Existence and regularity of weak solutions for the Navier-Stokes equations with partial slip boundary conditions

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Abstract

In this note we derive existence and regularity results on weak and strong solutions for instationary flows in the half-space $\mathbb{R}^n_+$ that are subject to partial slip (also known as Robin or Navier) boundary conditions. The results are well known in the case of Dirichlet boundary conditions, which represents a special case of our situation. We will show global existence of weak solutions, higher regularity for arbitrary weak solutions, and the existence of local strong solutions that exist even globally and coincide with every weak solution, if we assume that space dimension is two. All results are essentially based on the previous work [17] of the author, which includes in particular the maximal regularity of the Stokes operator with partial slip boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$.

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1 Introduction and main results

For $n \in \mathbb{N}$, $n \geq 2$, and $T \in (0, \infty]$ we consider the Navier-Stokes equations with partial slip boundary conditions

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &=& f & \text{in} \quad \mathbb{R}^n_+ \times (0,T), \\ \operatorname{div} u &=& 0 & \text{in} \quad \mathbb{R}^n_+ \times (0,T), \\ u(0) &=& u_0 & \text{in} \quad \mathbb{R}^n_+, \\ T_\alpha u &=& 0 & \text{on} \quad \partial\mathbb{R}^n_+ \times (0,T), \end{array} \right. \tag{1.1}$$

i.e. the trace operator T_α is given by

$$T_\alpha u := \left(\begin{array}{c} \alpha u' - \partial_n u' \\ u^n \end{array} \right) \Bigg|_{\partial\mathbb{R}^n_+}, \tag{1.2}$$

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where u' denotes the tangential part of u and $\alpha \in [0, \infty]$[1]. Observe, that the case $\alpha = 0$ or $\alpha = \infty$ corresponds to the classical Neumann or Dirichlet boundary conditions respectively. Note that in the literature these boundary conditions are also known as Robin or Navier boundary conditions. In a first step we will prove the existence and higher regularity of weak solutions for system (1.1). Next we will state a result on existence and uniqueness of local strong solutions. Finally, we show that the local strong solution extends uniquely to a global one in two space dimensions, i.e. if $n = 2$.

Note that for the special case of Dirichlet boundary conditions the above mentioned results are well known. The existence of weak solutions, for instance, is known since the poineering works of Leray [14] and Hopf [10]. The existence of uniquely determined global strong solutions in two space dimensions was proved in [11] (see also [12]). Since that time a huge literature developed concerning weak and local strong solutions for the Navier-Stokes equations with Dirichlet boundary conditions. Therefore, here we just refer to the monographs of Temam [21], Constantin and Foias [4], Galdi [8, 7], Sohr [20], and the references cited therein for more information on this topic. For a comprehensive approach on strong solutions see also the work of Amann [1] and for a result on higher regularity on weak solutions which is similar to our result for partial slip boundary conditions (Theorem 1.3) see the paper of Giga and Sohr [9].

The content of this article can be regarded as a generalization to partial slip boundary conditions of the above mentioned results for Dirichlet boundary conditions. Although the most common boundary conditions used in the fluid mechanics literature are the no slip boundary conditions, it is known that in some situations, e.g. for gas flows, non-Newtonian fluids, or moving contact lines, partial slip can occur (see e.g. [15], [3], and [5] respectively). Moreover, physico-chemical parameters as wetting, shear rate, surface charge, and surface roughness can influence the behavior of a fluid at the solid-liquid interface. We refer to [13] for a review on recent investigations on this subject and to the literature cited therein. This shows that in certain situations it might be more appropriate to assume partial slip boundary conditions. This was motivation enough for the author to examine the Navier-Stokes equations with partial slip boundary conditions, also in view of the lack of results on this problem in the existing literature.

