|v_{nt}\|_{L^6} \cdot \|\nabla \zeta\|_{L^3} \\
&\quad + \|\zeta\|_{L^\infty} \cdot \|\nabla v_{mt}\| + \|v_n\|_{L^\infty} \cdot \|\nabla \zeta_t\| \\
&\leq c \cdot \|\eta_t\| \cdot \{\Phi_2^{1/2} + \Psi_1^{1/2} \cdot \Phi_3^{1/2}\}
\end{aligned}$$

due to (0.5), (0.7), (3.25), (3.32), (3.47) for all $t \in [0, T)$. Moreover, from (1.12) and (3.4) we get

$$\begin{aligned}
(3.59) \quad \|\operatorname{rot} Q_n \eta_t\| &\leq c \cdot \|Q_n \eta_t\|_{H^1} \leq c \cdot \|P \operatorname{rot} Q_n \eta_t\| \\
&\leq c \cdot \|P \operatorname{rot} \eta_t\|,
\end{aligned}$$

observing $Q_n \eta_t \in F^1$.

The results (3.54) - (3.59) together lead to

$$(3.60) \quad |\langle Q_n(f_{mt} - f_{nt}), \eta_t \rangle| \leq c \cdot \|\eta_t\| \cdot \|P \operatorname{rot} \eta_t\| \cdot \{\Phi_2^{1/2} + \Psi_1^{1/2} \cdot \Phi_3^{1/2}\}.$$

From (3.49) with (3.53), (3.60), (0.6) we come to

$$(3.61) \quad \begin{aligned} \frac{d}{dt} \|\eta_t\|^2 + 2(1 - \mu) \|P \operatorname{rot} \eta_t\|^2 &\leq c_\mu \cdot \{\lambda_{n+1}^{-1} \cdot \Phi_1 \\ &\quad \cdot \|P \operatorname{rot} w_{mt}\|^2 + \|\eta_t\|^2 \\ &\quad \cdot (\Phi_2 + \Psi_1 \cdot \Phi_3)\}, \\ \|\eta_t(0)\|^2 &= \|(Q_m - Q_n)w_t(0)\|^2 \\ &= \psi_3. \end{aligned}$$

Again Lemma 1.1 (b) applies with $h = c_\mu \cdot (\Phi_2 + \Psi_1 \cdot \Phi_3)$, $f(t) = c_\mu \cdot \lambda_{n+1}^{-1} \cdot \Phi_1 \cdot \|P \operatorname{rot} w_{mt}\|^2$. Using the bound $\bar{\Phi}_1 = \sup_{\tau \in [0, t]} \Phi_1(\tau, \varphi_1)$ and inequality (3.31), from (1.33) we find

$$(3.62) \quad \begin{aligned} \|\eta_t(t)\|^2 &\leq (\psi_3 + c_\mu \cdot \lambda_{n+1}^{-1} \cdot \bar{\Phi}_1 \cdot \tilde{\Phi}_3(t, \varphi_1)) \cdot e^{\int_0^t h(\tau) d\tau} \\ &\leq \Psi_2(t, \psi_3, h, f) \end{aligned}$$

for all $t \in [0, T)$.

Due to the completeness of the orthonormal system (\tilde{e}_j) in F^0 , the norm $\|\eta_t(0)\| = \|(Q_m - Q_n)w_t(0)\| = \psi_3$ converges to zero with $m, n \rightarrow \infty$, the initial value $w_t(0) \in F^0$ being fixed by the differential equation (0.1) and the initial value $w(0) \in E^2$. Consequently inequality (3.62) shows the convergence $\|\eta_t(t)\|^2 \rightarrow 0$ with $m, n \rightarrow \infty$, since this implies $\lambda_n \rightarrow \infty$, $\bar{\Phi}_1(t)$, $\tilde{\Phi}_1(t, \varphi_1)$ being bounded uniformly on each compact interval $[0, T'] \subset [0, T)$. Thus the time derivatives $w_{kt}(t)$ being strongly $L^2(\Omega)$ -convergent uniformly on each $[0, T'] \subset [0, T)$, the w_{kt} converge to the time derivative w_t of the limit function $w(t) = \lim_{k \rightarrow \infty} w_k(t)$.

3.7 Convergence to the solution :

The estimate (3.62), (3.40) together with (3.23) and (3.14) show that the term $\Delta w_k(t)$ in (3.8) is converging in $L^2(\Omega)$ uniformly on each $[0, T'] \subset [0, T)$. Moreover the inequality

$$(3.63) \quad \|\Delta \eta\| \leq \|Q_m f_m - Q_n f_n\| + \|\eta_t\|$$

which we find from equation (3.41), by (1.8), (1.9) implies the convergences

$$(3.64) \quad \begin{aligned} \|\eta(t)\|_{H^2} &= \|w_m(t) - w_n(t)\|_{H^2} \rightarrow 0, \\ \|\zeta(t)\|_{H^3} &= \|v_m(t) - v_n(t)\|_{H^3} \rightarrow 0 \end{aligned}$$

uniformly on each $[0, T'] \subset [0, T]$. Thus the limit functions

$$w(t) = \lim_{k \rightarrow \infty} w_k(t) \in C^0([0, T], F^2) \cap C^1([0, T], F^0),$$

$$v(t) = \lim_{k \rightarrow \infty} v_k(t) \in C^0([0, T], E^3)$$

represent the solution of (0.1), (0.2), the solution being unique because of Theorem 2.1.

Corollary 3.1: *In case $w(0) = \text{rot } v(0) \in F^2, v(0) \in E^3$,*

$$(3.65) \quad \|P \text{rot } w(0)\|^2 < 2 \frac{1 - \mu}{c_\mu \cdot c^2},$$

the solution $w(t) = \text{rot } v(t)$ exists globally for all $t \geq 0$, and the convergences stated in Theorem 3.1 hold uniformly on each compact time intervall $[0, T], T > 0$. Thus we get $w \in C^0([0, \infty), F^2), w_t \in C^0([0, \infty), F^0), v \in C^0([0, \infty), E^3)$.

Proof: As we have seen from (3.14), (3.23), (3.31), (3.40), the interval of guaranteed existence of the strong solution $w(t)$ is fixed by the bound Φ_1 , which we have calculated from (3.13) without taking in account the second term in the sum of the left hand side. But because of $w_k = \text{rot } v_k \in F^2$, from (1.9) we find

$$\|P \text{rot } w_k\| \leq c \cdot \|\Delta w_k\|.$$

Thus (3.13) leads to

$$(3.66) \quad \begin{aligned} \frac{d}{dt} \|P \text{rot } w_k\|^2 &\leq \|P \text{rot } w_k\|^2 \cdot \{c_\mu \|P \text{rot } w_k\|^2 - 2 \frac{1-\mu}{c^2}\}, \\ \|P \text{rot } w_k(0)\|^2 &\leq \varphi_1. \end{aligned}$$

Therefore in the special case (3.65), the solution $\Phi_1(t, \varphi_1)$ to the differential equation (1.31) related to (3.66) remains globally bounded, which we can easily verify by elementary integration. Then from (3.62), (3.64), (3.40) we get the uniform convergence of the $(w_k(t)) \subset F^2, (v_k(t)) \subset E^3, (w_{kt}(t)) \subset F^0$ on each compact time interval $[0, T]$.

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