Before stating our main results let us introduce some notation and some basic facts used in this note. We will use standard notation throughout this paper. Let $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}$, $p, q \in (1, \infty)$, and X be a Banach space. By $L^p(\Omega, X)$ and $W^{m,p}(\Omega, X)$ we denote the X-valued Lebesgue and Sobolev space respectively with their canonical norms. If $X = \mathbb{C}^n$ or $X = \mathbb{R}^n$ we write $L^p(\Omega)$ and $W^{m,p}(\Omega)$. Furthermore, we do not distinguish between spaces of functions and spaces of vector fields in the sequel, i.e. we write also $L^p(\Omega)$ for $L^p(\Omega)^n$ for example. We also make use of the homogeneous Sobolev space $\widehat{W}^{m,p}(\Omega, X)$ which is defined as the space $\{v \in L^1_{loc}(\Omega, X) : \|\nabla^m v\|_{L^p(\Omega,X)} < \infty\}$ modulo

[1] The case $\alpha = \infty$ is to understand in the following sense: divide the first line in (1.2) by α and let $\alpha \to \infty$.

polynomials of order $m-1$. The subspace of solenoidal vector fields of $L^q(\mathbb{R}^n_+)$ we denote as usual by

$$L^q_\sigma(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{L^q},$$

where $C^\infty_{c,\sigma}(\mathbb{R}^n_+) := \{v \in C^\infty_c(\mathbb{R}^n_+) : \operatorname{div} v = 0\}$ and $C^\infty_c(\mathbb{R}^n_+)$ denotes the space of all smooth compactly supported functions in $\mathbb{R}^n_+$. $L^q_\sigma(\mathbb{R}^n_+)$ is known to be a complementary subspace of $L^q(\mathbb{R}^n_+)$ and the related Helmholtz projection operator, which maps $L^q(\mathbb{R}^n_+)$ onto $L^q_\sigma(\mathbb{R}^n_+)$ in the sequel we denote by P. For short we will often write $\|\cdot\|_q$ for the norm in $L^q(\mathbb{R}^n_+)$ and $\|\cdot\|_{q,p}$ for the norm in $L^p((0,T),L^q(\mathbb{R}^n_+))$, where $T \in (0,\infty]$. Finally, the L^q-$L^{q'}$ dual pairing we denote by $(\cdot,\cdot)$, where $1/q + 1/q' = 1$.

All the results for system (1.1) proved in this note are based on results of the author obtained in the works [17] and [19]. There it is proved that the Stokes operator with partial slip boundary conditions, defined by

$$A_q := -P\Delta, \quad \mathscr{D}(A_q) := \{v \in W^{2,q}(\mathbb{R}^n_+); \operatorname{div} v = 0, T_\alpha v = 0\},$$

is a sectorial operator on $L^q_\sigma(\mathbb{R}^n_+)$ for $q \in (1,\infty)$ which is selfadjoint for $q = 2$. It is even proved that A_q admits a bounded H^∞-calculus on the space $L^q_\sigma(\mathbb{R}^n_+)$. Since $(A_q)_{q\in(1,\infty)}$ is a compatible family, we will often omit the subscript q in the sequel, if no confusion seems likely. As a consequence we obtain the maximal regularity of A on $L^q_\sigma(\mathbb{R}^n_+)$, since by the Dore-Venni Theorem in the form given in [9] the class of all operators that admit a bounded H^∞-calculus on $L^q_\sigma(\mathbb{R}^n_+)$ is known to be a subclass of the class of all operators having maximal regularity. The maximal regularity means that for all data $u_0 \in \mathcal{I}^p_q := (L^q(\mathbb{R}^n_+), \mathscr{D}(A_q))_{1-1/p,p}$ (real interpolation space) and $f \in L^p((0,T), L^q_\sigma(\mathbb{R}^n_+))$ the Cauchy problem associated to A,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v + Av &= f, \quad t \in (0,T), \\ v(0) &= u_0 \end{cases} \tag{1.3}$$

has a unique solution u that satisfies (1.3) for almost all $t \in (0,T)$ and the estimate

$$\|\frac{\mathrm{d}}{\mathrm{d}t}u\|_{q,p} + \|A_q u\|_{q,p} \le C\left(\|u_0\|_{\mathcal{I}^p_q} + \|f\|_{q,p}\right) \tag{1.4}$$

with $C > 0$ independent of u and $T \in (0,\infty]$. Estimate (1.4) will play an essential role in the proof of all results presented in this note.

Another consequence of the results in [17] and [19] which will be frequently used here, are Sobolev estimates of the form

$$\|A^{k/2}v\|_q \le C_1(k)\|\nabla^k v\|_q \le C_2(k)\|A^{k/2}v\|_q \tag{1.5}$$

for $v \in \mathscr{D}(A^{k/2})$ and $k \in \mathbb{N}$ (see [19, Proposition 4.17]). This allows us to switch freely between the norms $\|A^{k/2}\cdot\|_q$ and $\|\nabla^k \cdot\|_q$, $k = 1, 2$.

We proceed by recalling the notion of a weak solution to system (1.1).

Definition 1.1. Let $n \in \mathbb{N}$, $n \ge 2$, and $T \in (0,\infty]$. We call u a weak solution of system (1.1), if u belongs to the Leray-Hopf class, i.e.

$$u \in L^\infty((0,T), L^2_\sigma(\mathbb{R}^n_+)) \cap L^2((0,T), \widehat{H}^1(\mathbb{R}^n_+))$$

and u satisfies

$$\int_0^T \left[-(u, \partial_t \phi) - (u, \Delta\phi) + \sum_{j=1}^n (\partial_j u, u^j \phi) \right] \mathrm{d}t = (u_0, \phi(0)) + \int_0^T (f, \phi) \mathrm{d}t$$

for all $\phi \in C_c^\infty([0,T), C_{c,\sigma}^\infty(\overline{\mathbb{R}_+^n}))$ so that

$$T_\alpha \phi(t) = 0, \quad t \in (0,T).$$

Our main results on weak solutions read as follows.

Theorem 1.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and $T \in (0,\infty]$. For each $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ and distribution f such that $A^{-1/2} f \in L^2((0,T), L^2_\sigma(\mathbb{R}^n_+))$ there exists a weak solution*

$$u \in L^\infty((0,T), L^2_\sigma(\mathbb{R}^n_+)) \cap L^2((0,T), \widehat{H}^1(\mathbb{R}^n_+))$$

of (1.1) satisfying the energy inequality

$$\|u\|^2_{2,\infty} + \|\nabla u\|^2_{2,2} \leq \|u_0\|_2^2 + \int_0^T (f(t), u(t)) \mathrm{d}t.$$

Theorem 1.3. *Let $n \in \mathbb{N}$, $n \geq 2$, and $T \in (0,\infty]$. Let u be any weak solution of (1.1) (not necessarily satisfying the energy inequality). Furthermore, assume that $u_0 \in \mathcal{I}^p_q := (L^q(\mathbb{R}^n_+), \mathscr{D}(A_q))_{1-1/p,p}$ and $f \in L^p((0,T), L^q(\mathbb{R}^n_+))$ with $1 < p, q < \infty$ such that*

$$n/q + 2/p = n + 1.$$

Then there exists a $C > 0$ independent of u_0 and f such that

$$\|\partial_t u\|_{q,p} + \|\nabla^2 u\|_{q,p} + \|\nabla p\|_{q,p} \leq C(\|u_0\|_{\mathcal{I}^p_q} + \|f\|_{q,p} + \|u\|^2_{2,\infty} + \|\nabla u\|^2_{2,2}). \quad (1.6)$$

The proof of these two results is given in Section 2. Concerning strong solutions we have

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $(n+2)/3 < q < \infty$, and $T \in (0,\infty]$. Then, for each $v_0 \in \mathcal{I}^q_q$ and $f \in L^q((0,T), L^q(\Omega(t)))$ there exists a $T^* \in (0,T)$ and a unique solution (u,p) of problem (1.1) such that*

$$\begin{aligned} u &\in W^{1,q}((0,T^*); L^q(\mathbb{R}^n_+)) \cap L^q((0,T^*); D(A_q)), \\ p &\in L^q((0,T^*); \widehat{W}^{1,q}(\mathbb{R}^n_+)). \end{aligned}$$

Since it is quite standard we will not demonstrate the proof of Theorem 1.4 in this note. For example it can be copied almost verbatim from [18, Theorem 1.2]. Essentially it is also a consequence of the maximal regularity of the Stokes operator A with partial slip boundary conditions on $L^q_\sigma(\mathbb{R}^n_+)$.

On the other hand, in two space dimensions we will present a proof of the following result in Section 3.

Theorem 1.5. *Let $T \in (0,\infty]$. Suppose that $u_0 \in L^2_\sigma(\mathbb{R}^2_+) \cap H^1(\mathbb{R}^2_+)$, and $f \in L^2((0,T), L^2_\sigma(\mathbb{R}^2_+))$ such that $A^{-1/2} f \in L^2((0,T), L^2(\mathbb{R}^2_+))$. Then the weak solution u of Theorem 1.2 is unique. Furthermore, u and the pressure p satisfy*

$$\nabla u \in L^\infty((0,T), L^2(\mathbb{R}^2_+)), \quad \partial_t u, \nabla^2 u, \nabla p \in L^2((0,T), L^2(\mathbb{R}^2_+)).$$

2 Existence and regularity of weak solutions

In order to construct weak solutions we follow the approach used in [2]. To this end we switch to the operatorial form of system (1.1), that is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u + Au + P(u \cdot \nabla)u &= f, \quad t \in (0,T), \\ u(0) &= u_0. \end{cases} \tag{2.1}$$

Next we set

$$J_k := (1 + \frac{1}{k}A)^{-1}, \quad k \in \mathbb{N}. \tag{2.2}$$

The $L^p - L^q$-estimates for the semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ in [17, Corollary 5.8] then yield

$$\begin{aligned} \|J_k u\|_p &= \|\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-tA/k}u\mathrm{d}t\|_p \\ &\leq C\int_0^\infty \mathrm{e}^{-t}(t/k)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\mathrm{d}t\|u\|_q \\ &\leq C(p,q,n,k)\|u\|_q, \quad u \in L^q_\sigma(\mathbb{R}^n_+), \end{aligned}$$

if $\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) < 1$. Thus, if we choose $N \in \mathbb{N}$ such that $N > 1 + n/4$, by an iterative application of the above estimate we can achieve that

$$\|J_k^N u\|_\infty \leq C(n,k)\|u\|_2, \quad u \in L^2_\sigma(\mathbb{R}^n_+). \tag{2.3}$$

Now, for each $k \in \mathbb{N}$ consider the approximate system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v + Av + P(J_k^N v \cdot \nabla)v &= J_k f =: f_k, \quad t \in (0,T), \\ v(0) &= J_k u_0 =: u_{0,k}. \end{cases} \tag{2.4}$$

Since A is the generator of a holomorphic semigroup in $L^q_\sigma(\mathbb{R}^n_+)$, solving (2.4) is equivalent to show that the mapping

$$Fv(t) := \mathrm{e}^{-tA}u_{0,k} + \int_0^t \mathrm{e}^{-(t-s)A}P(J_k^N v(s)\cdot\nabla)v(s)\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}f_k(s)\mathrm{d}s \tag{2.5}$$

has a unique fixed point.

Proposition 2.1. *Let $k \in \mathbb{N}$ and $T \in (0,\infty)$. There exists a unique solution*

$$u \in C([0,T],\mathscr{D}(A^{1/2})) \cap L^2((0,T),\mathscr{D}(A)) \cap H^1((0,T),L^2_\sigma(\mathbb{R}^n_+))$$

of (2.5) that satisfies (2.4) f.a.a. $t \in (0,T)$.

Proof. Since the proof is standard we will be brief in details. Fix $k \in \mathbb{N}$. First note that by definition $u_{0,k} \in \mathscr{D}(A^{1/2})$ and $f_k \in L^2((0,T);\mathscr{D}(A^{1/2}))$. Next define

$$B_M := \{v \in C([0,T],\mathscr{D}(A^{1/2})) : v(0) = u_{0,k}, \|v\|_T \leq M\},$$

where $\|v\|_T := \sup_{t\in[0,T]}(\|v(t)\|_2 + \|A^{1/2}v(t)\|_2)$. Observe that by (1.5) and (2.2),

$$\|P(J_k^N v(s)\cdot\nabla)v(s)\|_2 \leq C\|v(t)\|_2\|A^{1/2}v(t)\|_2, \quad t\in(0,T). \tag{2.6}$$

From this estimate we easily obtain that

$$\|Fv\|_T \leq C_1(\|u_{0,k}\|_{\mathscr{D}(A^{1/2})}\|f_k\|_{L^2((0,T),\mathscr{D}(A^{1/2}))} + M^2(T+T^{1/2}))$$

for $v\in B_M$. Similarly, by using

$$\begin{aligned}&\|P(J_k^N v(s)\cdot\nabla)v(s) - P(J_k^N w(s)\cdot\nabla)w(s)\|_2\\ &\leq C\left(\|v(t)-w(t)\|_2\|A^{1/2}v(t)\|_2 + \|A^{1/2}(v(t)-w(t))\|_2\|w(t)\|_2\right)\end{aligned}$$

for $t\in(0,\infty)$, we have that

$$\|Fv - Fw\|_T \leq C_2M(T+T^{1/2})\|v-w\|_T$$

for $v,w\in B_M$. Fixing M such that $C_1\|u_{0,k}\|_{\mathscr{D}(A^{1/2})}\|f_k\|_{L^2((0,T),\mathscr{D}(A^{1/2}))} \leq M/2$ and then $T>0$ so that $C_jM(T+T^{1/2})\leq 1/2$, $j=1,2$, we see that F is a contraction on B_M. The contraction mapping principle then yields a unique solution $u\in B_M$ for small $T>0$.

Next observe that estimate (2.6) for u implies that

$$P(J_k^N u\cdot\nabla)u \in L^2((0,T),L^2_\sigma(\mathbb{R}^n_+)).$$

By the maximal regularity of A on $L^2_\sigma(\mathbb{R}^n_+)$ we then obtain

$$u\in L^2((0,T),\mathscr{D}(A))\cap H^1((0,T),L^2_\sigma(\mathbb{R}^n_+)). \tag{2.7}$$

This proves the assertion for small $T>0$. In order to show that u exists uniquely for arbitrary $T>0$ we derive a priori bounds for $\|u\|_T$. Using the equations (2.4) and relation (2.7) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + \|A^{1/2}u(t)\|_2^2 = (f_k,u), \quad t\in(0,T). \tag{2.8}$$

Integrating over t yields

$$\frac{1}{2}\|u\|_{2\infty}^2 + \|A^{1/2}u\|_{2,2}^2 \leq \frac{1}{2}\|u_{0,k}\|_2^2 + \|A^{-1/2}f_k\|_{2,2}\|A^{1/2}u\|_{2,2}.$$

Again note that by $\|\cdot\|_{q,p}$ we always mean the norm on the full time interval $(0,T)$, i.e. here $\|\cdot\|_{2,2} = \|\cdot\|_{L^2((0,T),L^2(\mathbb{R}^n_+))}$. Applying $ab\leq a^2/2+b^2/2$ on the latter term and rearranging then results in

$$\|u\|_{2,\infty}^2 + \|A^{1/2}u\|_{2,2}^2 \leq \|u_{0,k}\|_2^2 + \|A^{-1/2}f_k\|_{2,2}^2. \tag{2.9}$$

Multiplying (2.4) by Au instead by u and employing once again (2.2) we can also obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u(t)\|_2^2 + 2\|Au(t)\|_2^2 \leq C\left(\|u(t)\|_2^2\|A^{1/2}u(t)\|_2^2 + \|f_k(t)\|_2^2\right) + \|Au(t)\|_2^2 \tag{2.10}$$

for $t \in (0,T)$. Now (2.9) implies that

$$\|u(t)\|_2^2 \le \|u_{0,k}\|_2^2 + \|A^{-1/2} f_k\|_{2,2}^2, \quad t \in (0,T).$$

Inserting this into (2.10) and rearranging the resulting estimate yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u(t)\|_2^2 \le C\|f_k(t)\|_2^2 + C\left(\|u_{0,k}\|_2^2 + \|A^{-1/2}f_k\|_{2,2}^2\right)\|A^{1/2}u(t)\|_2^2.$$

Integrating this expression finally results in

$$\begin{aligned}\|A^{1/2}u(t)\|_2^2 \quad &\le \quad C\left(\|A^{1/2}u_{0,k}\|_2^2 + \|f_k\|_{2,2}^2\right)\\ &\qquad + C\left(\|u_{0,k}\|_2^2 + \|A^{-1/2}f_k\|_{2,2}^2\right)\int_0^t \|A^{1/2}u(s)\|_2^2 \mathrm{d}s.\end{aligned}$$

By the lemma of Gronwall we therefore may conclude

$$\|A^{1/2}u(t)\|_2^2 \le C\left(\|A^{1/2}u_{0,k}\|_2^2 + \|f_k\|_{2,2}^2\right)\exp\left(C\left(\|u_{0,k}\|_2^2 + \|A^{-1/2}f_k\|_{2,2}^2\right)t\right) \tag{2.11}$$

for all $t > 0$. This shows that u exists uniquely on arbitrary intervals $[0,T]$ (as long as f exists of course) and that it admits the claimed regularity properties. □

In order to deal with the nonlinear term the next lemma will be useful.

Lemma 2.2. *Let $T \in (0,\infty]$. For $1 < p, q < \infty$ satisfying $2/q + n/p = n+1$ there exists a $C = C(p,q,n) > 0$ so that*

$$\|(v\cdot\nabla)u\|_{q,p} \le C\left(\|v\|_{2,\infty}^2 + \|\nabla v\|_{2,2}^2 + \|\nabla u\|_{2,2}^2\right)$$

for all v, u satisfying $v \in L^\infty((0,T), L^2_\sigma(\mathbb{R}^n_+))$ and $\nabla u, \nabla v \in L^2((0,T), L^2(\mathbb{R}^n_+))$.

Proof. Applying the Hölder inequality with $1/q = 1/r + 1/2$ to the term on the left hand side gives

$$\|(v(t)\cdot\nabla)u(t)\|_q \le C(n,q)\|v(t)\|_r\|\nabla u(t)\|_2, \quad t \in (0,T). \tag{2.12}$$

Next, the Gagliardo-Nirenberg inequality (see [6, Theorem 9.3], for the validity on $\mathbb{R}^n_+$ see also [16, Appendix A]) implies that

$$\|v(t)\|_r \le C(r,n)\|\nabla v(t)\|_2^\theta \|v(t)\|_2^{1-\theta},$$

where $1/r = 1/2 - \theta/n$. Inserting this into (2.12) and taking L^p-norm in t yields

$$\|(v\cdot\nabla)u\|_{q,p} \le C(n,q,p)\|v\|_{2,\infty}^{2-2/p}\|\nabla v\|_{2,2}^{2/p-1}\|\nabla u\|_{2,2},$$

where we took into account that $2/q + n/p = n+1$. By applying twice Young's inequality we end up with

$$\begin{aligned}\|(v\cdot\nabla)u\|_{q,p} \quad &\le \quad C(n,q,p)\left(\|v\|_{2,\infty}^2 + \|\nabla v\|_{2,2}^{2-p}\|\nabla u\|_{2,2}^p\right)\\ &\le \quad C(n,q,p)\left(\|v\|_{2,\infty}^2 + \|\nabla v\|_{2,2}^2 + \|\nabla u\|_{2,2}^2\right).\end{aligned}$$

□

We turn to the proof of our results on weak solutions for arbitrary space dimension $n \geq 2$.

Proof. (of Theorem 1.2). Since the right hand side of (2.9) is bounded by $\|u_0\|_{2,\infty}^2 + \|A^{-1/2} f\|_{2,2}^2$, we obtain that the sequence u_k is bounded and therefore has a weak limit u in the Leray-Hopf class. It remains to show that u is a weak solution of (1.1). To this end let v_k be the unique solution of

$$\left\{ \begin{array}{rcl} \partial_t v + Av &=& f_k, \quad t \in (0,T), \\ v(0) &=& u_{0,k}. \end{array} \right. \tag{2.13}$$

Observe that v_k even converges strongly in

$$L^\infty((0,T), L^2_\sigma(\mathbb{R}^n_+)) \cap L^2((0,T), \widehat{H}^1(\mathbb{R}^n_+)).$$

Then $w_k := u_k - v_k$ converges weakly in the same class. Moreover, w_k satisfies

$$\left\{ \begin{array}{rcl} \partial_t w_k + Aw_k &=& P(J_k^N u_k \cdot \nabla) u_k, \quad t \in (0,T), \\ w_k(0) &=& 0. \end{array} \right. \tag{2.14}$$

Note that Lemma 2.2 implies that the right hand side of (2.14) is bounded in $L^q((0,T), L^q(\mathbb{R}^n_+))$ for $q = (n+2)/(n+1)$. For finite $T > 0$ the maximal regularity of the operator A then yields that the sequence w_k is bounded in

$$W^{1,q}((0,T), L^q(\mathbb{R}^n_+)) \cap L^q((0,T), \mathscr{D}(A_q)).$$

Therefore we can apply [21, Chapter III, Theorem 2.1] to the result that w_k converges strongly in

$$L^2_{loc}(\overline{\mathbb{R}_+ \times \mathbb{R}^n_+}).$$

Thus, also $u_k = w_k + v_k$ converges strongly to u in $L^2_{loc}(\overline{\mathbb{R}_+ \times \mathbb{R}^n_+})$. In view of this fact, it is easy to see that u is indeed a weak solution of system (1.1), whereas the energy inequality for u is an obvious consequence of (2.8). □

Proof. (of Theorem 1.3). If u is a weak solution of (1.1) then Lemma 2.2 implies that

$$\|P(u \cdot \nabla)u\|_{q,p} \leq C\left(\|u\|_{2,\infty}^2 + \|\nabla u\|_{2,2}^2\right)$$

for $1 < p, q < \infty$ so that $2/q + n/p = n+1$. Now set $H := f - P(u \cdot \nabla)u$ and consider the system

$$\left\{ \begin{array}{rcl} \partial_t v + Av &=& H, \quad t \in (0,T), \\ v(0) &=& u_0. \end{array} \right. \tag{2.15}$$

Once again the maximal regularity of A implies that the solution to (2.15) is uniquely determined and satisfies

$$\|\partial_t v\|_{q,p} + \|\nabla^2 v\|_{q,p} \leq C(\|u_0\|_{\mathcal{I}_q^p} + \|f\|_{q,p} + \|u\|_{2,\infty}^2 + \|\nabla u\|_{2,2}^2).$$

It remains to show that $v = u$. Although v must not necessarily be a weak solution (in the L^2-sense) of (2.15) by forming the dual pairing with $\phi \in C_c^\infty((0,T), C_{c,\sigma}(\mathbb{R}^n_+))$ we obtain that v solves

$$\int_0^T \{-(v(t), \partial_t \phi(t)) + (v(t), A\phi(t))\} \,\mathrm{d}t = \int_0^T \{(F(t), \phi(t))\} \,\mathrm{d}t. \tag{2.16}$$

Since u obviously solves (2.16) as well, we have that

$$\int_0^T \{(u(t) - v(t), -\partial_t \phi(t) + A\phi(t))\} \,\mathrm{d}t = 0.$$

On the other hand for each $\psi \in C_c^\infty((0,T), C_{c,\sigma}(\mathbb{R}^n_+))$ it is well known that the dual system

$$\left\{ \begin{array}{rcll} -\partial_t \phi + A\phi & = & \psi, & t \in (0,T), \\ \phi(0) & = & 0, & \end{array} \right.$$

has a solution ϕ such that

$$\phi \in W^{1,r}((0,T), L^s(\mathbb{R}^n_+)) \cap L^r((0,T), W^{2,s}(\mathbb{R}^n_+))$$

for all $r, s \in (1, \infty)$. By a density argument this implies that

$$\int_0^\infty \int_{\mathbb{R}^n_+} (u - v)\psi \mathrm{d}t\mathrm{d}x = 0$$

for all $\psi \in C_c^\infty((0,T), C_{c,\sigma}(\mathbb{R}^n_+))$ which shows that $u = v$. The pressure gradient now can be recovered by

$$\nabla p = (I - P)(\Delta u - (u \cdot \nabla)u).$$

This shows that we can obtain estimate (1.6) also for ∇p and Theorem 1.3 is proved. □

3 Global regularity in two dimensions

Here we prove that the weak solution constructed in the proof of Theorem 1.2 is unique and regular for all times $t > 0$, if we assume that dimension $n = 2$. Besides the results in Section 1 on weak solutions, for the proof a basic ingredient will be the estimate

$$\|v\|_4 \leq C \|\nabla v\|_2^{1/2} \|v\|_2^{1/2}, \tag{3.1}$$

valid for all $v \in H^1(\mathbb{R}^2_+)$, and which is a special case of the Gagliardo-Nirenberg inequality.

Proof. (of Theorem 1.5). Let $T \in (0, \infty]$ and u_k be the approximate sequence constructed in Proposition 2.1. The proof of Theorem 1.5 includes two steps.

Step 1: Here we first prove that $\nabla u \in L^\infty((0,T), L^2(\mathbb{R}^2_+))$. For this purpose we show that relation (2.11) holds with a constant C independent of $k \in \mathbb{N}$. Indeed, employing (3.1) we can estimate the the nonlinear term by

$$\begin{aligned} |(P(J_k^N u_k(t) \cdot \nabla)u_k(t), Au_k(t))| &\leq C\|u_k(t)\|_4 \|\nabla u_k(t)\|_4 \|Au_k(t)\|_2 \\ &\leq C\|u_k(t)\|_2^{1/2} \|\nabla u_k(t)\|_2 \|Au_k(t)\|_2^{3/2} \\ &\leq C\|u_k(t)\|_2^2 \|\nabla u_k(t)\|_2^4 + \frac{1}{2}\|Au_k(t)\|_2^2. \end{aligned}$$

Hence similar to (2.10) we deduce

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u_k(t)\|_2^2 + 2\|Au_k(t)\|_2^2 \\ &\leq C\left(\|u_k(t)\|_2^2 \|A^{1/2}u_k(t)\|_2^4 + \|f_k(t)\|_2^2\right) + \|Au_k(t)\|_2^2, \end{aligned} \tag{3.2}$$

for $t \in (0,T)$ and $k \in \mathbb{N}$, but now with a constant $C > 0$ independent of k. We set $\phi_k(t) := \|u_k(t)\|_2^2 \|A^{1/2}u_k(t)\|_2^2$. Rearranging (3.2) we therefore obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u_k(t)\|_2^2 \leq C\left(\phi_k(t)\|A^{1/2}u_k(t)\|_2^2 + \|f_k(t)\|_2^2\right).$$

Hence, having in mind that

$$\int_0^t \phi_k(s)\mathrm{d}s \leq \left(\|u_0(t)\|_2^2 + \|A^{-1/2}f\|_{2,2}^2\right)^2$$

for $t \in (0,T)$ and $k \in \mathbb{N}$, completely analogous to (2.11) we deduce

$$\|A^{1/2}u_k(t)\|_2^2 \leq C\left(\|A^{1/2}u_0\|_2^2 + \|f\|_{2,2}^2\right)\exp\left(C\left(\|u_0\|_2^2 + \|A^{-1/2}f\|_{2,2}^2\right)^2\right)$$

valid for all $t \in (0,T)$ and $k \in \mathbb{N}$. Consequently, we have that

$$A^{1/2}u_k \to A^{1/2}u \quad \text{weakly in} \quad L^\infty((0,T), L^2(\mathbb{R}_+^2)).$$

Note that integrating (3.2) over t also yields

$$\begin{aligned} &\|A^{1/2}u_k\|_{2,\infty}^2 + \|Au_k(t)\|_2^2 \\ &\leq C\left(\|u_k\|_{2,\infty}^2 \|A^{1/2}u_k(t)\|_{2,\infty}^2 \|A^{1/2}u_k(t)\|_{2,2}^2 + \|f_k(t)\|_2^2\right). \end{aligned}$$

By the uniform boundedness of the right hand side we therefore see that $Au \in L^2((0,T), L^2(\mathbb{R}_+^2))$. This in turn implies that also $P(u \cdot \nabla)u \in L^2((0,T), L^2(\mathbb{R}_+^2))$. In fact, applying the Hölder inequality and (3.1) gives us

$$\|P(u \cdot \nabla)u\|_{2,2} \leq \left(\|u\|_{2,\infty}\|\nabla u\|_{2,2}\|\nabla u\|_{2,\infty}\|Au\|_{2,\infty}\right)^{1/2},$$

and we deduce by using the equations (1.1) and (2.1) that also $\partial_t u, \nabla p \in L^2((0,T), L^2(\mathbb{R}_+^2))$.

Step 2: Uniqueness. Let v be any other weak solution of (1.1). According to Theorem 1.3 and Lemma 2.2 the difference $w := u - v$ satisfies

$$\left\{ \begin{array}{rcll} \partial_t w + Aw + P(w \cdot \nabla)u + P(v \cdot \nabla)w & = & 0, & t \in (0,T), \\ w(0) & = & 0, & \end{array} \right. \tag{3.3}$$

in $L^{4/3}((0,T), L^{4/3}(\mathbb{R}^2_+))$. On the other hand we infer from (3.1) that

$$\|w\|_{4,4} \le C\|w\|_{2,\infty}^{1/2}\|\nabla w\|_{2,2}^{1/2},$$

which means that w is an element of the dual space of $L^{4/3}((0,T), L^{4/3}(\mathbb{R}^2_+))$. Thus we may form the dual pairing of w and the single terms in (3.3) which yields

$$\begin{array}{rcl} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_2^2 + \|A^{1/2}w(t)\|_2^2 & = & \sum_{j=1}^{2}(\partial_j u, w^j w) \\ & \le & C\|A^{1/2}u(t)\|_2\|w(t)\|_4^2 \\ & \le & C\|A^{1/2}u(t)\|_{2,\infty}^2\|w(t)\|_2^2 + \frac{1}{2}\|A^{1/2}w(t)\|_2^2, \end{array}$$

and therefore that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_2^2 \le C\|w(t)\|_2^2, \quad t \in (0,T).$$

The lemma of Gronwall then implies $w = 0$ and the assertion follows. □

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